New blocks for the conformal bootstrap

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Outline

1. Back to the bootstrap?
2. Conformal blocks and radial quantization
3. Applications and outlook

Based on 1303.1111 and 1305.1321 with Hugh Osborn and Slava Rychkov.

Why study CFTs at all?

- Many interesting theories (statistical models, QCD in the IR, SCFTs) are strongly coupled
- Beyond perturbation theory: lattice, gauge-gravity, . . . ?
- Often: these theories are (almost) conformal $\rightarrow$ use this!
- Cf. 4d picture: proof of a-theorem [Komargodski-Schwimmer 2011] and RG asymptotics [Luty-Polchinski-Rattazzi 2012]

[ε expansion]

Success story for $d = 2$

- In 2d, emergence of Virasoro group
  $\rightarrow$ severe constraints on spectrum
- Many solvable models
- Constraints coming from conformal symmetry, unitarity and modular invariance
- Can we do the same for $d \geq 3$?

CFT_d without Lagrangians (I)

- Recall: conformal map defined as $g_{\mu\nu} \mapsto e^{-2\sigma(x)} g_{\mu\nu}$.

- CFT $=$ set of local* operators $\{\mathcal{O}_i\}_i$
  carrying spin ℓ and a scaling dimension Δ.
- Can have more symmetry (Lie or discrete groups).
- Every **primary** operator $\mathcal{O}$ has **descendants**:

$$\partial_\mu \mathcal{O}, \quad \partial^2 \mathcal{O}, \quad \partial_\mu \partial_\nu \mathcal{O}, \quad \ldots$$

  $=$ conformal multiplet.
- Same for other Lorentz reps: currents, tensors, fermions, ...

CFT$_d$ without Lagrangians (II)

- Two-pt functions $\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = \delta_{ij}\,|x-y|^{-2\Delta}$ are fixed
- Three-pt functions depend on theory:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \lambda_{123}\,|x_{12}|^{-\delta_{123}}|x_{23}|^{-\delta_{231}}|x_{31}|^{-\delta_{312}}$$

for $x_{ab} = x_a - x_b$, $\delta_{abc} = \Delta_a + \Delta_b - \Delta_c$.

- n-pt functions: weak constraints.

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle = (\text{scale factors}) \times g(u,v)$$

and u, v are cross ratios, e.g. $u = x_{12}^2 x_{34}^2 / x_{13}^2 x_{24}^2$.

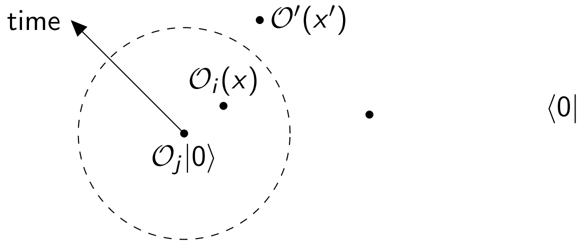
- Spectrum $\{\Delta_i\}$ + coefficients λ_{ijk} define the theory.

Consistency conditions? (I)

- Does any choice of $\{\Delta_i, \lambda_{ijk}\}$ define a CFT?
- Use operator product expansion

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum \lambda_{ijk} C(x, \partial_x)\mathcal{O}_k(0).$$

Proof: $\mathbb{R}^d \to \mathbb{R} \times S^{d-1}$, Hamiltonian $H|\Delta\rangle = \Delta|\Delta\rangle$.

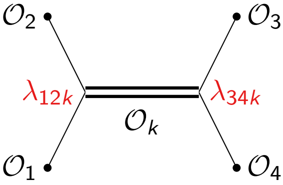


Consistency conditions? (2)

- OPE reduces n-pt functions to $(n-1)$-pt functions.
- $3 \mapsto 2$: nothing new.
- $4 \mapsto 3 \mapsto 2$: decomposition

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle \propto g(u,v) = \sum_k \lambda_{12k} \lambda_{34k} G_{\Delta_k, \ell_k}(u,v)$$

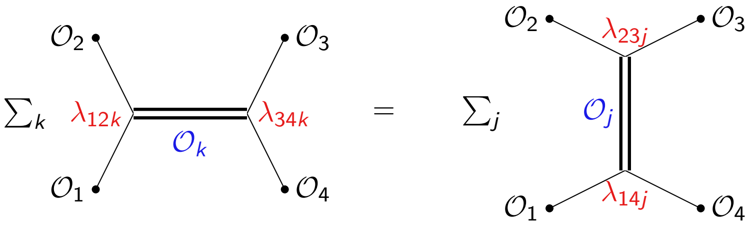
$$g(u,v) = \sum_k$$



- $G_{\Delta_k, \ell_k}(u,v)$ are *conformal partial waves* = *conformal blocks*.

Consistency conditions? (3)

- Can take OPE in any order – must be **crossing symmetric**:



- $\Rightarrow$ functional constraints!
  Take $\mathcal{O}_1 = \ldots = \mathcal{O}_4$ with dim. Δ, then

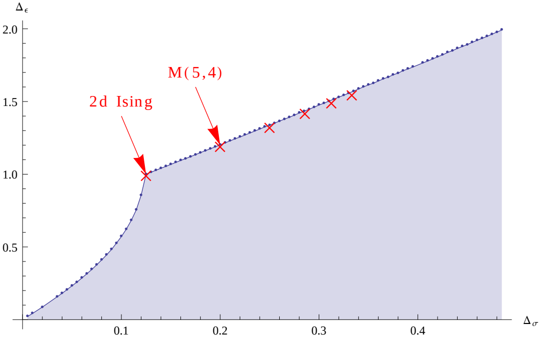
$$v^\Delta \sum_i \lambda_i^2 G_{\Delta_i,\ell_i}(u,v) = u^\Delta \sum_i \lambda_i^2 G_{\Delta_i,\ell_i}(v,u), \quad \lambda_i^2 \geq 0.$$

[Polyakov 1974, Rattazzi-Rychkov-Tonni-Vichi 2008]

Example results: bounds for 2d CFTs

- Bootstrap equation can be used to **exclude spectra**.
- Consider unitary 2d CFT with scalar σ:

$$\sigma \times \sigma \sim 1 + \varepsilon + \ldots + T_{\mu\nu} + \ldots$$



Constraint from $\langle\sigma\sigma\sigma\sigma\rangle$. Only global symmetry $= SL(2,\mathbb{C})$ used.

[Plot: El-Showk & Paulos hep-th/1211.2810.]

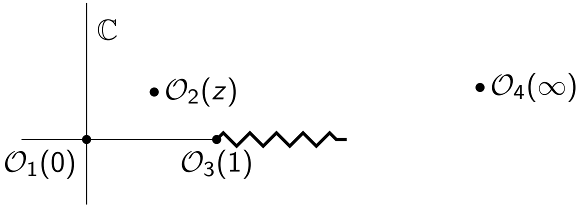
Sketch of method

bootstrap eqn: $v^{\Delta} \sum_i \lambda_i^2 G_{\Delta_i,\ell_i}(u,v) = u^{\Delta} \sum_i \lambda_i^2 G_{\Delta_i,\ell_i}(v,u), \quad \lambda_i^2 \geq 0.$

- Compute conformal blocks in d dimensions
- Cutoff for **scaling dimensions** $\Delta \leq \Delta^*$ in the spectrum
  $+$ discretize [not in state-of-the-art methods]
- Cutoff for **spins** ℓ
- Taylor expand around a point in (u,v) space
  $\Rightarrow$ linear programming problem:
  given a spectrum $\{\Delta_i, \ell_i\}$, can a consistent CFT exist?
- How harmful are these cutoffs?

Which geometry for the 4-pt function?

- Wick rotate to Euclidean space, put points on plane:



- Change of coordinates for cross ratios:

$$u \to z\bar{z}, \quad v \to (1-z)(1-\bar{z}).$$

Conformal blocks: what is known?

- Old question dating back to [Ferrara-Gatto-Grillo-Parisi 1972]
- Breakthrough: Dolan & Osborn (2001-2011);
  closed-form solutions for even d.
  If $\Delta_{ab} = \Delta_a - \Delta_b$, then $d = 4$ CBs are:

$$G_{\Delta,\ell}(z,\bar{z}) = \frac{1}{\ell+1}\frac{z\bar{z}}{z-\bar{z}}\left[k_{\Delta+\ell}(z)k_{\Delta-\ell-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right]$$

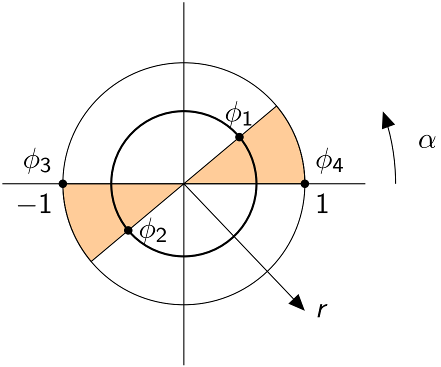
$$k_\beta(z) = z^{\beta/2}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}(\beta+\Delta_{12}), \frac{1}{2}(\beta-\Delta_{34}) \\ \beta\end{matrix}; z\right].$$

- Closed-form solutions for $d = 3, 5, \ldots$ might not exist.
  Keep on looking?

Use new "ρ" coordinate

[Pappadopulo-Rychkov-Espin-Rattazzi 2012, MH-Rychkov 2013]

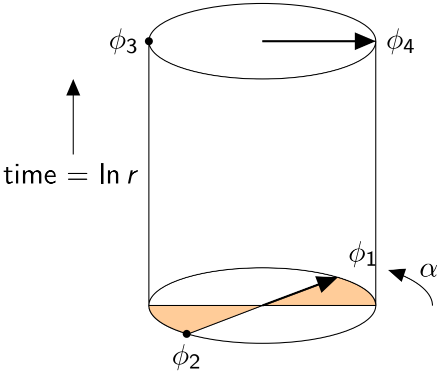
- Use conformal freedom to place points symmetrically around origin:



- replace $u, v \to \rho = re^{i\alpha}$.

Radial quantization in the ρ coordinate

- Want to calculate $\langle \phi_4 \phi_3 \phi_2 \phi_1 \rangle = \langle 0 | \phi_4 \phi_3 \phi_2 \phi_1 | 0 \rangle$.
- Do radial quantization in this new coordinate ($\rho = re^{i\alpha}$):



Rad. qtization in the ρ coordinate (II)

- Insert identity at some intermediate time:

$$\langle 0|\phi_4\phi_3\phi_2\phi_1|0\rangle = \sum_{\text{states } s} \langle 0|\phi_4\phi_3|s\rangle\langle s|\phi_2\phi_1|0\rangle.$$

- Matrix elements for a state with energy E and spin ℓ given by

$$r^E \cdot \text{Gegenbauer}_\ell^{(d)}(\cos\alpha).$$

- Hilbert space factorizes over primaries and descendants; conformal blocks thus given by

$$G_{\mathcal{O}}(\rho = re^{i\alpha}) = \sum_{\text{desc. of } \mathcal{O}} B_{E,\ell}\, r^E \text{Gegenbauer}_\ell^{(d)}(\cos\alpha).$$

Properties of CBs in the ρ coordinate

- Previous slide: representation

$$G_{\mathcal{O}}(\rho = re^{i\alpha}) = \sum_{\text{desc. of } \mathcal{O}} B_{E,\ell}\, r^E \text{Gegenbauer}_\ell^{(d)}(\cos\alpha).$$

- Good convergence: we use $\rho \simeq 0.17$, so few descendants contribute:

$$G_{\Delta,\ell} = r^\Delta [1 \pm 3\% \text{ error}].$$

  Can be shown to be 'best possible' radial coordinate.

- Coefficients $B_{E,\ell}$ can be calculated recursively.
  Depend analytically on $d \to$ non-perturbative def. of theories in fractional dimensions.

Truncating the CB expansion

- Recall CB expansion

$$\langle \phi\phi\phi\phi \rangle \sim g(u,v), \qquad g(u,v) = \sum (\lambda_{\mathcal{O}})^2 G_{\Delta,\ell}(\rho).$$

- Now discard contribution of all operators with dimension $\Delta \geq E_*$ and fix ρ.

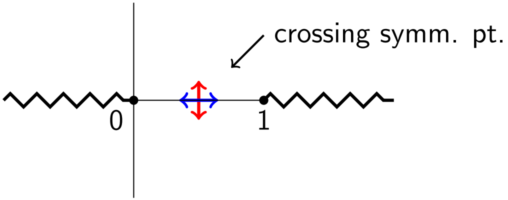
$$\text{error } \lesssim \text{cst. } E_*^{4\Delta_\phi} |\rho|^{E_*}$$

  $\rightarrow$ tail vanishes exponentially ($\rho \simeq 0.17$).
- In a unitary theory, high spin $\Rightarrow$ high dimension: $\Delta \geq \ell + d - 2$. Can thus also discard high spin operators.
- Depends on kinematics, additional results in Minkowski space [Komargodski-Zhiboedov 2012, Fitzpatrick et al. 2012]

Efficient algorithm to calculate CBs

[MH-Osborn-Rychkov 2013]

- Still need to calculate coefficients $B_{E,\ell}$ up to arbitrary order.
- Limit to real line $\rho = r$ or $z = \bar{z}$ (= *diagonal*):



- Crucial ingredient: quadratic and quartic Casimirs

$$L_{AB}L^{AB}, \quad L_{AB}L^{BC}L_{CD}L^{DA}$$

of the conformal algebra $SO(d+1,1)$.

- Generalizes closed-form results in terms of ${}_4F_3$ hypergeometrics.

Summary and consequences for bootstrap

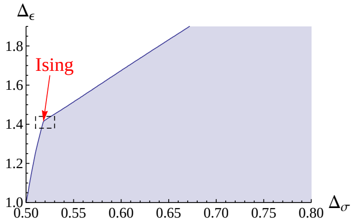
- Developed new, quickly converging, representation for conformal blocks in *d* dimensions
- Tools in place to analyze mixed correlation functions, i.e.

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle, \quad \phi_i \neq \phi_j.$$

- Significant speed-up for numerics
- Numerical method put on solid footing: errors under control
- First fully analytical and rigorous proof of existence of bootstrap bounds
  Method: **toy bootstrap**, ignores distinction between primaries/descendants $\Rightarrow$ very weak in practice.

Improving Ising 3d results?

- Extracting spectrum for Ising 3d efficiently and to high precision would be milestone for bootstrap method
- OPE: $\sigma \times \sigma \sim 1 + \varepsilon + \ldots$. Bootstrap results in agreement with lattice/RG data [El-Showk et al. hep-th/1203.6064]:



- Can analyze $\langle \varepsilon\varepsilon\varepsilon\varepsilon \rangle$ and $\langle \sigma\sigma\varepsilon\varepsilon \rangle$ as well (coherently). How does this change bounds?

Extra: some (selected) work in progress

- Stress tensor/conserved current correlators [Dymarsky, talk at BttB3]

$$\langle TTTT \rangle, \quad \langle JJJJ \rangle. \qquad (T = T_{\mu\nu}, \quad J = J_\mu)$$

- Bootstrap with $\mathcal{N} = 2$ or $\mathcal{N} = 4$ susy [Beem-Rastelli-Van Rees, ...]
- For non-scalar operators, bootstrap still wide open.
  Better understanding of spinning/susy/fermionic/... conformal blocks will be crucial to make progress.