# *FILLING SOME GAPS*

*Lecture 7*
*January 2013*

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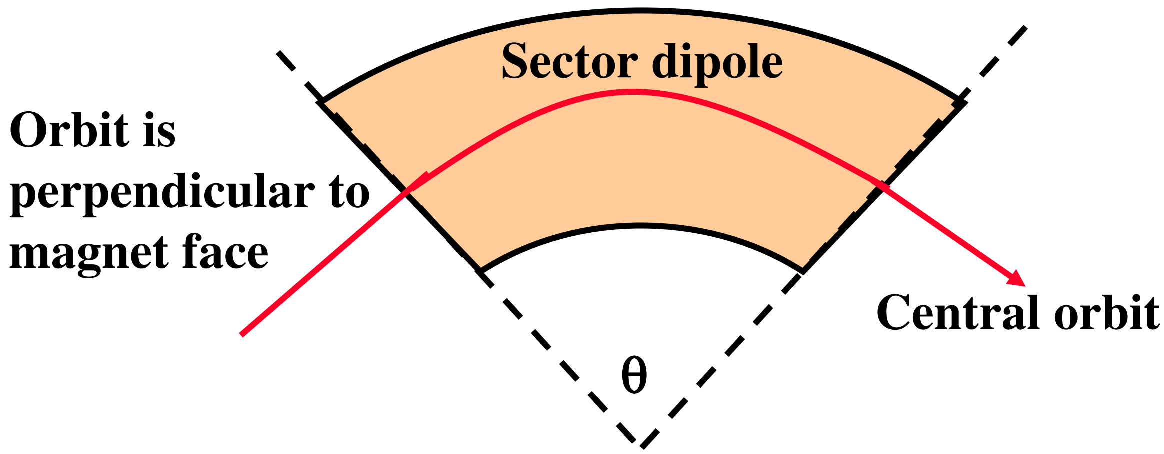
# *Introduction*

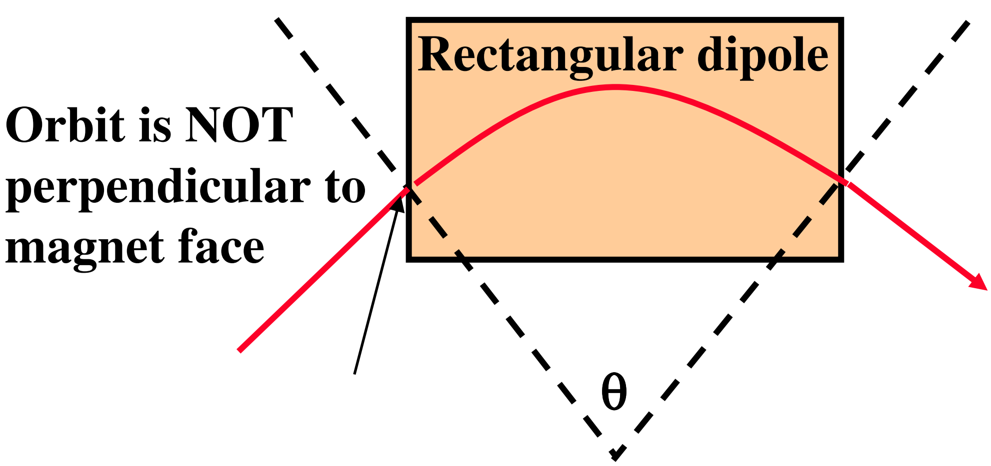
- **The first 5 lectures contain essential information with simple and concise derivations of basic formulæ and the 6th lecture uses this introduction to tackle accelerator design.**
- **In this lecture, we will fill some of the gaps:**
  - **Edge focusing**
  - **Skew quadrupoles**
  - **Solenoids**
  - **Normalised phase space**
  - **Beam steering**
  - **Position & angle measurement**
  - **Half-wavelength bump**
  - **3-magnet bump**
  - **4-magnet bump**
- **Of course, there is still much more.**

# *Edge focusing*

- The theory in Lecture 1 is based on the hard-edge model of a *sector dipole.*

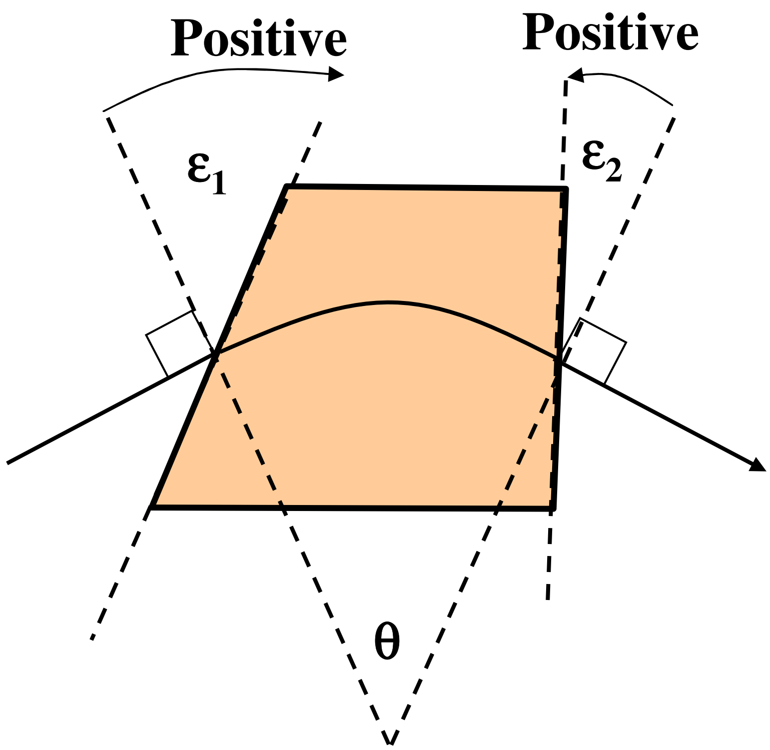



- Modern lattices usually have *rectangular dipoles* and in some cases dipoles with edges inclined at a general angle. These cases excite *edge focusing.*



# *Edge focusing continued*

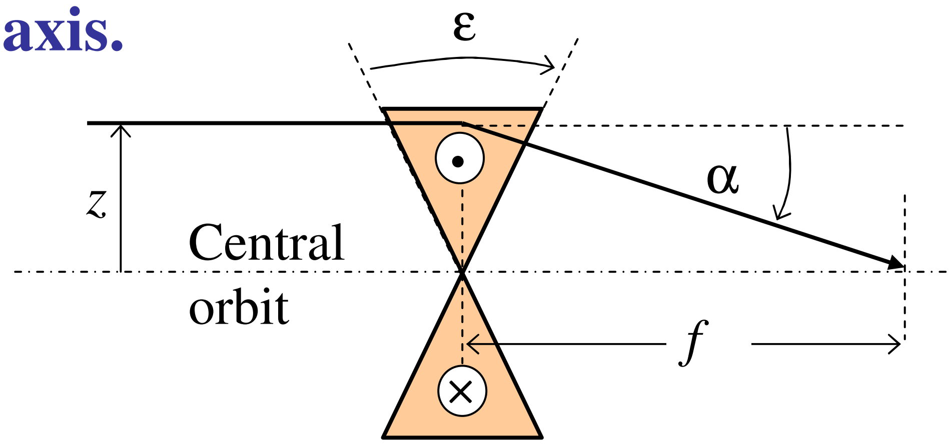
- The edge angles of a dipole are measured with respect to the basic sector magnet. The sign convention used here designates the edge angles as positive when the field integral is reduced with respect to the sector dipole on the outer side of the bend (i.e. defocusing).




- For the special case of a rectangular dipole the edge angles are equal to half the bending angle.

# *Edge focusing continued*

**An edge angle is modelled by a *'hard-edge' field wedge* set on the boundary of a sector dipole. The field in the wedge is made equal in magnitude to that in the dipole and the wedge is arranged to add to the field integral on one side of the central orbit and to subtract on the other. Since the width of the wedge is proportional to the distance from the axis, the angular deviation, α, suffered by an ion will be proportional to its distance from the axis.**




$$\alpha = \frac{B\,z\tan\varepsilon}{B\rho} = \frac{z\tan\varepsilon}{\rho} \qquad (1)$$

***z* is the coordinate in the plane of bending that can be *x* or *y*.**

# Edge focusing continued

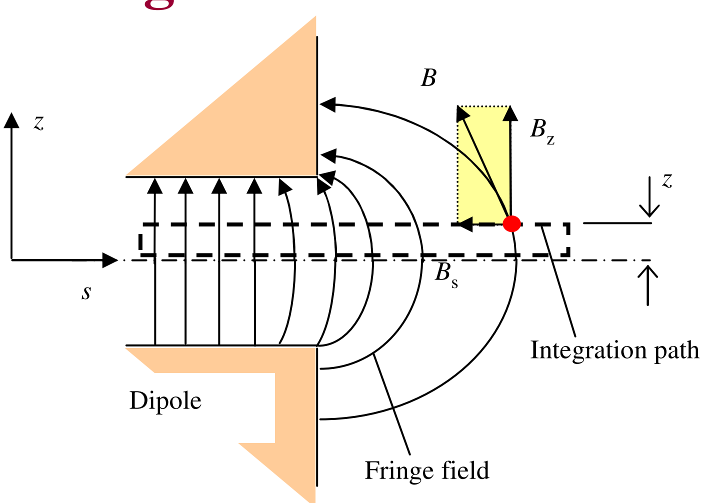
- The effect of the wedge is local and relatively weak, so it is safe to regard the wedge as a thin lens. The transfer matrix of this thin lens **in the plane of bending** will be,

$$\begin{pmatrix} z \\ z' \end{pmatrix}_2 = \begin{pmatrix} 1 & 0 \\ \mp\dfrac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} z \\ z' \end{pmatrix}_1 = \begin{pmatrix} 1 & 0 \\ \pm\dfrac{\tan\varepsilon}{|\rho|} & 1 \end{pmatrix} \begin{pmatrix} z \\ z' \end{pmatrix}_1 \quad (2)$$

- The wedge will focus (positive $f$ and negative $k$) in the plane of bending when it adds to the field integral on the outside of the bend and subtracts from the field integral on the inside of the bend. **Thus focusing in the plane of bending corresponds to a negative edge angle.**
- This convention is independent of whether the dipole is bending to the left or right or upwards or downwards.
- However, an F-type quadrupole is universally accepted as horizontally focusing and vertically defocusing, which requires the upper signs for horizontally bending dipoles and the lower signs for vertically bending dipoles.

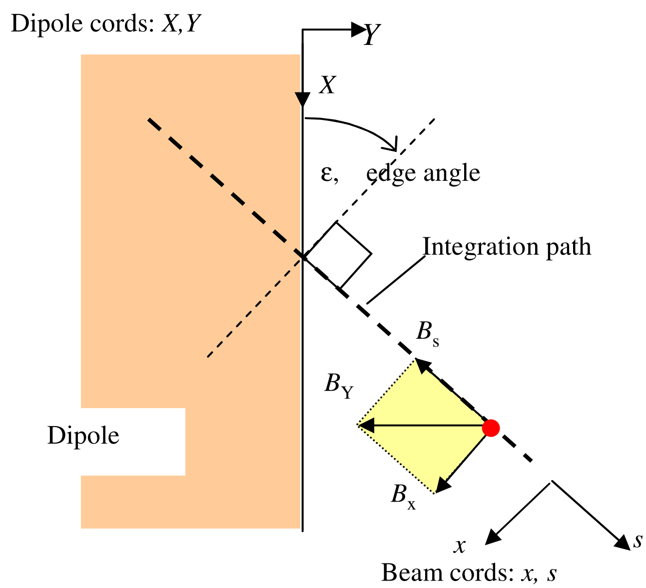
# Edge focusing continued

- So far, only the gradient seen in the plane of bending has been evaluated.
- There must also be a gradient in the orthogonal plane, but to demonstrate this requires some extra explanation that invokes the focusing action of the longitudinal field components that arise above and below the median plane.
- Note this is a departure from the strict ‘hard-edge’ model.




Side view

Plan view



# *Edge focusing continued*

- **An integration loop is drawn along the beam path with one side in the median plane and the return at a height *z* above it. The orbit's curvature has been neglected. The vertical sides are either deep inside the magnet and 'see' only a vertical field $B_0$, or far outside and 'see' zero field. The integral is zero, since no current is passing through the loop.**

$$\oint \boldsymbol{B}\cdot \mathrm{d}\ell = B_0 z + \int_{\text{Upper side of path}} B_s \, \mathrm{d}s = 0$$

- **The beam crosses the magnet edge with angle ε. The fringe field component $B_Y$ is resolved into $B_s$, parallel to the path, and the $B_x$, perpendicular to the path. These two components are related by,**

$$\int_{\text{Upper side of path}} B_x \, \mathrm{d}s = \int_{\text{Upper side of path}} B_s \tan\varepsilon \, \mathrm{d}s$$

- **Combining these equations gives an angular kick, α, in the plane perpendicular to the main bending that is independent of the fringe field shape, linear in *z* and of opposite sign.**

$$\alpha = \frac{1}{B_0\rho_0}\int_{\text{Fringefield}} B_x \mathrm{d}s = -\frac{\tan\varepsilon}{\rho_0} z \quad (3)$$

# *Edge focusing continued*

- **Thus the 'hard-edge' wedge of field that physically represents the edge focusing in the bending plane can be replaced by a thin quadrupole lens that represents the additional focusing in both planes.**

- **For accurate work, one more step can be taken by using the *dipole fringe-field correction*** **(Ref D.C. Carey, The optics of charged particle beams, (Harwood Academic Publishers, 1987), ISBN 3-7186-0350-0).** **This accounts for the shape of the fringe field and the curve of the beam path that was hitherto neglected. The effect can be significant and is expressed as an effective edge angle.**

- **Less well-known and rarely used is a *quadrupole fringe-field correction*** **(Ref P. Krejcik, Nonlinear quadrupole end-field effects in the CERN antiproton accumulators, 1987 Part. Accel. Conf., Washington D.C., March 16-19, (IEEE)).**

# Tilted elements

- Any element can be rotated (tilted) about its longitudinal axis by applying a rotation matrix at the entry and a compensating rotation at the exit.

$$\underbrace{\begin{pmatrix} C & 0 & -S & 0 \\ 0 & C & 0 & -S \\ S & 0 & C & 0 \\ 0 & S & 0 & C \end{pmatrix}}_{\substack{\text{Compensating rotation} \\ \text{at exit}}} \underbrace{\begin{pmatrix} hh_{11} & hh_{12} & hv_{11} & hv_{12} \\ hh_{21} & hh_{22} & hv_{21} & hv_{22} \\ vh_{11} & vh_{12} & vv_{11} & vv_{12} \\ vh_{21} & vh_{22} & vv_{21} & vv_{22} \end{pmatrix}}_{\text{Basic element transfer matrix}} \underbrace{\begin{pmatrix} C & 0 & S & 0 \\ 0 & C & 0 & S \\ -S & 0 & C & 0 \\ 0 & -S & 0 & C \end{pmatrix}}_{\text{Rotation at entry}} \quad (4)$$

where $C = \cos\theta$, $S = \sin\theta$ and $\theta$ is the angle of rotation in the anticlockwise direction when viewed in the beam direction.

- For a $6 \times 6$ matrix, the rotation becomes,

$$\left(\begin{array}{cccc|cc} C & 0 & S & 0 & 0 & 0 \\ 0 & C & 0 & S & 0 & 0 \\ -S & 0 & C & 0 & 0 & 0 \\ 0 & -S & 0 & C & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \quad (5)$$

# *Skew quadrupoles*

- Skew quadrupoles are normal quadrupoles rotated by π/4, so that $C = S = 1/\sqrt{2}$. These are relatively common elements, so it is worth multiplying (4) out,

$$\begin{pmatrix} \frac{1}{2}(hh_{11}+vv_{11}) & \frac{1}{2}(hh_{12}+vv_{12}) & \frac{1}{2}(hh_{11}-vv_{11}) & \frac{1}{2}(hh_{12}-vv_{12}) \\ \frac{1}{2}(hh_{21}+vv_{21}) & \frac{1}{2}(hh_{22}+vv_{22}) & \frac{1}{2}(hh_{21}+vv_{21}) & \frac{1}{2}(hh_{22}-vv_{22}) \\ \frac{1}{2}(hh_{11}-vv_{11}) & \frac{1}{2}(hh_{12}-vv_{12}) & \frac{1}{2}(hh_{11}+vv_{11}) & \frac{1}{2}(hh_{12}+vv_{12}) \\ \frac{1}{2}(hh_{21}-vv_{21}) & \frac{1}{2}(hh_{22}-vv_{22}) & \frac{1}{2}(hh_{21}+vv_{21}) & \frac{1}{2}(hh_{22}+vv_{22}) \end{pmatrix}$$

which becomes

$$\begin{pmatrix} \frac{1}{2}(\cos\varphi+\cosh\varphi) & \frac{\ell}{2\varphi}(\sin\varphi+\sinh\varphi) & \frac{1}{2}(\cos\varphi-\cosh\varphi) & \frac{\ell}{2\varphi}(\sin\varphi-\sinh\varphi) \\ \frac{\varphi}{2\ell}(\sinh\varphi-\sin\varphi) & \frac{1}{2}(\cos\varphi+\cosh\varphi) & -\frac{\varphi}{2\ell}(\sin\varphi+\sinh\varphi) & \frac{1}{2}(\cos\varphi-\cosh\varphi) \\ \frac{1}{2}(\cos\varphi-\cosh\varphi) & \frac{\ell}{2\varphi}(\sin\varphi-\sinh\varphi) & \frac{1}{2}(\cos\varphi+\cosh\varphi) & \frac{\ell}{2\varphi}(\sin\varphi+\sinh\varphi) \\ -\frac{\varphi}{2\ell}(\sin\varphi+\sinh\varphi) & \frac{1}{2}(\cos\varphi-\cosh\varphi) & \frac{\varphi}{2\ell}(\sinh\varphi-\sin\varphi) & \frac{1}{2}(\cos\varphi+\cosh\varphi) \end{pmatrix} \quad (6)$$

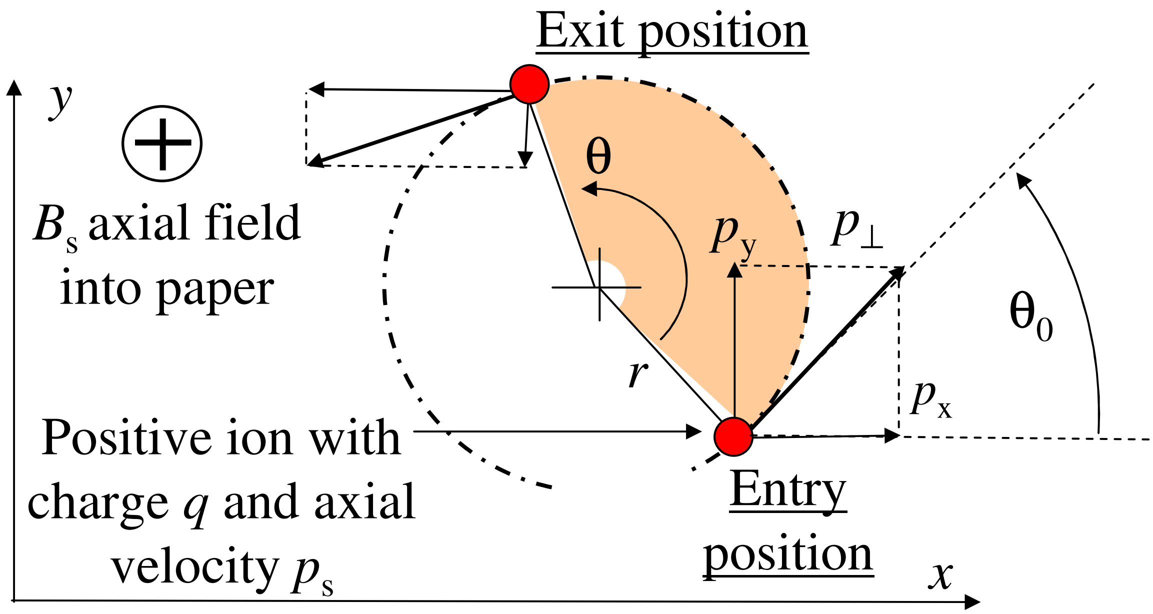
where $\varphi = \sqrt{|k|}\,\ell$

# Solenoids

- Solenoids with their main field aligned with the central axis have long been used as focusing devices at low beam energies where the ability to focus in both planes over a short distance is particularly useful.
- At high beam energies, quadrupole focusing schemes are more power-efficient, but solenoids find a new application as particle physics detectors.
- To derive the transfer matrix of a solenoid, it is necessary to consider separately the motion in the central region and in the two ends.

# *Solenoids continued*

- **Motion in the uniform axial field in the central part of the solenoid.**




- **The angular divergences with respect to the longitudinal axis at the entry position are,**

$$x_1' = \frac{p_\perp}{p_s}\cos\theta_0 \quad \textbf{and} \quad y'_1 = \frac{p_\perp}{p_s}\sin\theta_0 \quad (7)$$

**and, at the exit position,**

$$x_2' = \frac{p_\perp}{p_s}\cos(\theta_0+\theta) \quad \textbf{and} \quad y'_2 = \frac{p_\perp}{p_s}\sin(\theta_0+\theta) \quad (8)$$

# *Solenoids continued*

- **Substitute the entry position equations (7) into the exit position equations (8) to get,**

$$\begin{pmatrix} x' \\ y' \end{pmatrix}_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}_1 \qquad (9)$$

- **The particle positions at the entry and exit are similarly derived from the geometry of the circle diagram,**

$$x_2 = x_1 - r\sin\theta_0 + r\sin(\theta_0 + \theta)$$
$$y_2 = y_1 + r\cos\theta_0 - r\cos(\theta_0 + \theta) \qquad (10)$$

- **Expanding and substituting from (7) gives**

$$x_2 = x_1 + r\frac{p_s}{p_\perp}\sin\theta\, x_1' + r\frac{p_s}{p_\perp}(\cos\theta - 1)\, y'_1$$

$$y_2 = y_1 + r\frac{p_s}{p_\perp}(1 - \cos\theta)\, x_1' + r\frac{p_s}{p_\perp}\sin\theta\, y'_1 \qquad (11)$$

# *Solenoids continued*

- **For a given transit time t, the precession angle θ and the length of the solenoid l are related by,**

$$\frac{p_\perp}{p_s} = \frac{v_\perp}{v_s} = \frac{r\theta/\tau}{\ell/\tau} = \frac{r\theta}{\ell}$$

**so that**

$$r\frac{p_s}{p_\perp} = \frac{\ell}{\theta} \qquad (13)$$

- **Substitute (13) into (12) and combine with (9), to get transfer matrix for the central region of a solenoid,**

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_2 = \begin{pmatrix} 1 & \frac{\ell}{\theta}\sin\theta & 0 & -\frac{\ell}{\theta}(1-\cos\theta) \\ 0 & \cos\theta & 0 & -\sin\theta \\ 0 & \frac{\ell}{\theta}(1-\cos\theta) & 1 & \frac{\ell}{\theta}\sin\theta \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_1 \qquad (14)$$

# Solenoids continued

❖ **It has been convenient to use the angle of precession θ in this derivation, but in order to relate to more basic quantities and to be in line with the literature, we define**

$$M = \frac{B_s}{|B\rho|} = \frac{\theta}{\ell} \qquad (15)$$

❖ **With (14) this yields,**

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_2 = \begin{pmatrix} 1 & \frac{1}{M}\sin M\ell & 0 & -\frac{1}{M}(1-\cos M\ell) \\ 0 & \cos M\ell & 0 & -\sin M\ell \\ 0 & \frac{1}{M}(1-\cos M\ell) & 1 & \frac{1}{M}\sin M\ell \\ 0 & \sin M\ell & 0 & \cos M\ell \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_1 \qquad (16)$$

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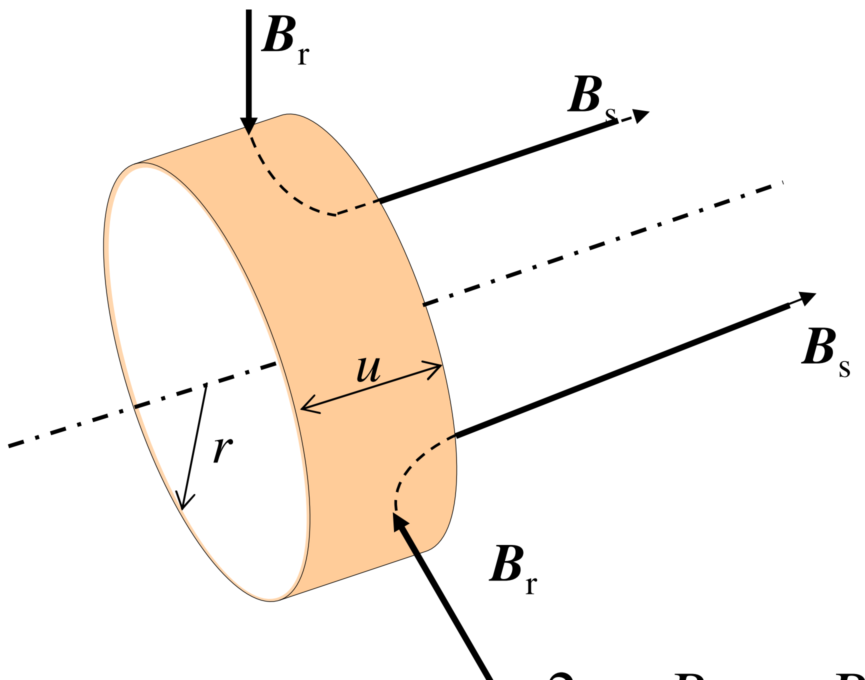
**Derivation of (15):**

Rewrite the cyclotron motion $B_s e v_\perp = \frac{m v_\perp^2}{r}$ as $p_\perp = B_s e r$ and substitute into (13) to get (15).

Tacitly (15) assumes that the transverse momentum is negligible with respect to the axial component, so that we can equate the axial momentum to total momentum and the magnetic rigidity $|B\rho|$.

# *Solenoids continued*

- **Motion in the end fields.**
- **The end field of a solenoid is usually concentrated in an iron end plate with a circular hole for the beam to pass through.**




$$2\pi r u B_r = -B_s \pi r^2$$

**Transverse kick.**

$$\delta = \frac{B_r u}{|B\rho|} = -\frac{1}{2}\frac{B_s r}{|B\rho|} = -\frac{M}{2} r \qquad (17)$$

# Solenoids continued

- **These kicks can be represented by thin lenses with the transfer matrices,**
- **Entry**

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{M}{2} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{M}{2} & 0 & 0 & 1 \end{pmatrix} \qquad (18)$$

- **Exit,**

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{M}{2} & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{M}{2} & 0 & 0 & 1 \end{pmatrix} \qquad (19)$$

# Solenoids continued

- **The end matrices must now be multiplied with the central matrix to get the final transfer matrix of a solenoid.**

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_2 = \begin{pmatrix} \cos^2\varphi & \frac{\ell}{\varphi}\sin\varphi\cos\varphi & -\sin\varphi\cos\varphi & -\frac{\ell}{\varphi}\sin^2\varphi \\ \frac{\varphi}{\ell}\sin\varphi\cos\varphi & \cos^2\varphi & \frac{\varphi}{\ell}\sin^2\varphi & -\sin\varphi\cos\varphi \\ \sin\varphi\cos\varphi & \frac{\ell}{\varphi}\sin^2\varphi & \cos^2\varphi & \frac{\ell}{\varphi}\sin\varphi\cos\varphi \\ -\frac{\varphi}{\ell}\sin^2\varphi & \sin\varphi\cos\varphi & \frac{\varphi}{\ell}\sin\varphi\cos\varphi & \cos^2\varphi \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_1 \qquad (20)$$

- **where** $\varphi = \frac{M\ell}{2}$

**and** $$M\ell = \frac{1}{|B\rho|}\int_{-\infty}^{\infty} B_s \,\mathrm{d}s$$

**relates the 'hard-edge' model to the real world.**

# *Normalised phase space*

- **From Lecture 4, Eqn (14),**

$$z(s) = A\beta^{1/2}(s)\cos\left(\mu(s) + B\right)$$
$$z'(s) = -A\beta^{-1/2}\left[\alpha\cos\left(\mu + B\right) + \sin\left(\mu + B\right)\right]$$

- **The phase terms can be isolated and used to define new co-ordinates $Z(\mu)$ and $Z'(\mu)$ that are known as *normalised co-ordinates*.**

$$Z(\mu) = A\cos\left(\mu + B\right) \quad = z(s)\beta^{-1/2} \qquad (21)$$
$$Z'(\mu) = -A\sin\left(\mu + B\right) \quad = z(s)\alpha\beta^{-1/2} + z'(s)\beta^{1/2}$$

- **Real-space co-ordinates use lower case and normalised co-ordinates upper case.**
- **Real-space co-ordinates use $s$ as the independent variable and normalised co-ordinates use the phase advance $\mu$.**
- **Real-space coordinates show the phase-space motion as an ellipse, while normalised coordinates show a circle.**

# *Normalised phase space continued*

- **The transformations between the two systems are conveniently expressed in matrix form as:**

$$\begin{pmatrix} Z \\ \mathrm{d}Z/\mathrm{d}\mu \end{pmatrix} = \begin{pmatrix} \beta^{-1/2} & 0 \\ \alpha\beta^{-1/2} & \beta^{1/2} \end{pmatrix} \begin{pmatrix} z \\ \mathrm{d}z/\mathrm{d}s \end{pmatrix} \qquad (22)$$

$$\begin{pmatrix} z \\ \mathrm{d}z/\mathrm{d}s \end{pmatrix} = \begin{pmatrix} \beta^{1/2} & 0 \\ -\alpha\beta^{-1/2} & \beta^{-1/2} \end{pmatrix} \begin{pmatrix} Z \\ \mathrm{d}Z/\mathrm{d}\mu \end{pmatrix} \qquad (23)$$

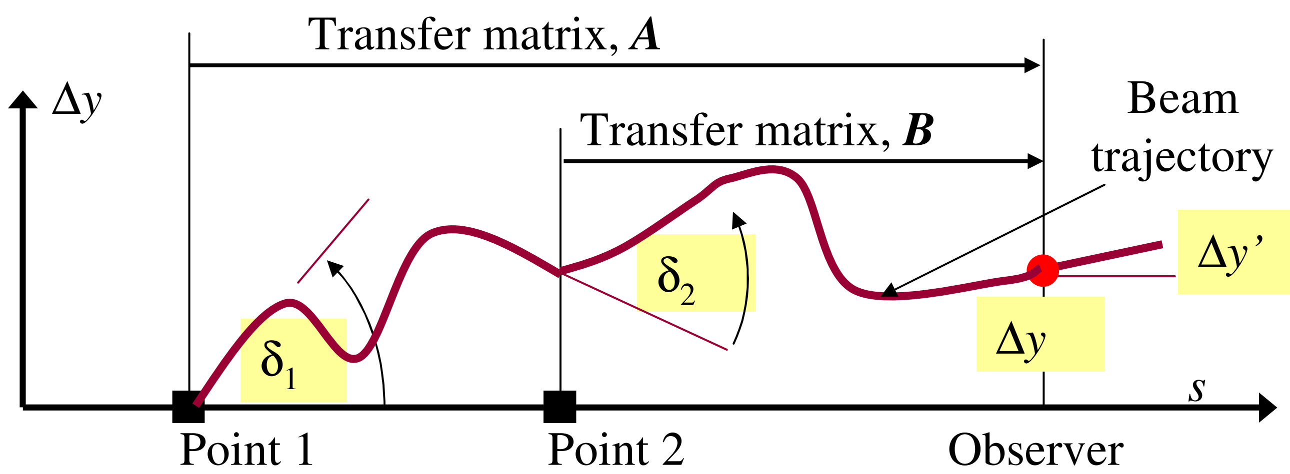
- **Similarly, there is a normalised form of the dispersion function ($D_N$, $D'_N$), which is also frequently used,**

$$\begin{pmatrix} D_N \\ \mathrm{d}D_N/\mathrm{d}\mu \end{pmatrix} = \begin{pmatrix} \beta^{-1/2} & 0 \\ \alpha\beta^{-1/2} & \beta^{1/2} \end{pmatrix} \begin{pmatrix} D \\ \mathrm{d}D/\mathrm{d}s \end{pmatrix} \qquad (24)$$

$$\begin{pmatrix} D \\ \mathrm{d}D/\mathrm{d}s \end{pmatrix} = \begin{pmatrix} \beta^{1/2} & 0 \\ -\alpha\beta^{-1/2} & \beta^{-1/2} \end{pmatrix} \begin{pmatrix} D_N \\ \mathrm{d}D_N/\mathrm{d}\mu \end{pmatrix} \qquad (25)$$

# *Beam steering*

- **How to change the position and angle ($\Delta y$, $\Delta y'$) at a given point in a transfer line using two upstream dipole kicks ($\delta_1$, $\delta_2$).**




- **The system is linear, so the effect of each kick at the 'Observer' can be calculated and the effects added.**

$$\begin{pmatrix} \Delta y \\ \Delta y' \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 \\ \delta_1 \end{pmatrix}}_{\text{Effect of } \delta_1 \text{ at 'Observer'}} + \underbrace{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} 0 \\ \delta_2 \end{pmatrix}}_{\text{Effect of } \delta_2 \text{ at 'Observer'}}$$

# *Beam steering continued*

- which can be rewritten as,

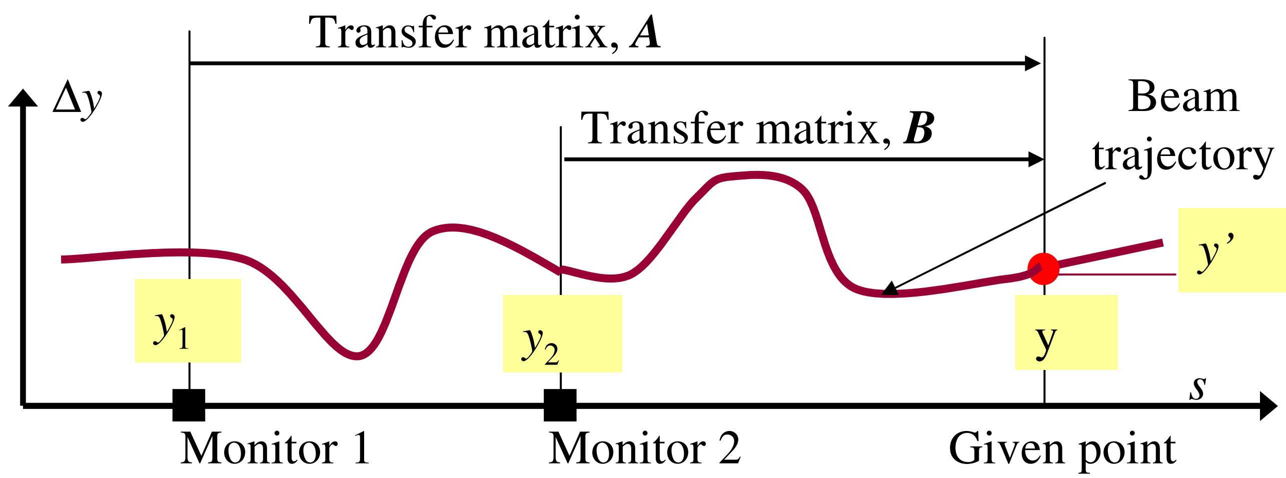
$$\begin{pmatrix}\Delta y \\ \Delta y'\end{pmatrix} = \begin{pmatrix} a_{12} & b_{12} \\ a_{22} & b_{22}\end{pmatrix}\begin{pmatrix}\delta_1 \\ \delta_2\end{pmatrix} \qquad (26)$$

where Δ indicates the changes in the position and angle seen by the 'Observer'. Inverting (26) gives the kicks required for the position and angle changes,

$$\begin{pmatrix}\delta_1 \\ \delta_2\end{pmatrix} = \frac{1}{(b_{22}a_{12} - b_{12}a_{22})}\begin{pmatrix} b_{22} & -b_{12} \\ -a_{22} & a_{12}\end{pmatrix}\begin{pmatrix}\Delta y \\ \Delta y'\end{pmatrix}_{\text{Observer}} \qquad (27)$$

- A similar reasoning can be used to transform two independent orbit measurements into a position and angle measurement at a given point.

# *Position & angle measurement*




- **Transfers to given point from each monitor,**

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y'_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} y_2 \\ y'_2 \end{pmatrix}$$

- **Eliminate $y_1'$ and $y_2'$**

$$y_1\left(\frac{a_{11}}{a_{12}} - \frac{a_{21}}{a_{22}}\right) = \frac{y}{a_{12}} - \frac{y'}{a_{22}} \quad \text{and} \quad y_2\left(\frac{b_{11}}{b_{12}} - \frac{b_{21}}{b_{22}}\right) = \frac{y}{b_{12}} - \frac{y'}{b_{22}}$$

- **Solve for $y$ and $y'$**

$$y = \frac{1}{(a_{22}b_{12} - a_{12}b_{22})}(y_1 b_{12} - y_2 a_{12})$$

$$y' = \frac{1}{(a_{22}b_{12} - a_{12}b_{22})}(y_1 b_{22} - y_2 a_{22}) \qquad (28)$$

Remember: $|A| = a_{11}a_{22} - a_{12}a_{21} = 1$ and $|B| = b_{11}b_{22} - b_{12}b_{21} = 1$.

# *Half-wavelength bump*

- **From Lecture 4, Eqn (14), we have,**

$$z(s) = A\beta(s)^{1/2}\cos[\mu(s) + B]$$

$$z'(s) = -A\beta^{-1/2}[\alpha\cos(\mu + B) + \sin(\mu + B)]$$

- **If the first kick $\delta_1 (= z'_1)$ is put at $\mu_1 = -\pi/2$, then $B = 0$ and $A = \delta_1\beta_1^{1/2}$. At $\mu = \pi/2$, just half a wavelength later, the excursion will again be zero. At this point the oscillation can be killed by a second kick $\delta_2$, which is equal and opposite to the trajectory slope at this point, so that $\delta_2 = -z'_2$, which gives the conditions,**

**Imposed condition:** $\mu_2 - \mu_1 = \pi$

**Derived condition:** $\delta_1\beta_1^{1/2} = \delta_2\beta_2^{1/2} \quad (29)$

- **The bump height can be controlled at any point by scaling the kicks, but the angle of the trajectory is a feature of the lattice geometry and cannot be controlled.**

# 3-magnet bump

- **It is rare that magnets can be placed with a phase separation of exactly π. Even when possible, this makes the lattice inflexible for future developments.**

- **It is therefore useful to know how to correct the residual error of an imperfect 2-magnet bump with a third dipole.**

- **Using Lecture 4, Eqn (8), track forwards from kick 1 to kick 2.**

$$\begin{cases} z_{2,\mathrm{f}} = (\beta_1\beta_2)^{1/2}\sin(\Delta\mu_{2,1})\delta_1 \\ z'_{2,\mathrm{f}} = (\beta_1/\beta_2)^{1/2}\left[\cos(\Delta\mu_{2,1}) - \alpha_2\sin(\Delta\mu_{2,1})\right]\delta_1 \end{cases}$$

- **Also using Lecture 4, Eqn (8) track backwards from kick 3 to kick 2.**

$$\begin{cases} z_{2,\mathrm{b}} = (\beta_3\beta_2)^{1/2}\sin(-\Delta\mu_{2,3})\delta_3 \\ z'_{2,\mathrm{b}} = (\beta_3/\beta_2)^{1/2}\left[\cos(-\Delta\mu_{2,3}) - (-\alpha_2)\sin(-\Delta\mu_{2,3})\right]\delta_3 \end{cases}$$

# *3-magnet bump continued*

- **The forward- and back-track amplitudes at $\delta_2$ must be identical and the difference in the derivatives must be matched by the dipole kick $\delta_2$, i.e.**

$$y_{2,\mathrm{f}} = y_{2,\mathrm{b}} \quad \text{and} \quad \delta_2 = -y'_{2,\mathrm{b}} - y'_{2,\mathrm{f}}$$

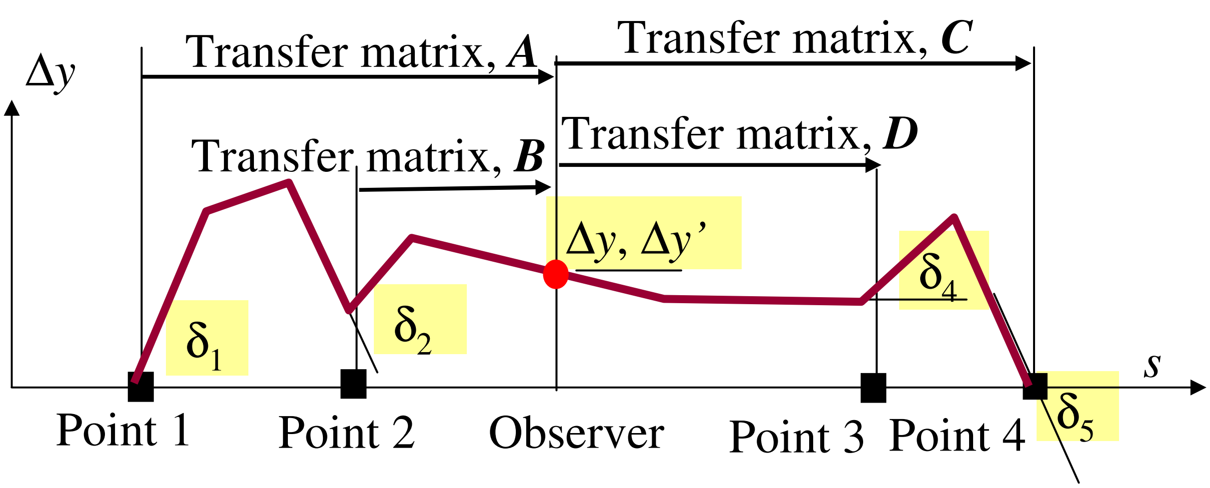
- **Some manipulation of the above equations yields,**

$$\frac{\delta_1 \beta_1^{1/2}}{\sin(\Delta\mu_{3,2})} = \frac{\delta_2 \beta_2^{1/2}}{\sin(\Delta\mu_{1,3})} = \frac{\delta_3 \beta_3^{1/2}}{\sin(\Delta\mu_{2,1})} \qquad (30)$$

- **As with the 2-magnet, half-wavelength bump, the excursion of the trajectory can be controlled at any point by scaling the kicks, but the angle of the trajectory is a feature of the lattice geometry and cannot be controlled.**

# 4-magnet bump

- Often a local bump is required that controls both the position and angle of the beam at some particular position. This requires four magnets with one pair upstream of the control point and one pair downstream.




- Calculate the kicks $\delta_1$ and $\delta_2$ to achieve the displacement $\Delta y$, $\Delta y'$ at the Observer position by using the steering equation (27).

$$\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}=\frac{1}{(b_{22}a_{12}-b_{12}a_{22})}\begin{pmatrix}b_{22} & -b_{12}\\ -a_{22} & a_{12}\end{pmatrix}\begin{pmatrix}\Delta y\\ \Delta y'\end{pmatrix}_{\text{Observer}} \qquad (27)$$

# 4-magnet bump continued

- **Eqn (27) can also be used to specify the downstream kicks, but because the transfer matrices and kicks are defined in the beam direction, the downstream kicks that close the bump are found by back-tracking.**
- **Remember,**

$$C^{-1} = \frac{1}{|C|}\begin{pmatrix} c_{22} & -c_{12} \\ -c_{21} & c_{11} \end{pmatrix} = \begin{pmatrix} c_{22} & -c_{12} \\ -c_{21} & c_{11} \end{pmatrix} \quad \text{since} \quad |C| = 1$$

$$D^{-1} = \frac{1}{|D|}\begin{pmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{pmatrix} = \begin{pmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{pmatrix} \quad \text{since} \quad |D| = 1$$

- **This gives,**

$$\begin{pmatrix} \Delta y \\ -\Delta y' \end{pmatrix}_{\text{Observer}} = \underbrace{\begin{pmatrix} c_{22} & -c_{12} \\ -c_{21} & c_{11} \end{pmatrix}\begin{pmatrix} 0 \\ -\delta_4 \end{pmatrix}}_{\text{Effect of } \delta_4 \text{ at 'Observer'}} + \underbrace{\begin{pmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{pmatrix}\begin{pmatrix} 0 \\ -\delta_3 \end{pmatrix}}_{\text{Effect of } \delta_3 \text{ at 'Observer'}}$$

$$\begin{pmatrix} \Delta y \\ -\Delta y' \end{pmatrix}_{\text{Observer}} = \begin{pmatrix} -c_{12} & -d_{12} \\ c_{11} & d_{11} \end{pmatrix}\begin{pmatrix} -\delta_4 \\ -\delta_3 \end{pmatrix} \qquad (31)$$