RF Engineering Basic Concepts: The Smith Chart

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Abstract
The Smith chart is a very valuable and important tool that facilitates interpretation of S-parameter measurements. This paper will give a brief overview on why and more importantly on how to use the chart. Its definition as well as an introduction on how to navigate inside the cart are illustrated. Useful examples show the broad possibilities for usage of the chart in a variety of applications.

1 Motivation

With the equipment at hand today, it has become rather easy to measure the reflection factor Γ even for complicated networks. In the "good old days" though, this was done measuring the electric field strength[1] at of a coaxial measurement line with a slit at different positions in axial direction (Fig. 1). A

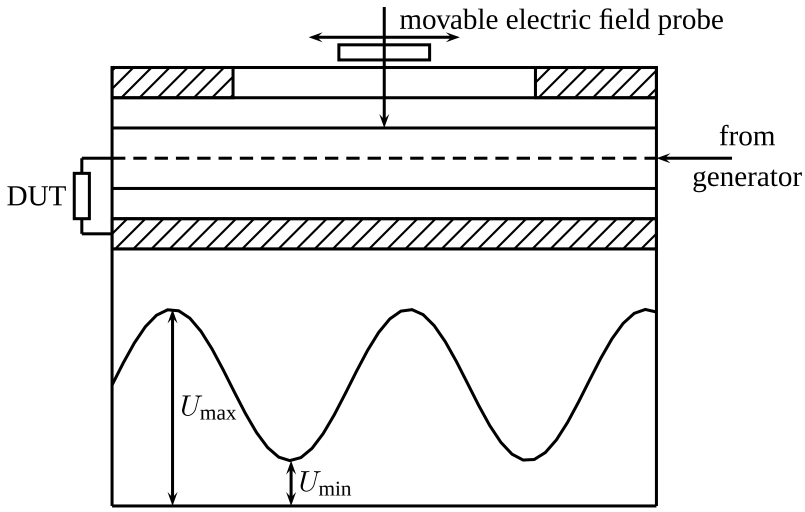

Fig. 1: Schematic view of a measurement setup used to determine the reflection coefficient as well as the voltage standing wave ratio of a device under test (DUT) [1].

small electric field probe, protruding into the field region of the coaxial line near the outer conductor, was moved along the line. Its signal was picked up and displayed on a micro–voltmeter after rectification via a microwave diode. While moving the probe, field maxima and minima as well as their position and spacing could be found. From this the reflection factor Γ and the *V*oltage *S*tanding *W*ave *R*atio (VSWR or SWR) could be determined using following definitions:

- Γ is defined as the ratio of the electrical field strength E of the reflected wave over the forward traveling wave:

$$\Gamma = \frac{E \text{ of reflected wave}}{E \text{ of forward traveling wave}} \tag{1}$$

- The VSWR is defined as the ratio of maximum to minimum measured voltage:

$$\text{VSWR} = \frac{U_{\text{max}}}{U_{\text{min}}} = \frac{1+|\Gamma|}{1-|\Gamma|} \tag{2}$$

[1] The electrical field strength was used since it can be measured considerably easier than the magnetic field strength.

Although today this measurements are far easier to conduct, the definitions of the aforementioned quantities are still valid. Also their importance has not diminished in the field of microwave engineering and so the reflection coefficient as well as the VSWR are still a vital part of the everyday life of a microwave engineer be it for simulations or measurements.

A special diagram is widely used to visualize and to facilitate the determination these quantities. Since it was invented in 1939 by the engineer Phillip Smith, it is simply known as the Smith chart [2].

2 Definition of the Smith Chart

The Smith chart provides a graphical representation of Γ that permits the determination of quantities such as the VSWR or the terminating impedance of a device under test (DUT). It uses a bilinear Moebius transformation, projecting the complex impedance plane onto the complex Γ plane:

$$\Gamma = \frac{Z - Z_0}{Z + Z_0} \text{ with } Z = R + \mathrm{j}\, X \tag{3}$$

As can be seen in Fig. 2 the half plane with positive real part of impedance Z is mapped onto the interior of the unit circle of the Γ plane. For a detailed calculation see Appendix A.

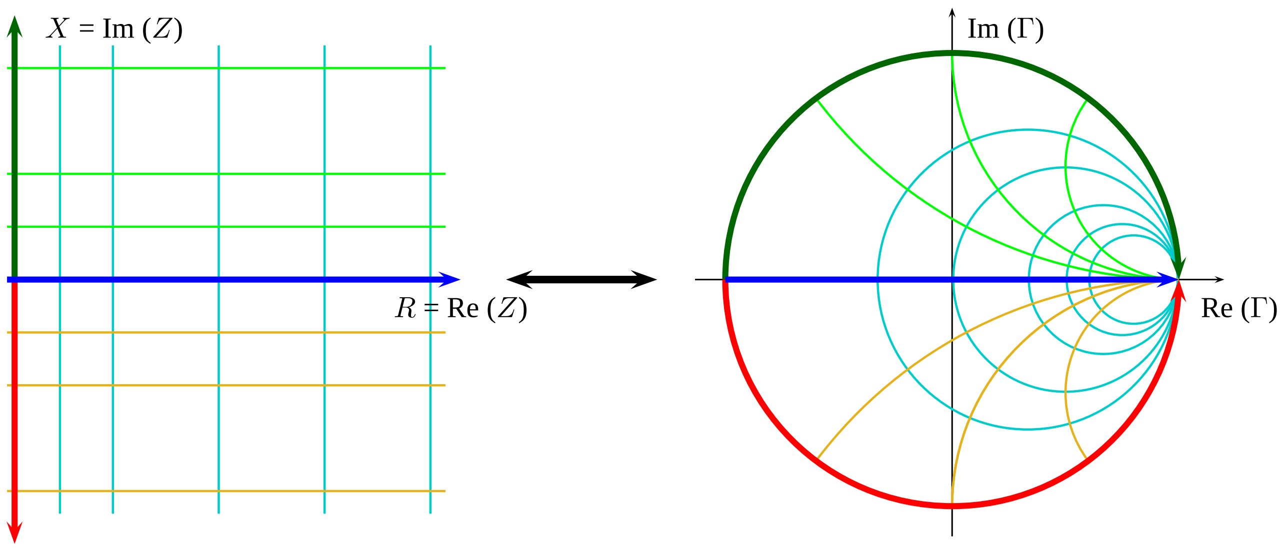

Fig. 2: Illustration of the Moebius transform from the complex impedance plane to the Γ plane commonly known as Smith chart.

2.1 Properties of the Transformation

In general, this transformation has two main properties:

- generalized circles are transformed into generalized circles (note that a straight line is nothing else than a circle with infinite radius and is therefore mapped as a circle in the Smith chart)
- angles are preserved locally

Fig. 3 illustrates how certain basic shapes transform from the impedance to the Γ plane.

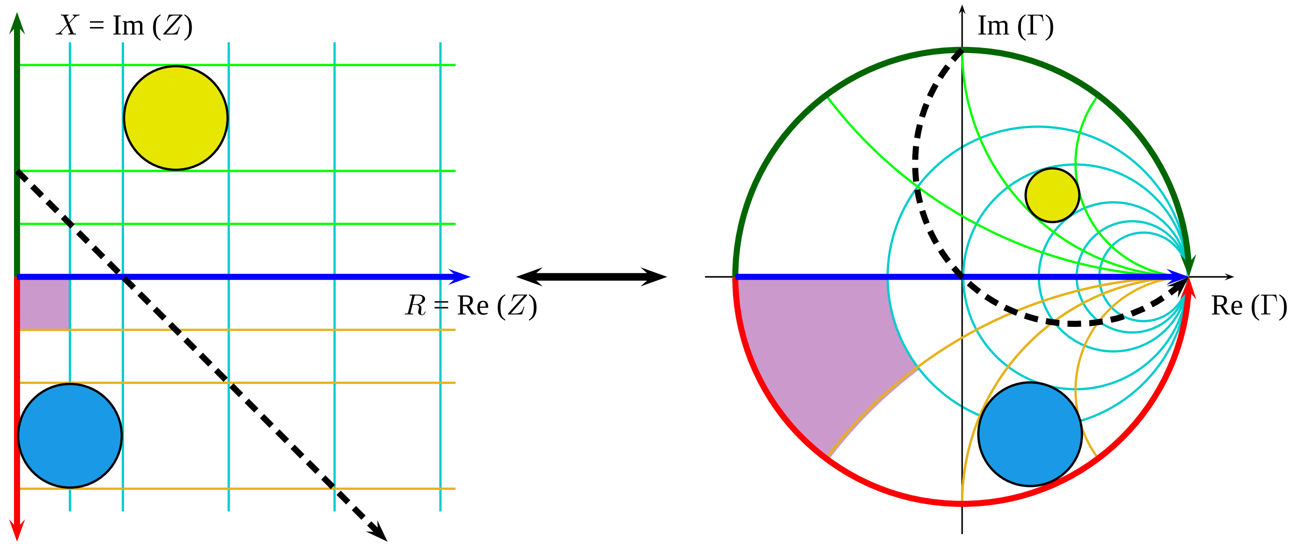

Fig. 3: Illustration of the transformation of basic shapes from the Z to the Γ plane.

2.2 Normalization

The Smith chart is usually normalized to a terminating impedance Z_0 (= real):

$$z = \frac{Z}{Z_0} \tag{4}$$

This leads to a simplification of the transform:

$$\Gamma = \frac{z-1}{z+1} \Leftrightarrow z = \frac{1+|\Gamma|}{1-|\Gamma|} \tag{5}$$

Although Z = 50 Ω is the most common reference impedance (characteristic impedance of coaxial cables) and many applications use this normalization, there is any other real and positive value possible. *Therefore it is crucial to check the normalization before using any chart.*

Commonly used charts that map the impedance plane onto the Γ plane always look confusing at first, as many circles are depicted (Fig. 4). Keep in mind that all of them can be calculated as shown in Appendix A and that this representation is the same as shown in all figures before — it just contains more circles.

2.3 Admittance plane

The Moebius transform that generates the Smith chart provides also a mapping of the complex admittance plane ($Y = \frac{1}{Z}$ or normalized $y = \frac{1}{z}$) into the same chart:

$$\Gamma = -\frac{y-1}{y+1} = -\frac{Y-Y_0}{Y+Y_0} = -\frac{1/Z-1/Z_0}{1/Z+1/Z_0} = \frac{Z-Z_0}{Z+Z_0} = \frac{z-1}{z+1} \tag{6}$$

Using this transformation, the result is the same chart, only mirrored at the center of the Smith chart (Fig. 5). Often both mappings, the admittance and the impedance plane, are combined into one chart, which looks even more confusing (see last page). For reasons of simplicity all illustrations in this paper will use only the mapping from the impedance to the Γ plane.

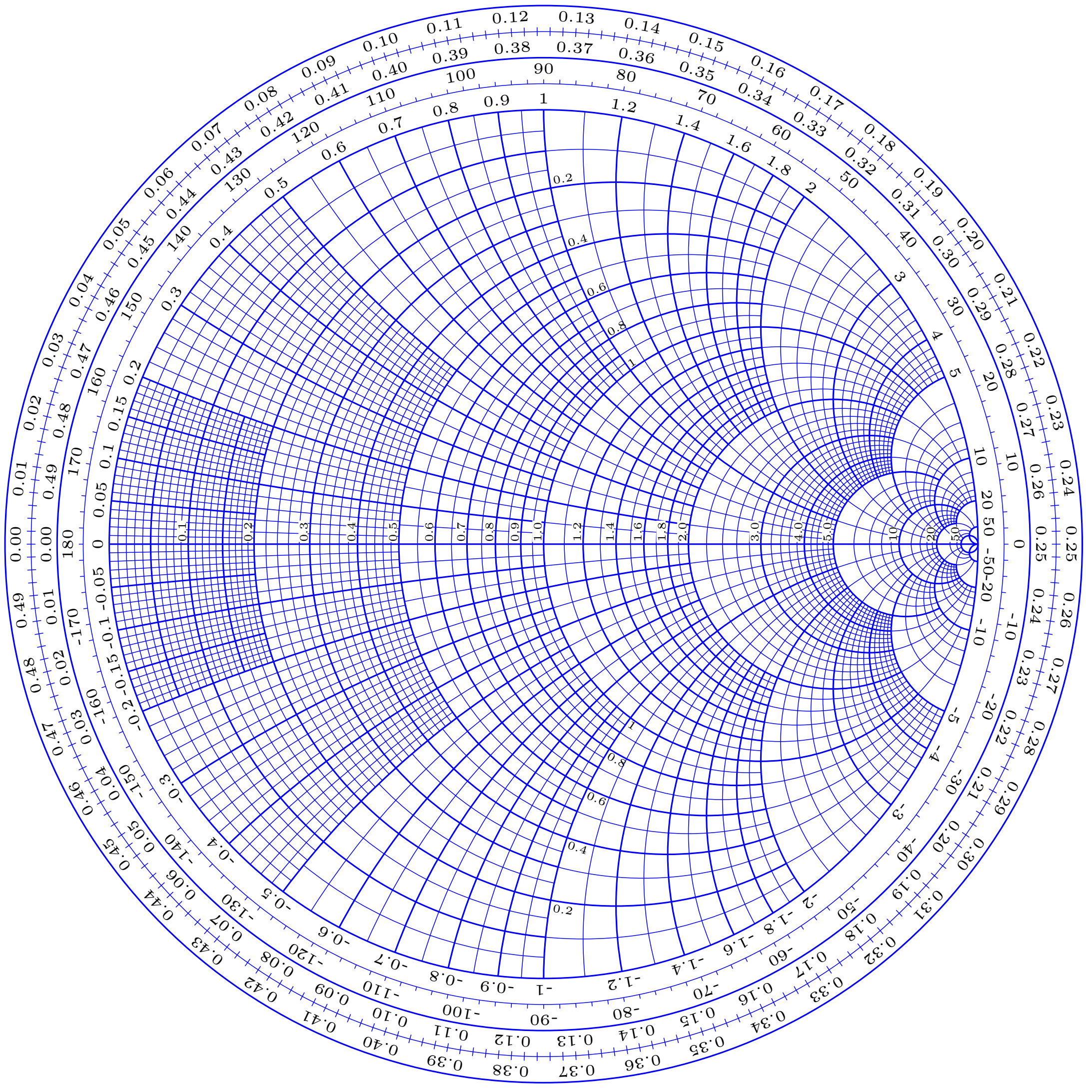

Fig. 4: Example for a commonly used Smith chart.

3 Navigation in the Smith chart

The representation of circuit elements in the Smith chart is discussed in this chapter starting with the important points inside the chart. Then several examples of circuit elements will be given and their representation in the chart will be illustrated.

3.1 Important points

There are three important points in the chart:

1. Open circuit with $\Gamma = 1, z \to \infty$
2. Short circuit with $\Gamma = -1, z = 0$
3. Matched load with $\Gamma = 0, z = 1$

They all are located on the real axis at the beginning, the end and the center of the circle (Fig. 6). The upper half of the chart is inductive, since it corresponds to the positive imaginary part of the impedance. The lower half is capacitive as it is corresponding to the negative imaginary part of the impedance.

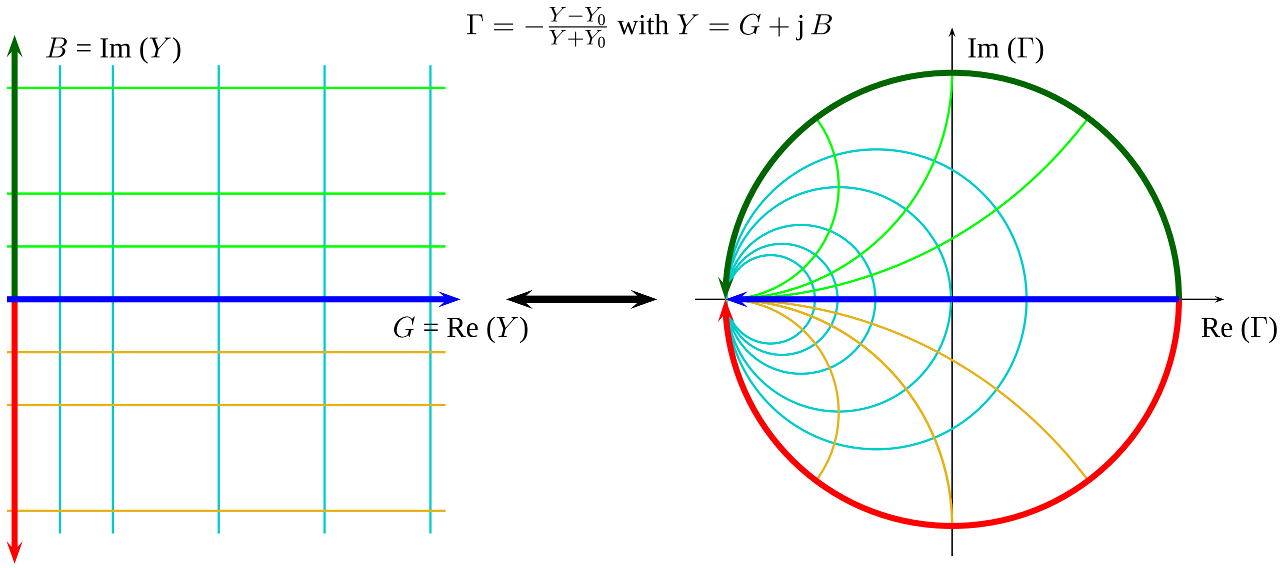

Fig. 5: Mapping of the admittance plane into the Γ plane.

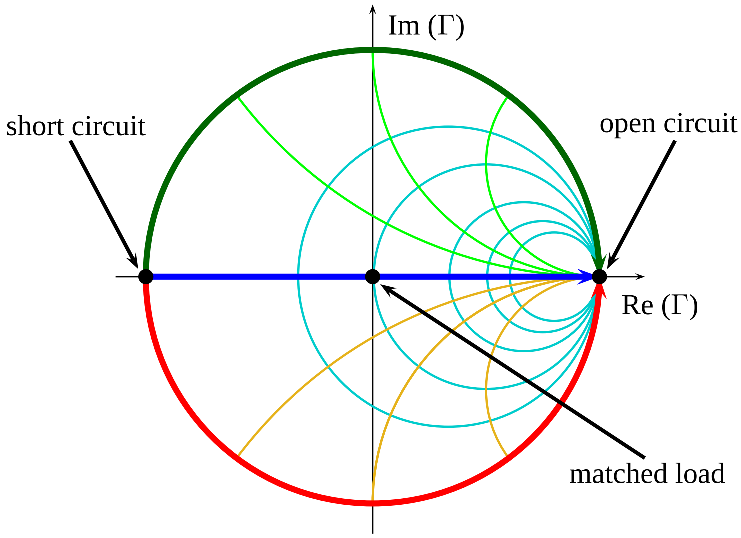

Fig. 6: Important points in the Smith chart.

Concentric circles around the diagram center represent constant reflection factors (Fig. 7). Their radius is directly proportional to the magnitude of Γ, therefore a radius of 0.5 corresponds to reflection of 3 dB (half of the signal is reflected) whereas the outermost circle (radius = 1) represents full reflection. Therefore matching problems are easily visualized in the Smith chart since a mismatch will lead to a reflection coefficient larger than 0 (equation (7)).

$$\text{Power into the load = forward power - reflected power: } P = \frac{1}{2}\left(|a|^2 - |b|^2\right) = \frac{|a|^2}{2}\left(1 - |\Gamma|^2\right) \quad (7)$$

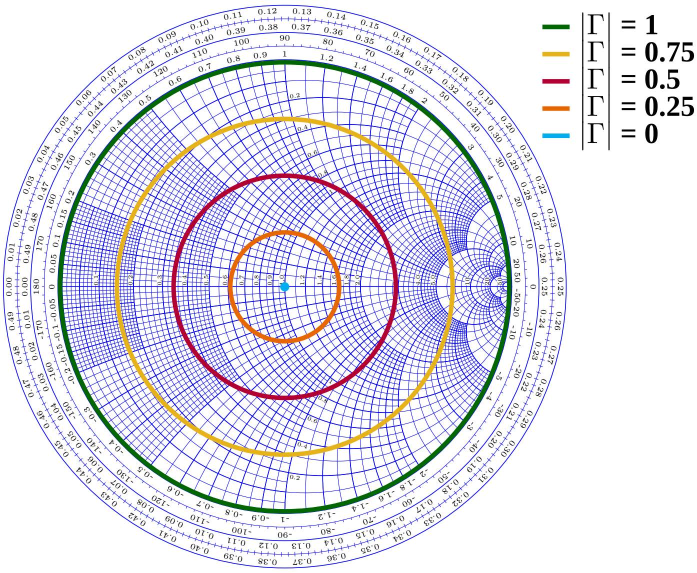

Fig. 7: Illustration of circles representing a constant reflection factor.

In equation (7) the European notation[2] is used, where power = $\frac{|a|^2}{2}$. Furthermore $(1 - |\Gamma|^2)$ corresponds to the mismatch loss.

Although only the mapping of the impedance plane to the Γ plane is used, one can easily use it to determine the admittance since

$$\Gamma(\frac{1}{z}) = \frac{\frac{1}{z} - 1}{\frac{1}{z} + 1} = \frac{1 - z}{1 + z} = \left(\frac{z - 1}{z + 1}\right) \text{ or } \Gamma(\frac{1}{z}) = -\Gamma(z) \tag{8}$$

In the chart this can be visualized by rotating the vector of a certain impedance by 180° (Fig. 8).

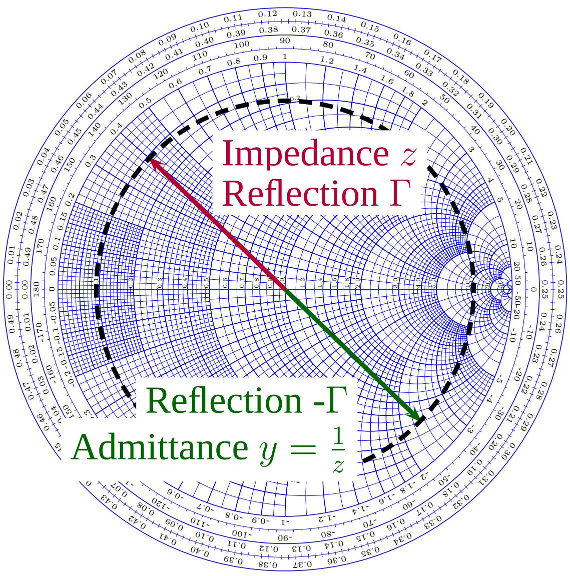

Fig. 8: Conversion of an impedance to the corresponding emittance in the Smith chart.

[2]The commonly used notation in the US is power = $|a|^2$. These conventions have no impact on S–parameters but they are relevant for absolute power calculation. Since this is rarely used in the Smith chart, the used definition is not critical for this paper.

3.2 Adding impedances in series and parallel (shunt)

A lumped element with variable impedance connected in series as an example of a simple circuit. The corresponding signature of such a circuit for a variable inductance and a variable capacitor is a circle. Depending on the type of the impedance this circle is passed through clockwise (inductance) or counterclockwise (Fig. 9). If a lumped element is added in parallel, the situation is the same as it is for an

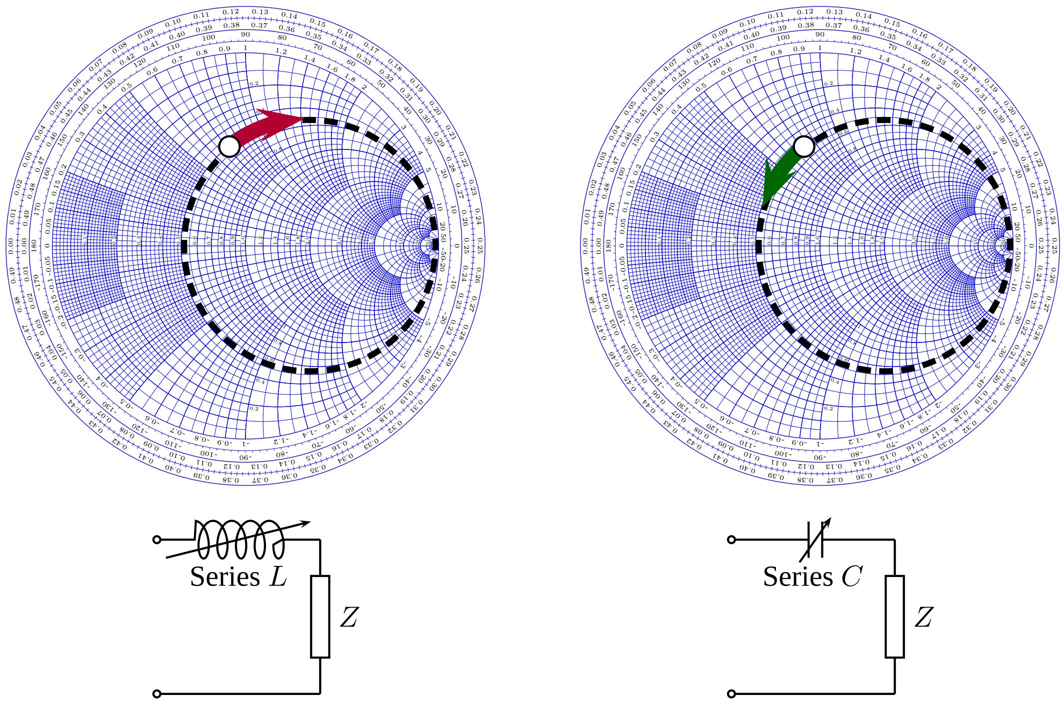

Fig. 9: Traces of circuits with variable impedances connected in series.

element connected in series mirrored by 180° (Fig. 10). This corresponds to taking the same points in the admittance mapping. Summarizing both cases, one ends up with a simple rule for navigation in the Smith chart:

1 *For elements connected in series use the circles in the impedance plane. Go clockwise for an added inductance and anticlockwise for an added capacitor. For elements in parallel use the circles in the admittance plane. Go clockwise for an added capacitor and anticlockwise for an added inductance.*

This rule can be illustrated as shown in Fig. 11

3.3 Impedance transformation by transmission line

The S–matrix of an ideal, lossless transmission line of length l is given by

$$S = \begin{bmatrix} 0 & \mathrm{e}^{-\mathrm{j}\beta l} \\ \mathrm{e}^{-\mathrm{j}\beta l} & 0 \end{bmatrix} \tag{9}$$

where $\beta = \frac{2\pi}{\lambda}$ is the propagation coefficient with the wavelength λ ($\lambda = \lambda_0$ for $\epsilon_r = 1$).

When adding a piece of coaxial line, we turn clockwise on the corresponding circle leading to a transformation of the reflection factor Γ_{load} (without line) to the new reflection factor $\Gamma_{\text{in}} = \Gamma_{\text{load}}\mathrm{e}^{-\mathrm{j}2\beta l}$. Graphically spoken, this means that the vector corresponding to Γ_{in} is rotated clockwise by an angle of $2\beta l$ (Fig. 12).

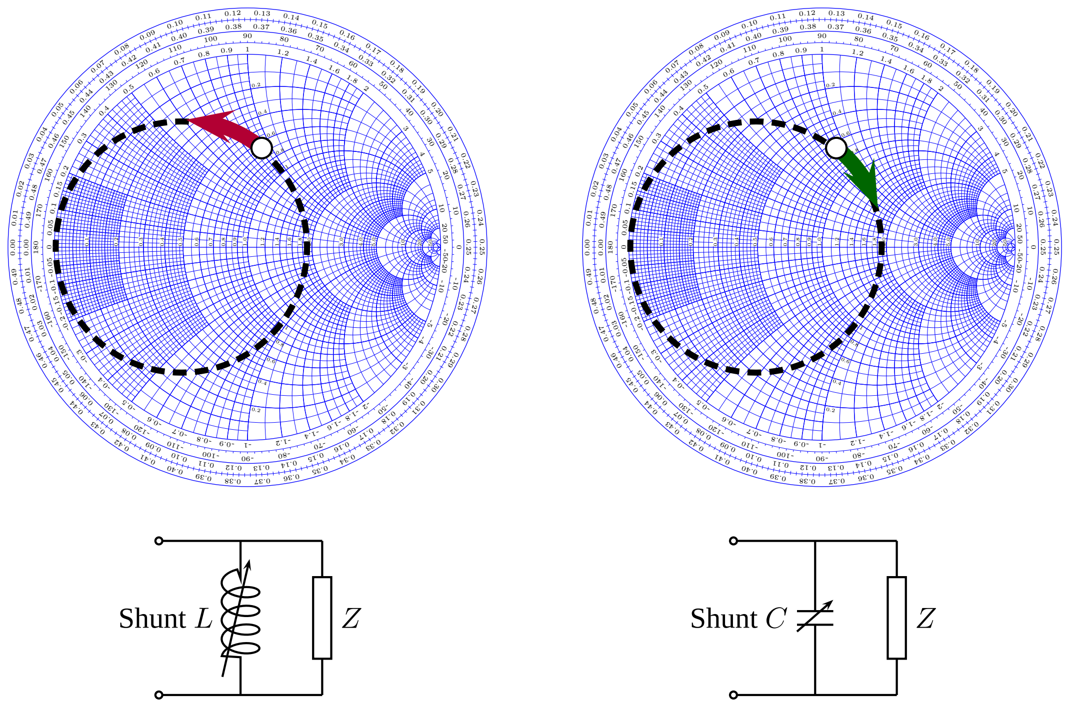

Fig. 10: Traces of circuits with variable impedances connected in parallel.

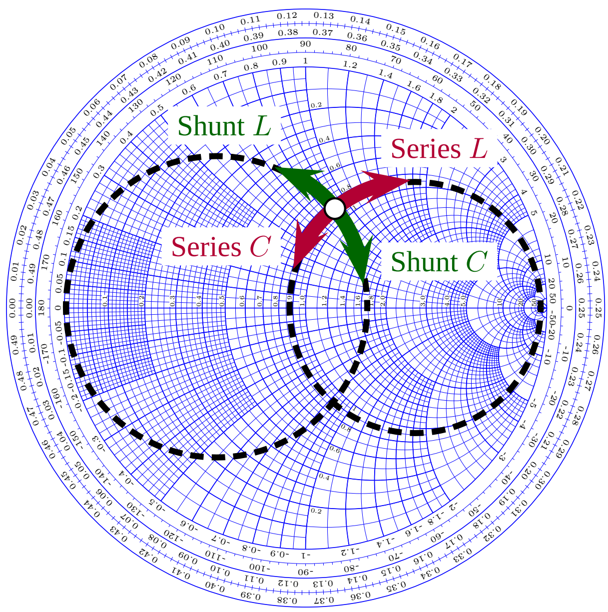

Fig. 11: Illustration of navigation in the Smith chart when adding lumped elements.

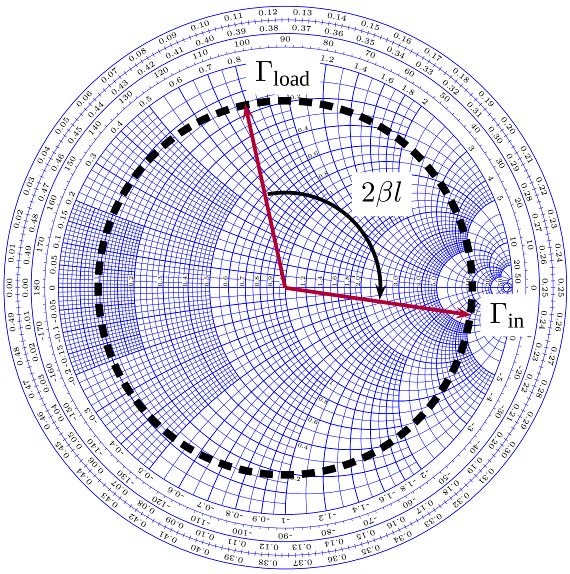

Fig. 12: Illustration of adding a transmission line of length l to an impedance.

The peculiarity of a transmission line is, that it behaves either as an inductance, a capacitor or a resistor depending on its length. The impedance of such a line (if lossless!) is given by

$$Z_{\text{in}} = \mathrm{j}Z_0 \tan(\beta l) \tag{10}$$

The function in equation (10) has a pole at a transmission line length of $\frac{\lambda}{4}$ (Fig. 13). Therefore adding a

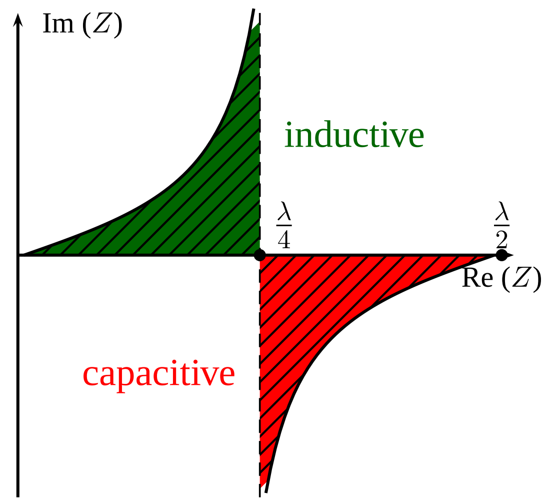

Fig. 13: Impedance of a transmission line as a function of its length l.

transmission line with this length results in a change of Γ by a factor -1:

$$\Gamma_{\text{in}} = \Gamma_{\text{load}}\mathrm{e}^{-\mathrm{j}2\beta l} = \Gamma_{\text{load}}\mathrm{e}^{-\mathrm{j}2(\frac{2\pi}{\lambda})l} \overset{l=\frac{\lambda}{4}}{=} \Gamma_{\text{load}}\mathrm{e}^{-\mathrm{j}\pi} = -\Gamma_{\text{load}} \tag{11}$$

Again this is equivalent to changing the original impedance z to it's admittance $\frac{1}{z}$ or the clockwise movement of the impedance vector by 180°. Especially when starting with a short circuit (at -1 in the Smith chart), adding a transmission line of length $\frac{\lambda}{4}$ transforms it into an open circuit (at $+1$ in the Smith chart).

A line that is shorter than $\frac{\lambda}{4}$ behaves as an inductance, while a line that is longer acts a capacitor. Since these properties of transmission lines are used very often, the Smith chart usually has a ruler around its border, where one can read $\frac{l}{\lambda}$ — it is the parametrization of the outermost circle.

3.4 Examples of different 2–ports

In general, the reflection coefficient when looking through a 2–port Γ_{in} is given via the S–matrix of the 2–port and the reflection coefficient of the load Γ_{load}:

$$\Gamma_{\text{in}} = S_{11} + \frac{S_{12}S_{21}\Gamma_{\text{load}}}{1 - S_{22}\Gamma_{\text{load}}} \tag{12}$$

In general, the outer circle of the Smith chart as well as its real axis are mapped to other circles and lines.

In the following three examples of different 2–ports are given along with their S–matrix, and their representation in the Smith chart is discussed. For illustration, a simplified Smith chart consisting of the outermost circle and the real axis only is used for simplicity reasons.

3.4.1 Transmission line $\frac{\lambda}{16}$

The S–matrix of a $\frac{\lambda}{16}$ transmission line is

$$S = \begin{bmatrix} 0 & e^{-j\frac{\pi}{8}} \\ e^{-j\frac{\pi}{8}} & 0 \end{bmatrix} \tag{13}$$

with the resulting reflection coefficient

$$\Gamma_{\text{in}} = \Gamma_{\text{load}} e^{-j\frac{\pi}{4}} \tag{14}$$

This corresponds to a rotation of the real axis of the Smith chart by an angle of 45° (Fig. 14) and hence a change of the reference plane of the chart (Fig. 14). Consider for example a transmission line terminated by a short and hence $\Gamma_{\text{load}} = -1$. The resultinig reflection coefficient is then equal to $\Gamma_{\text{in}} = e^{-j\frac{\pi}{4}}$.

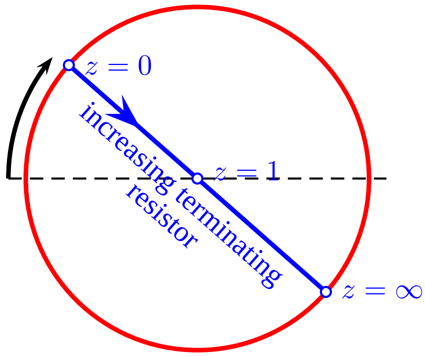

Fig. 14: Rotation of the reference plane of the Smith chart when adding a transmission line.

3.4.2 Attenuator 3dB

The S–matrix of an attenuator is given by

$$S = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \tag{15}$$

The resulting reflection coefficient is

$$\Gamma_{\text{in}} = \frac{\Gamma_{\text{load}}}{2} \tag{16}$$

In the Smith chart, the connection of such an attenuator causes the outermost circle to shrink to a radius of 0.5[3] (Fig. 15).

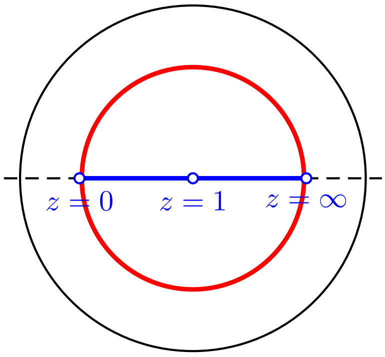

Fig. 15: Illustration of the appearance of an attenuator in the Smith chart.

3.4.3 Variable load resistor

Adding a variable load resistor ($0 < z < \infty$) is the simplest case that can be depicted in the Smith chart. It means moving through the chart along its real axis (Fig. 16).

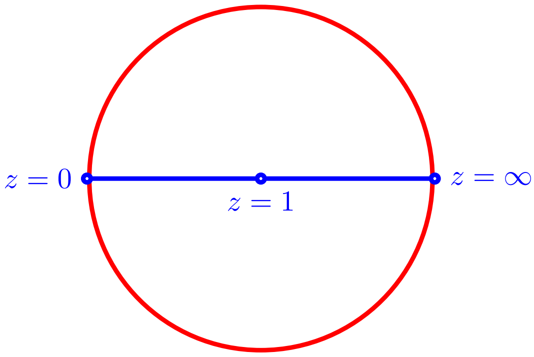

Fig. 16: A variable load resistor in the simplified Smith chart. Since the impedance has a real part only, the signal remains on the real axis of the Γ plane.

4 Advantages of the Smith chart – a summary

- The diagram offers a compact and handy representation of all passive impedances[4] from 0 to ∞. Impedances with negative real part such as reflection amplifier or any other active device would show up outside the Smith chart.
- Impedance mismatch is easily spotted in the chart.
- Since the mapping converts impedances or admittances ($y = \frac{1}{z}$) into reflection factors and vice versa, it is particularly interesting for studies in the radio frequency and microwave domain. For convenience reasons electrical quantities are usually expressed in terms of direct or forward waves and reflected or backwards waves in these frequency ranges instead of voltages and currents used at lower frequencies.

[3] An attenuation of 3 dB corresponds to a reduction by a factor 2 in power.
[4] Passive impedances are impedances with positive real part.

– The transition between impedance and admittance in the chart is particularly easy: $\Gamma(\mathrm{y} = \frac{1}{z}) = -\Gamma(z)$
– Furthermore the reference plane in the Smith chart can be moved very easily by adding a transmission line of proper length (3.4.1).
– Many Smith charts have rulers below the complex Γ plane from which a variety of quantities such as e. g. the return loss can be determined. For a more detailed discussion see Appendix B.

5 Examples for applications of the Smith chart

In this section two practical examples of common problems are given, where the use of the Smith chart facilitates their solving very much.

5.1 A step in characteristic impedance

Consider a junction between two infinitely short cables, one with a characteristic impedance of 50Ω and the other with 75Ω (Fig. 17). The waves running into each port are denoted with a_i ($i = 1, 2$) whereas

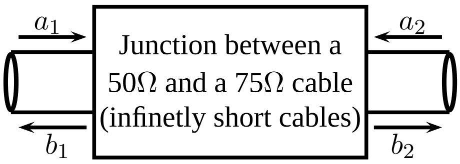

Fig. 17: Illustration of the junction between a coaxial cable with 50Ω characteristic impedance and another with 75Ω characteristic impedance respectively. Infinetly short cables are assumed – only the junction is considered.

the waves coming out of every point are denoted with b_i. The reflection coefficient for port 1 is then calculated as:

$$\Gamma_1 = \frac{Z - Z_1}{Z + Z_1} = \frac{75 - 50}{75 + 50} = 0.2 \tag{17}$$

Thus the voltage of the reflected wave at port 1 is 20% of the incident wave ($a_2 = a_1 \cdot 0.2$) and the reflected power at port 1 is 4%[5]. From conservation of energy, the transmitted power has to be 96% and it follows that b_2^2 = 0.96.

A peculiarity here is that the transmitted energy is *higher* than the energy of the incident wave, since E_{incident} = 1, $E_{\text{reflected}}$ = 0.2 and therefore $E_{\text{transmitted}} = E_{\text{incident}} + E_{\text{reflected}}$ = 1.2. The transmission coefficient t thus is t = 1 + Γ. Also note that this structure is not symmetric ($S_{11} \neq S_{22}$), but only reciprocal ($S_{21} = S_{12}$).

The visualization of this structure in the Smith chart is considerably easy, since all impedances are real and thus all vectors are located on the real axis (Fig. 18).

As stated before, the reflection coefficient is defined with respect to voltages. For currents its sign inverts and thus a positive reflection coefficient in terms of voltage definition gets negative when defined with respect to current.

For a more general case, e.g. Z_1 = 50 Ω and Z_2 = 50 + j80 Ω, the vectors in the chart are depicted in Fig. 19.

[5]Power is proportional to Γ^2 and thus $0.2^2 = 0.04$

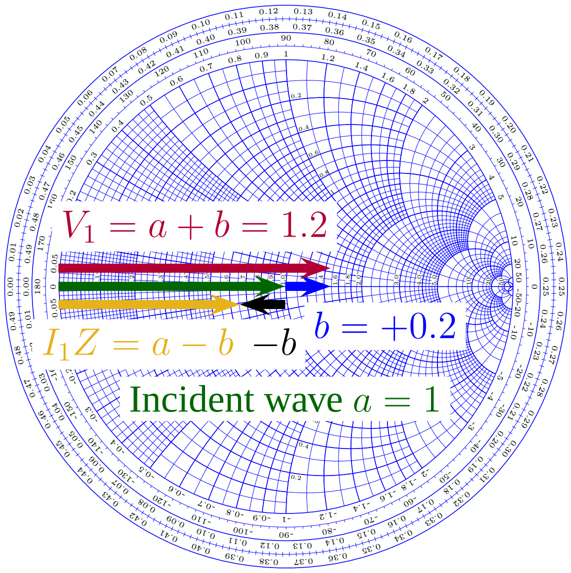

Fig. 18: Visualization of the two port formed by the two cables of different characteristic impedance.

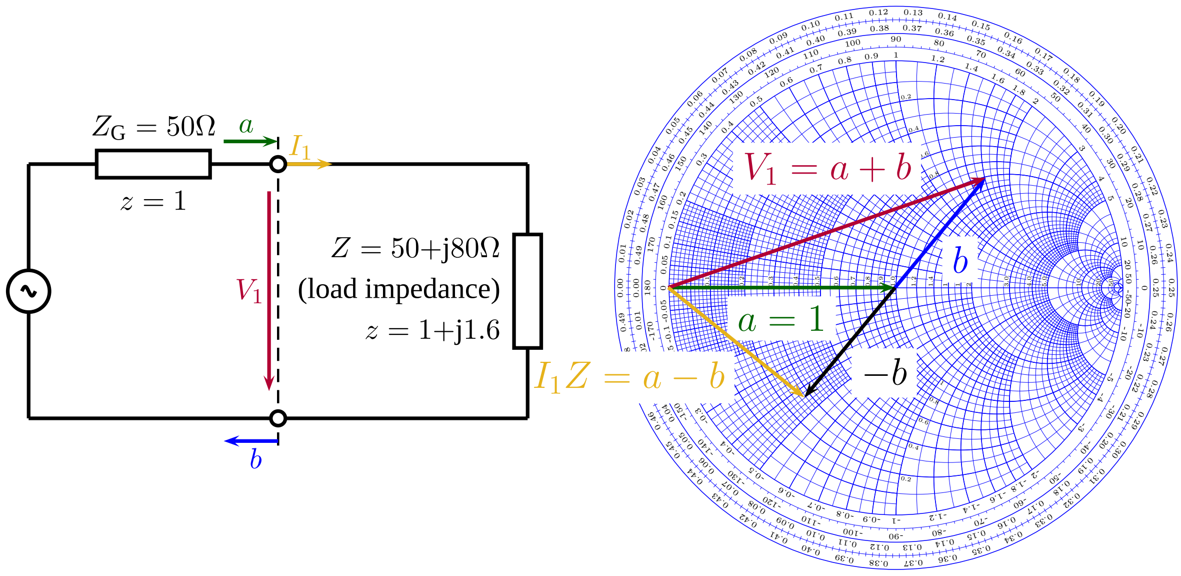

Fig. 19: Visualization of the two port depicted on the left in the Smith chart.

5.2 Determination of the Q–factors of a cavity

One of the most common cases where the Smith chart is used is the determination of the quality factor of a cavity. This section is dedicated to the illustration of this task.

A cavity can be described by a parallel RLC circuit (Fig. 20) where the resonance condition is given when:

$$\omega L = \frac{1}{\omega C} \tag{18}$$

This leads to the resonance frequency of

$$\omega_{\text{res}} = \frac{1}{\sqrt{LC}} \qquad \text{or} \qquad f_{\text{res}} = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}. \tag{19}$$

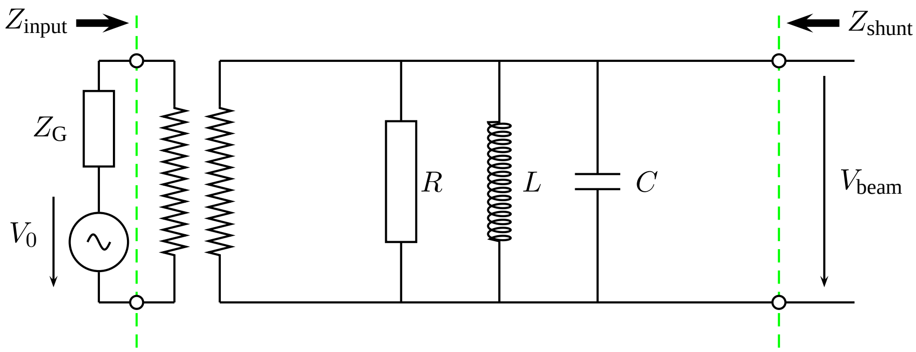

Fig. 20: The equivalent circuit that can be used to describe a cavity. The transformer is hidden in the coupling of the cavity ($Z \approx 1\,\text{M}\Omega$, seen by the beam) to the generator (usually $Z = 50\Omega$).

The Impedance Z of such an equivalent circuit is given by:

$$Z(\omega) = \frac{1}{\frac{1}{R} + \mathrm{j}\omega C + \frac{1}{\mathrm{j}\omega L}} \tag{20}$$

The 3dB bandwidth Δf refers to the points where Re(Z) = Im (Z) which corresponds to two vectors with an argument of 45° (Fig. 21) and an impedance of $|Z_{(-3\text{dB})}| = 0.707R = R/\sqrt{2}$.

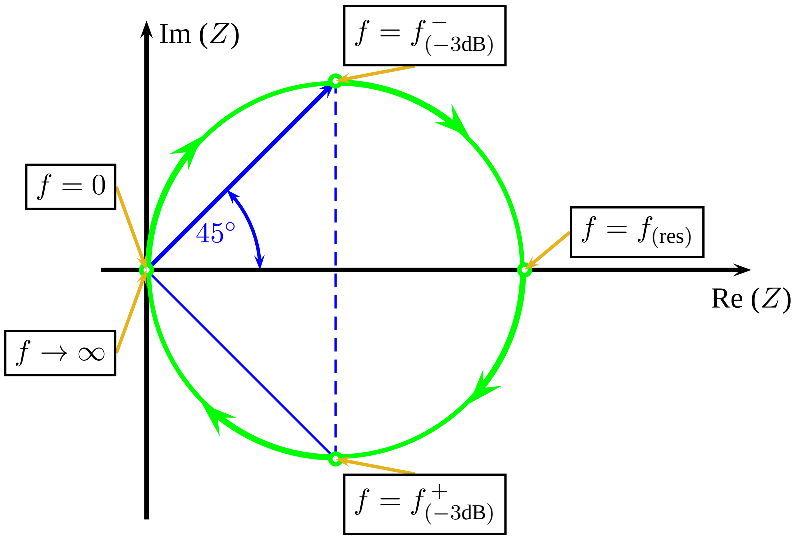

Fig. 21: Schematic drawing of the 3 dB bandwidth in the Impedance plane.

In general, the quality factor Q of a resonant circuit is defined as the ration of the stored energy W over the energy dissipated in one cycle P:

$$Q = \frac{\omega W}{P} \tag{21}$$

The Q factor for a resonance can be calculated via the 3dB bandwidth and the resonance frequency:

$$Q = \frac{f_{\text{res}}}{\Delta f} \tag{22}$$

For a cavity three different quality factors are defined:

– Q_0 (Unloaded Q): Q factor of the unperturbed system, i. e. the stand alone cavity
– Q_L (Loaded Q): Q factor of the cavity when connected to generator and measurement circuits
– Q_{ext} (External Q): Q factor that describes the degeneration of Q_0 due to the generator and diagnostic impedances

All these Q factors are connected via a simple relation:

$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{ext}} \tag{23}$$

The coupling coefficient β is then defined as:

$$\beta = \frac{Q_0}{Q_{ext}} \tag{24}$$

This coupling coefficient is not to be confused with the propagation coefficient of transmission lines which is also denoted as β.

In the Smith chart, a resonant circuit shows up as a circle (Fig. 22, circle shown in the detuned short position). The larger the circle is, the stronger the coupling. Three types of coupling are defined depending on the range of $beta$ (= size of the circle, assuming the circle is in the detuned short position):

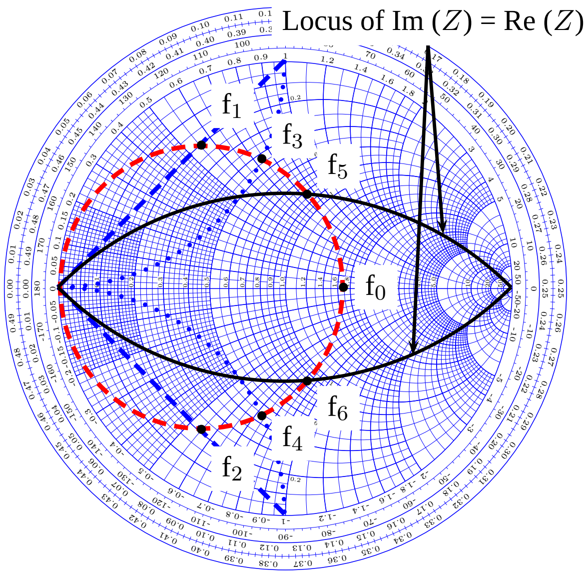

Fig. 22: Illustration of how to determine the different Q factors of a cavity in the Smith chart.

– Undercritical coupling ($0 < \beta < 1$): The radius of resonance circle is smaller than 0.25. Hence the center of the chart lies outside the circle.
– Critical coupling ($\beta = 1$): The radius of the resonance circle is exactly 0.25. Hence the circle touches the center of the chart.
– Overcritical coupling ($1 < \beta < \infty$): The radius of the resonance circle is larger than 0.25. Hence the center of the chart lies inside the circle

In practice, the circle may be rotated around the origin due to the transmission lines between the resonant circuit and the measurement device.

From the different marked frequency points in Fig. 22 the 3dB bandwidth and thus the quality factors Q_0, Q_L and Q_{ext} can be determined as follows:

– The unloaded Q can be determined from f_5 and f_6. The condition to find these points is Re(Z) = Im(Z) with the resonance circle in the detuned short position.
– The loaded Q can be determined from f_1 and f_2. The condition to find these points is $|\mathrm{Im}(S_{11})| \rightarrow$ max.
– The external Q can be calculated from f_3 and f_4. The condition to determine these points is Z = ±j.

To determine the points f_1 to f_6 with a network analyzer, the following steps are applicable:

– f_1 and f_2: Set the marker format to Re(S_{11}) + j Im(S_{11}) and determine the two points, where Im(S_{11}) = max.
– f_3 and f_4: Set the marker format to Z and find the two points where $Z = \pm$j.
– f_5 and f_6: Set the marker format to Z and locate the two points where Re(Z) = Im(Z)

References

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Appendices

A Transformation of lines with constant real or imaginary part from the impedance plane to the Γ plane

This section is dedicated to a detailed calculation of the transformation of coordinate lines form the impedance to t the Γ plane. The interested reader is furthermore referred to [4] for a more detailed study.

Consider a coordinate system in the complex impedance plane. The real part R of each impedance is assigned to the vertical axis and the imaginary part X of each impedance to the horizontal axis (Fig. A.1) For reasons of simplicity, all impedances used in this calculation are normalized to an impedance

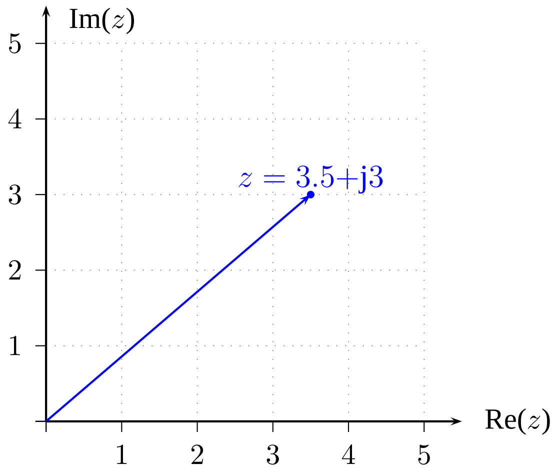

Fig. A.1: The complex impedance plane.

Z_0. This leads to the simplified transformation between impedance and Γ plane:

$$\Gamma = \frac{z-1}{z+1} \tag{A.1}$$

Γ is a complex number itself: $\Gamma = a$+jc. Using this identity and substituting$z = R$+ jX in equation (A.1) one obtains:

$$\Gamma = \frac{z-1}{z+1} = \frac{R+\mathrm{j}X-1}{R+\mathrm{j}X+1} = a+\mathrm{j}c \tag{A.2}$$

From this the real and the imaginary part of Γ can be calculated in terms of a, c, R and X:

$$\text{Re: } a(R+1) - cX = R-1 \tag{A.3}$$

$$\text{Im: } c(R+1) + aX = X \tag{A.4}$$

A.1 Lines with constant real part

To consider lines with constant real part, one can extract an expression for X from equation (A.4)

$$X = c\frac{1+R}{1-a} \tag{A.5}$$

and substitute this into equation (A.3):

$$a^2 + c^2 - 2a\frac{R}{1+R} + \frac{R-1}{R+1} = 0 \tag{A.6}$$

Completing the square, one obtains the equation of a circle:

$$\left(a - \frac{R}{1+R}\right)^2 + c^2 = \frac{1}{(1+R)^2} \tag{A.7}$$

From this equation following properties can be deduced:

– The center of each circle lies on the real a – axis.
– Since $\frac{R}{1+R} \geq 0$, the center of each circle lies on the positive real a – axis.
– The radius ρ of each circle follows the equation $\rho = \frac{1}{(1+R)^2} \leq 1$.
– The maximal radius is 1 for R = 0.

A.1.1 Examples

Here the circles for different R values are calculated and depicted graphically to illustrate the transformation from the z to the Γ plane.

1. R = 0: This leads to the center coordinates (c_a/c_c) = $\left(\frac{0}{1+0}/0\right) = (0/0)$, $\rho = \frac{1}{1+0} = 1$
2. R = 0.5: (c_a/c_c) = $\left(\frac{0.5}{1+0.5}/0\right) = (\frac{1}{3}/0)$, $\rho = \frac{1}{1+0.5} = \frac{2}{3}$
3. R = 1: (c_a/c_c) = $\left(\frac{1}{1+1}/0\right) = (\frac{1}{2}/0)$, $\rho = \frac{1}{1+1} = \frac{1}{2}$
4. R = 2: (c_a/c_c) = $\left(\frac{2}{1+2}/0\right) = (\frac{2}{3}/0)$, $\rho = \frac{1}{1+2} = \frac{1}{3}$
5. $R = \infty$: (c_a/c_c) = $\left(\frac{\infty}{1+\infty}/0\right) = (1/0)$, $\rho = \frac{1}{1+\infty} = 0$

This leads to the circles depicted in Fig. A.2.

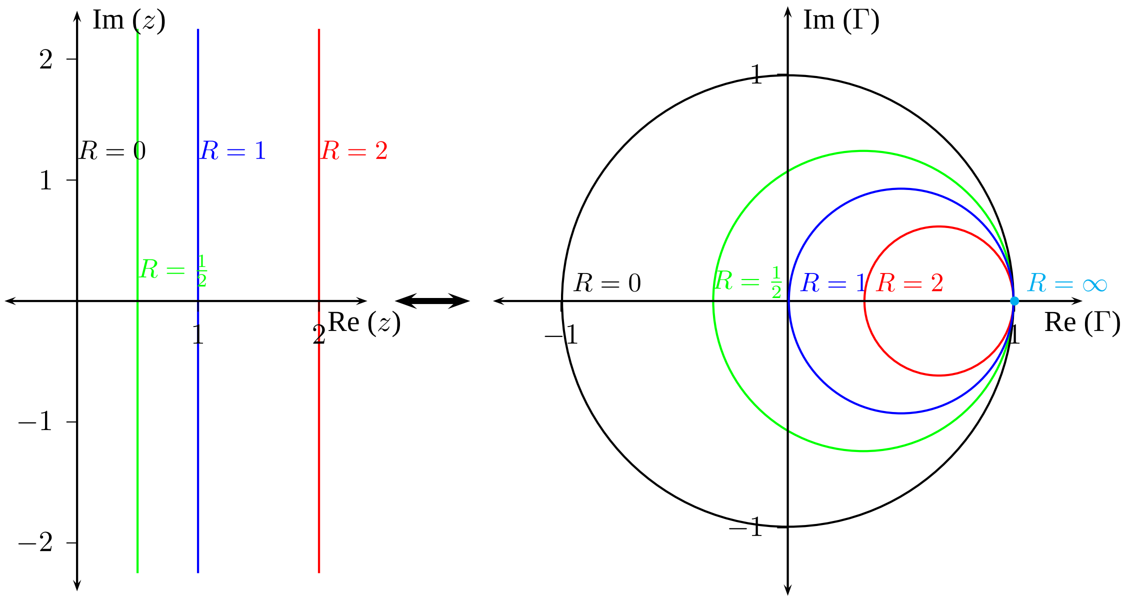

Fig. A.2: Lines of constant real part transformed into the Γ plane.

A.2 Lines with constant imaginary part

To calculate the circles in the Smith chart that correspond to the lines of constant imaginary part in the impedance plane, the formulas (A.3) and (A.4) are used again. Only this time an expression for R and R + 1 is calculated from formula (A.3)

$$R = \frac{a+1-cX}{1-a} \text{ and } 1+R = \frac{2-cX}{1-a} \tag{A.8}$$

and substituted into equation (A.4):

$$a^2 - 2a + 1 + c^2 - 2\frac{c}{X} = 0 \tag{A.9}$$

Completing the square again leads to an equation of a circle:

$$(a-1)^2 + \left(c - \frac{1}{X}\right)^2 = \frac{1}{X^2} \tag{A.10}$$

Examining this equation, the following properties can be deduced:

- The center of each circle lies on an axis parallel to the imaginary axis in a distance of 1.
- The first coordinate of each circle center is 1.
- The second coordinate of each circle center is $\frac{1}{X}$. It can be smaller or bigger than 0 depending on the value of X.
- No circle intersects the real a – axis.
- The radius ρ of each circle is $\rho = \frac{1}{|X|}$.
- All circles contain the point (1/0).

A.2.1 *Examples*

In the following, examples for different X values are calculated and depicted graphically to illustrate the transformation of the lines with constant imaginary part in the impedance plane to the corresponding circles in the Γ plane.

1. X = -2: (c_a/c_c) = $\left(1/\frac{1}{-2}\right) = (1/-0.5)$, $\rho = \frac{1}{|-2|} = 0.5$
2. X = -1: (c_a/c_c) = $\left(1/\frac{1}{-1}\right) = (1/-1)$, $\rho = \frac{1}{|-1|} = 1$
3. X = -0.5: (c_a/c_c) = $\left(1/\frac{1}{-0.5}\right) = (1/-2)$, $\rho = \frac{1}{|-2|} = 2$
4. X = 0: (c_a/c_c) = $\left(1/\frac{1}{0}\right) = (1/\infty)$, $\rho = \frac{1}{|0|} = \infty$ = real a – axis
5. X = 0.5: (c_a/c_c) = $\left(1/\frac{1}{0.5}\right) = (1/2)$, $\rho = \frac{1}{|-2|} = 2$
6. X = 1: (c_a/c_c) = $\left(1/\frac{1}{1}\right) = (1/1)$, $\rho = \frac{1}{|1|} = 1$
7. X = 2: (c_a/c_c) = $\left(1/\frac{1}{2}\right) = (1/0.5)$, $\rho = \frac{1}{|2|} = 0.5$
8. X = ∞: (c_a/c_c) = $\left(1/\frac{1}{\infty}\right) = (1/0)$, $\rho = \frac{1}{|\infty|} = 0$

A graphical representation of the circles corresponding to these values is given in Fig. A.3.

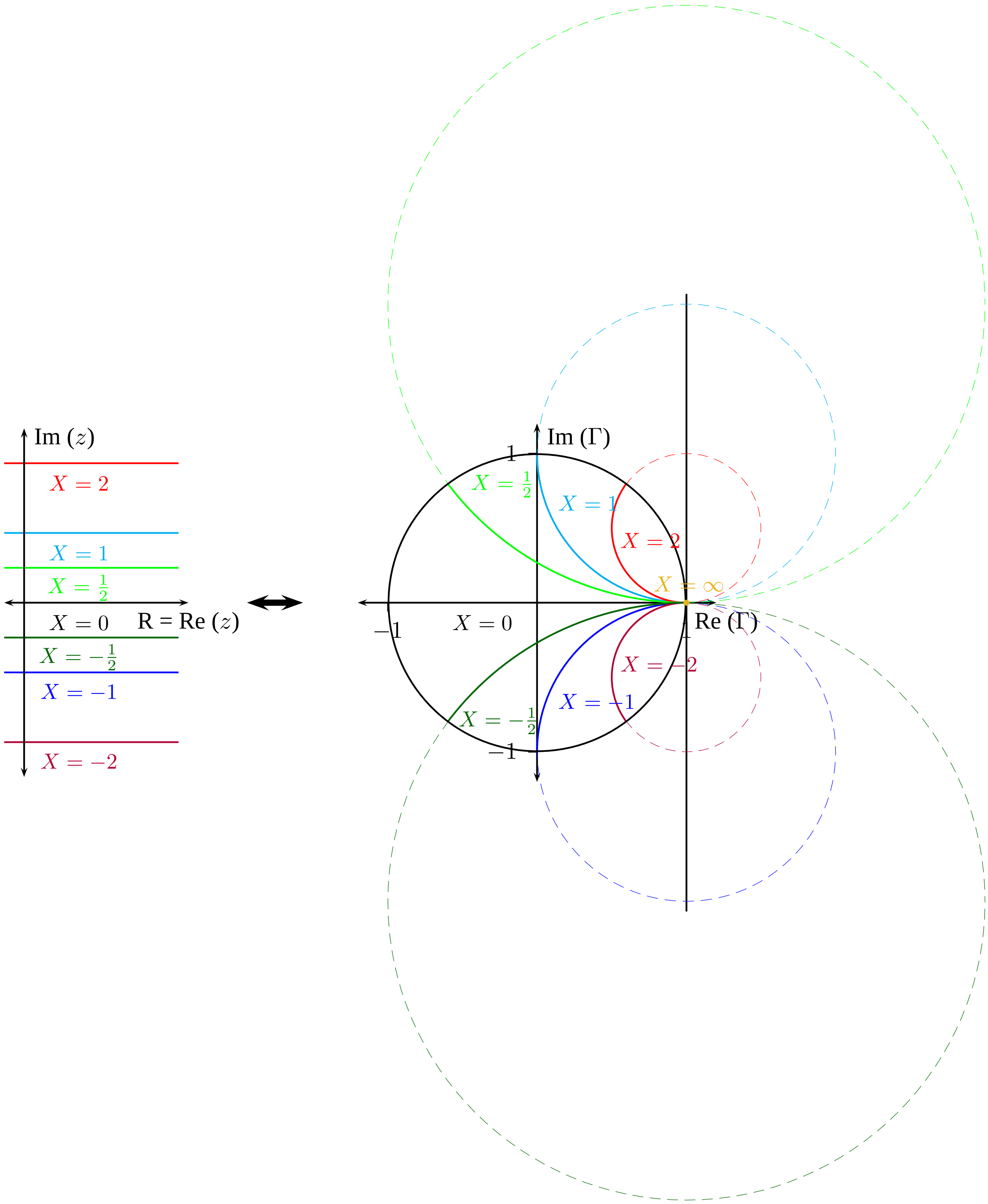

Fig. A.3: Lines of constant imaginary part transformed into the Γ plane.

B Rulers around the Smith chart

Some Smith charts provide rulers at the bottom to determine other quantities besides the reflection coefficient such as the return loss, the attenuation, the reflection loss etc. A short instruction of how to use these rulers as well as a specific example for such a set of rulers is given here.

B.1 Instruction how to use the rulers

First, one has to take the modulus (= distance between the center of the Smith chart and the point in the chart referring to the impedance in question) of the reflection coefficient of an impedance either with a conventional ruler or better using a compass. Then refer to the coordinate denoted to CENTER and go to the left or for the other part of the rulers to the right (except for the lowest line which is marked ORIGIN at the left which is the reference point of this ruler). The value in question can then be read from the corresponding scale.

B.2 Example of a set of rulers

A commonly used set of rulers that can be found below the Smith chart is depicted in Fig. B.1. For

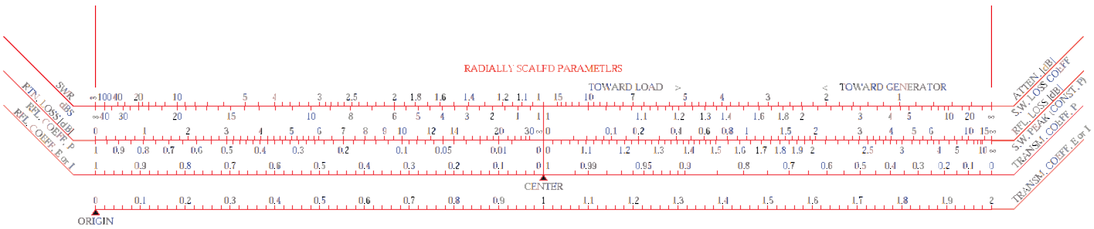

Fig. B.1: Example for a set of rulers that can be found underneath the Smith chart.

further discussion, this ruler is split along the line marked center, to a left (Fig. B.2) and a right part (Fig. B.3) since they will be discussed separately for reasons of simplicity. The upper part of the first

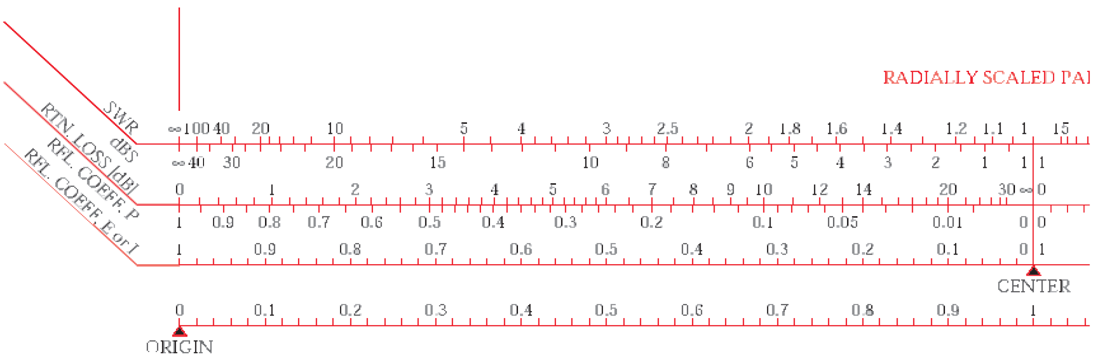

Fig. B.2: Left part of the rulers depicted in Fig. B.1.

ruler in Fig. B.2 is marked SWR which refers to the Voltage Standing Wave Ratio. The range of values is between one and infinity. One is for the matched case (center of the Smith chart), infinity is for total reflection (boundary of the SC). The upper part is in linear scale, the lower part of this ruler is in dB, noted as dBS (dB referred to Standing Wave Ratio). Example: SWR = 10 corresponds to 20 dBS, SWR = 100 corresponds to 40 dBS (voltage ratios, not power ratios).

The second ruler upper part, marked as RTN.LOSS = return loss in dB. This indicates the amount of reflected wave expressed in dB. Thus, in the center of the Smith chart nothing is reflected and the return loss is infinite. At the boundary we have full reflection, thus a return loss of 0 dB. The lower part

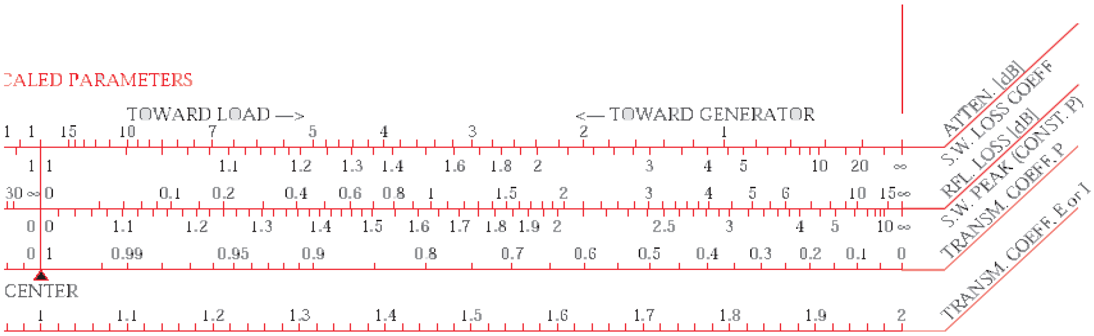

Fig. B.3: Right part of the rulers depicted in Fig. B.1.

of the scale denoted as RFL.COEFF. P = reflection coefficient in terms of POWER (proportional $|\Gamma|^2$). There is no reflected power for the matched case (center of the Smith chart), and a (normalized) reflected power = 1 at the boundary.

The third ruler is marked as RFL.COEFF,E or I. Whit this, the modulus (= absolute value) of the reflection coefficient can be determined in linear scale. Note that since we have the modulus we can refer it both to voltage or current as we have omitted the sign, we just use the modulus. Obviously in the center the reflection coefficient is zero, while at the boundary it is one.

The fourth ruler is the Voltage transmission coefficient. Note that the modulus of the voltage (and current) transmission coefficient has a range from zero, i.e. short circuit, to +2 (open = 1+$|\Gamma|$ with $|\Gamma|$=1). This ruler is only valid for Z_{load} = real, i.e. the case of a step in characteristic impedance of the coaxial line.

The upper part of the first ruler in Fig. B.3, denoted as ATTEN. in dB assumes that an attenuator (that may be a lossy line) is measured which itself is terminated by an open or short circuit (full reflection). Thus the wave is travelling twice through the attenuator (forward and backward). The value of this attenuator can be between zero and some very high number corresponding to the matched case. The lower scale of this ruler displays the same situation just in terms of VSWR. Example: a 10dB attenuator attenuates the reflected wave by 20dB going forth and back and we get a reflection coefficient of Γ=0.1 (= 10% in voltage).

The upper part of the second ruler, denoted as RFL.LOSS in dB is referring to the reflection loss. This is the loss in the transmitted wave, not to be confounded with the return loss referring to the reflected wave. It displays the relation $P_t = 1 - |\Gamma|^2$ in dB. Example: If $|\Gamma| = 1/\sqrt{2} = 0.707$ the transmitted power is 50% and thus the loss is 50% = 3dB.

Third ruler / right, marked as TRANSM.COEFF.P refers to the transmitted power as a function of mismatch and displays essentially the relation $P_t = 1 - |\Gamma|^2$. Thus, in the center of the Smith chart there is a full match and all the power is transmitted. At the boundary there is total reflection and e.g. for a Γ value of 0.5 75% of the incident power is transmitted.

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