Introduction to Transverse Beam Dynamics

Lecture 1: Magnetic fields and particle trajectories

Andrea Latina (CERN)

JUAS 2013

14th January 2013

Luminosity run of a typical storage ring

Storage ring: Protons are accelerated and stored for ~ 12 hours

distance of particles traveling at nearly the speed of light, $v \approx c$

$$d = 12 \times 10^{11} \text{ km}$$

$\rightarrow$ it's about 86 times the Earth-Sun distance !

Introduction and basic ideas

It's a circular machine: we need a transverse deflecting force→the Lorentz force

$$\vec{F} = q \cdot \left(\vec{E} + \vec{v} \wedge \vec{B} \right)$$

where, in high energy machines, $|\vec{v}| \approx c \approx 3 \cdot 10^8$ m/s. Usually there is no electric field, and the transverse deflection is given by a magnetic field only.

Example

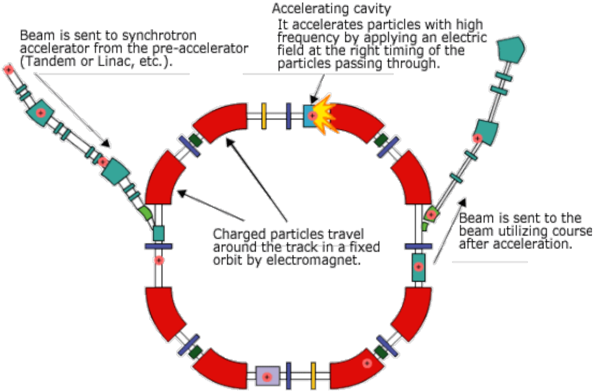
$$B = 1 \text{ T} \rightarrow \quad \begin{aligned} F &= q \cdot 3 \cdot 10^8 \ \frac{m}{s} \cdot 1 \ \text{T} \\ &= q \cdot 3 \cdot 10^8 \ \frac{m}{s} \cdot 1 \ \frac{Vs}{m^2} \\ &= q \cdot 300 \ \frac{MV}{m} \end{aligned}$$

Notice that there is a technical limit for an electric field:

$$E \lesssim 1 \ \frac{MV}{m}$$

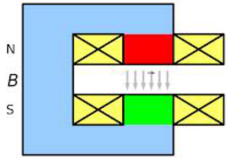
A general rule is : **in an accelerator, use magnetic fields wherever it's possible**

$$\left.\begin{array}{lll} \text{Lorentz force} & F_L & = evB \\ \text{Centrifugal force} & F_{\text{centr}} & = \dfrac{\gamma m v^2}{\rho} \\ & \dfrac{\gamma m v\!\!\!/^{\,2}}{\rho} & = e v\!\!\!/ B \end{array}\right\} \qquad \frac{p}{q} = B\rho \qquad B\rho = \text{"beam ridigity"}$$



Dipole magnets: the magnetic guide

- Dipole magnets:
  - define the ideal orbit
  - in a homogeneous field created by two flat pole shoes, $B = \frac{\mu_0 nI}{h}$



- Normalise magnetic field to momentum:

$$\frac{p}{e} = B\rho \quad \Rightarrow \quad \frac{1}{\rho} = \frac{eB}{p} \qquad B = [T] = \left[\frac{Vs}{m^2}\right] \quad \text{p} = \left[\frac{GeV}{c}\right]$$

- Example: the LHC

$$\left.\begin{aligned} B &= 8.3\ \text{T} \\ p &= 7000\ \tfrac{GeV}{c} \end{aligned}\right\} \quad \frac{1}{\rho} = \text{e}\frac{8.3\ \frac{Vs}{m^2}}{7000 \cdot 10^9\ \frac{eV}{c}} = \frac{8.3\text{s}\ \cdot 3 \cdot 10^8\ \frac{m}{s}}{7000 \cdot 10^9\ \text{m\^{}2}} =$$

$$= 0.333 \cdot \frac{8.3}{7000}\ \frac{1}{m} = \frac{1}{2.53}\ \frac{1}{km}$$

Dipole magnets: the magnetic guide

In the LHC, $\rho = 2.53$ km. The circumference $2\pi\rho = 17.6$ km $\approx 66\%$ of the entire LHC.

The field B is $\approx 1 \ldots 8$ T

Rule of thumb:

$$\boxed{\frac{1}{\rho\ [m]} \approx 0.3 \frac{B\ [T]}{p\ [GeV/c]}}$$

which is the "normalised bending strength"

Quadrupole magnets: the focusing force

Quadrupole magnets are required to keep the trajectories in vicinity of the ideal orbit

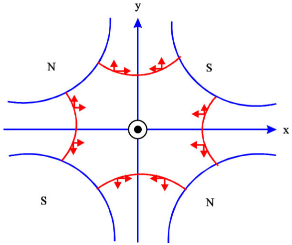
They exert a linearly increasing Lorentz force, thru a linearly increasing magnetic field:

$$\begin{aligned} B_x &= -gy \\ B_y &= -gx \end{aligned} \quad \Rightarrow \quad \begin{aligned} F_x &= -evgx \\ F_y &= evgy \end{aligned}$$

Gradient of a quadrupole magnet:

$$g = \frac{2\mu_0 nI}{r^2} \left[\frac{T}{m}\right] = \frac{B_{\text{poles}}}{r_{\text{aperture}}} \left[\frac{T}{m}\right]$$

- LHC main quadrupole magnets: $g \approx 25 \ldots 220$ T/m



the arrows show the force exerted on a particle

Focusing strength:

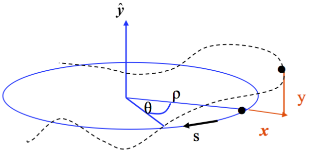
$$k = \frac{g}{p/e} \left[m^{-2}\right]$$

A simple rule: $k \left[m^{-2}\right] \approx 0.3 \frac{g\ [T/m]}{p\ [GeV/c]}$.

The equation of motion

Linear approximation:

- the ideal particle ⇒ stays on the **design orbit**
- any other particle ⇒ has coordinates x, y
  - which are small quantities $x, y \ll \rho$
- only linear terms in x and y of B are taken into account



Taylor expansion of the B field:

$$B_y(x) = B_{y0} + \frac{\partial B_y}{\partial x} x + \frac{1}{2} \frac{\partial^2 B_y}{\partial x^2} x^2 + \frac{1}{3!} \frac{\partial^3 B_y}{\partial x^3} x^3 + \ldots$$

if we normalise to the momentum $p/e = B\rho$

$$\frac{B(x)}{p/e} = \frac{B_0}{B_0 \rho} + \frac{g}{p/e} x + \frac{1}{2} \frac{eg'}{p/e} x^2 + \frac{1}{3!} \frac{eg''}{p/e} x^3 + \ldots$$

The equation of motion

$$\frac{B(x)}{p/e} = \frac{B_0}{B_0\rho} + \frac{g}{p/e}x + \frac{1}{2}\frac{eg'}{p/e}x^2 + \frac{1}{3!}\frac{eg''}{p/e}x^3 + \dots$$

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + kx + \frac{1}{2}mx^2 + \frac{1}{3!}nx^3 + \dots$$

In the linear approximation, only the terms linear in x and y are taken into account:

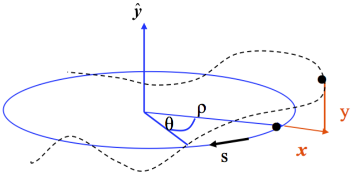
- dipole fields
- quadrupole fields

It is more practical to use "separate function" machines:

- split the magnets and optimise them regarding their function
  - bending
  - focusing, etc.

The equation of motion in radial coordinates

Let's consider a local segment of one particle's trajectory:



the radial acceleration is $a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2\rho}{dt^2} - \rho\omega^2$. In our case, for the ideal orbit: $\rho = \text{const} \Rightarrow \frac{d\rho}{dt} = 0$

$\Rightarrow$the force is

$$F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$$

$$F = mv^2/\rho$$

For a general trajectory:

$$\rho \to \rho + x : \qquad F = m\frac{d^2}{dt^2}(\rho + x) - \frac{mv^2}{\rho + x} = eB_y v$$

$$F = \underbrace{m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho + x)}_{\text{term 1}} - \underbrace{\frac{mv^2}{\rho + x}}_{\text{term 2}} = eB_y v$$

- Term 1. As $\rho =$const...

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\rho + x) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}x$$

- Term 2. Remember: $x \approx$ mm whereas $\rho \approx$ m $\rightarrow$ we develop for small x

remember

$$\frac{1}{\rho + x} \approx \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right)$$

Taylor expansion:

$$f(x) = f(x_0) +$$
$$+ (x - x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \cdots$$

$$m\frac{d^2x}{\mathrm{d}t^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = eB_y v$$

The guide field in linear approximation $B_y = B_0 + x\frac{\partial B_y}{\partial x}$

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = ev\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\} \qquad \text{let's divide by } m$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t} - \frac{v^2}{\rho}\left(1-\frac{x}{\rho}\right) = \frac{evB_0}{m} + \frac{evxg}{m}$$

Independent variable: $t \to s$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}$$

$$\frac{d^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\underbrace{\frac{\mathrm{d}x}{\mathrm{d}s}}_{x'}\underbrace{\frac{\mathrm{d}s}{\mathrm{d}t}}_{v}\right)\frac{\mathrm{d}s}{\mathrm{d}t}$$

$$\frac{d^2x}{\mathrm{d}t^2} = x''v^2 + \cancel{\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}v}{\mathrm{d}s}}v$$

$$x''v^2 - \frac{v^2}{\rho}\left(1-\frac{x}{\rho}\right) = \frac{evB_0}{m} + \frac{evxg}{m} \qquad \text{let's divide by } v^2$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{eB_0}{mv} + \frac{exg}{mv}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{xg}{p/e}$$

$$x'' \cancel{-\frac{1}{\rho}} + \frac{x}{\rho^2} = \cancel{-\frac{1}{\rho}} + kx$$

Remember:

$$mv = p$$

Normalise to the momentum of the particle:

$$\frac{g}{p/e}\frac{B_0}{p/e} = -\frac{1}{\rho}; \quad \frac{g}{p/e} = k.$$

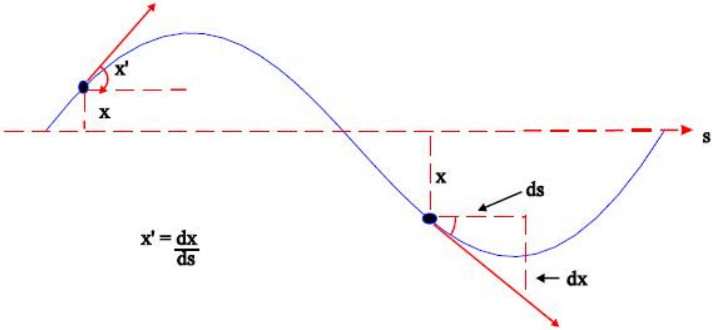
$$x'' + x\left(\frac{1}{\rho^2} - k\right) = 0$$

Equation for the vertical motion

- $\frac{1}{\rho^2} = 0$ usually there are not vertical bends
- $k \longleftrightarrow -k$ quadrupole field changes sign

$$y'' + ky = 0$$

Coordinates



Remarks

- Weak focusing:

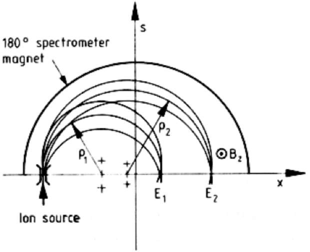
$$x'' + \left(\frac{1}{\rho^2} - k\right) x = 0$$

there is a focusing force even without a quadrupole gradient

$$k = 0 \quad \Rightarrow \quad x'' = -\frac{1}{\rho^2} x$$

even without quadrupoles there is retrieving force (focusing) in the bending plane of the dipole magnets

- In large machine this effect is very weak...



Mass spectrometer: particles are separated according to their energy and focused due to the $1/\rho$ effect of the dipole

Fringe fields

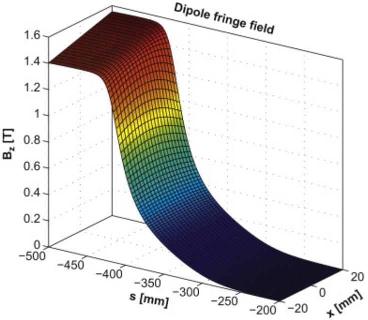
- Hard edge model:

$$x'' + \left(\frac{1}{\rho^2} - k\right) x = 0$$

this equation is not really correct

- Bending and focusing forces -even inside a magnet- depend on the position s

$$x''(s) + \left\{\frac{1}{\rho^2(s)} - k(s)\right\} x(s) = 0$$



Dipole fringe field. Quadrupole and sextupole field components can be seen by looking at the transverse slope and curvature, respectively.

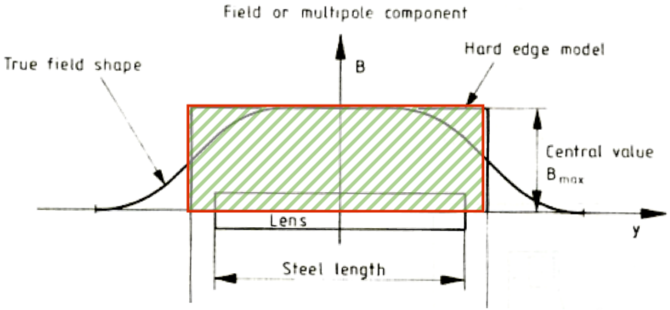
But still: inside the magnet the focusing properties hold:

$$\frac{1}{\rho} = const$$

$$k = const$$

Effective length

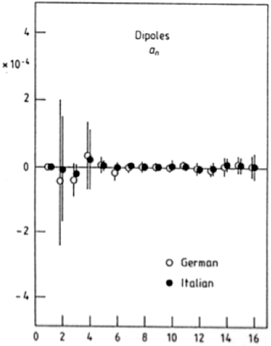
$$B \cdot L_{eff} = \int_0^{l_{mag}} B ds$$



Multipoles

Taylor expansion of the B field:

$$B_y(x) = B_{y0} + \frac{\partial B_y}{\partial x}x + \frac{1}{2}\frac{\partial^2 B_y}{\partial x^2}x^2 + \frac{1}{3!}\frac{\partial^3 B_y}{\partial x^3}x^3 + \dots \qquad \text{divide by } B_{y0}$$



Multipole coefficients:

- *divide by the main field to get the relative error contribution*

Solution of the trajectory equations: focusing quadrupole

Definition:

$$\left.\begin{array}{rrl} \text{horizontal plane} & K & = 1/\rho^2 - k \\ \text{vertical plane} & K & = k \end{array}\right\} \quad x'' + Kx = 0$$

This is the differential equation of a harmonic oscillator ... with spring constant K. We make an ansatz:

$$x(s) = a_1 \cos(\omega s) + a_2 \sin(\omega s)$$

General solution: a linear combination of two independent solutions:

$$x'(s) = -a_1 \omega \sin(\omega s) + a_2 \omega \cos(\omega s)$$

$$x''(s) = -a_1 \omega^2 \cos(\omega s) + a_2 \omega^2 \sin(\omega s) = -\omega^2 x(s) \quad \rightarrow \quad \omega = \sqrt{K}$$

General solution, for $K > 0$:

$$x(s) = a_1 \cos\left(\sqrt{K} s\right) + a_2 \sin\left(\sqrt{K} s\right)$$

We determine a_1, a_2 by imposing the following boundary conditions:

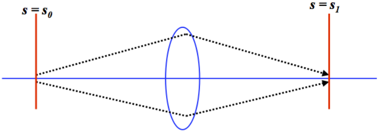
$$s = 0 \quad \rightarrow \quad \begin{cases} x(0) = x_0, & a_1 = x_0 \\ x'(0) = x_0', & a_2 = \frac{x_0'}{\sqrt{K}} \end{cases}$$

Horizontal focusing quadrupole, $K > 0$:

$$x(s) = x_0 \cos\left(\sqrt{K}s\right) + x_0' \frac{1}{\sqrt{K}} \sin\left(\sqrt{K}s\right)$$
$$x'(s) = -x_0 \sqrt{K} \sin\left(\sqrt{K}s\right) + x_0' \cos\left(\sqrt{K}s\right)$$

For convenience we can use a matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M_{foc} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}_{s_0}$$



Where:

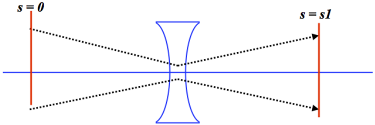
$$M_{\text{foc}} = \begin{pmatrix} \cos\left(\sqrt{K}s\right) & \frac{1}{\sqrt{K}} \sin\left(\sqrt{K}s\right) \\ -\sqrt{K} \sin\left(\sqrt{K}s\right) & \cos\left(\sqrt{K}s\right) \end{pmatrix}$$

Defocusing quadrupole

The equation of motion is

$$x'' + Kx = 0$$

with $K < 0$



Remember:
$$f(s) = \cosh(s)$$
$$f'(s) = \sinh(s)$$

Now the solution is in the form:

$$x(s) = a_1 \cosh(\omega s) + a_2 \sinh(\omega s)$$

and the transfer matrix:

$$M_{\text{defoc}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}s\right) & \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|}s\right) \\ \sqrt{|K|} \sinh\left(\sqrt{|K|}s\right) & \cosh\left(\sqrt{|K|}s\right) \end{pmatrix}$$

Notice that for a drift space, when $K = 0 \rightarrow M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$

Summary of the transfer matrices

- Focusing quad, $K > 0$

$$M_{\text{foc}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix}$$

- Defocusing quad, $K < 0$

$$M_{\text{defoc}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix}$$

- Drift space, $K = 0$

$$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

With the assumptions we have made, the motion in the horizontal and vertical planes is independent: "... the particle motion in x and y is uncoupled"

Thin-lens approximation

When the focal length, f, of the lens is much bigger than the length of the magnet L

$$f = \frac{1}{K \cdot L} \qquad \gg L$$

we can derive the limit for $L \to 0$ while we keep $K \cdot L = \text{const.}$

The transfer matrices are

$$M_x = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \qquad M_y = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

focusing, and defocusing respectively.

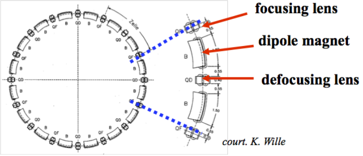
This approximation (yet quite accurate, in large machines) is useful for fast calculations... and for the guided studies !

Transformation through a system of lattice elements

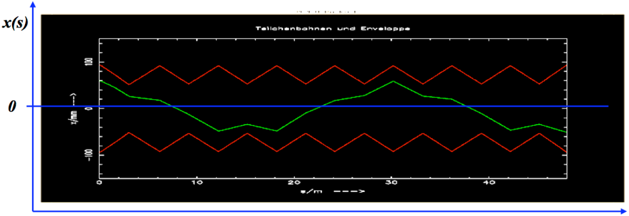
One can compute the solution of a system of elements, by multiplying the matrices of each single element:

$$M_{\text{total}} = M_{\text{QF}} \cdot M_{\text{D}} \cdot M_{\text{Bend}} \cdot M_{\text{D}} \cdot M_{\text{QD}} \cdot \quad \cdots$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = M_{s_1 \to s_2} \cdot M_{s_0 \to s_1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$



In each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator.



...typical values are:

$$x \approx \text{mm}$$

$$x' \leq \text{mrad}$$

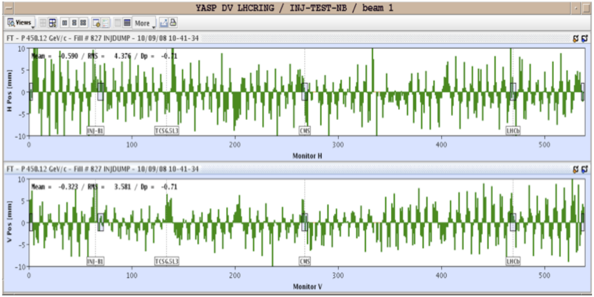
Orbit and Tune

Tune: the number of oscillations per turn.

Example:

64.31

59.32

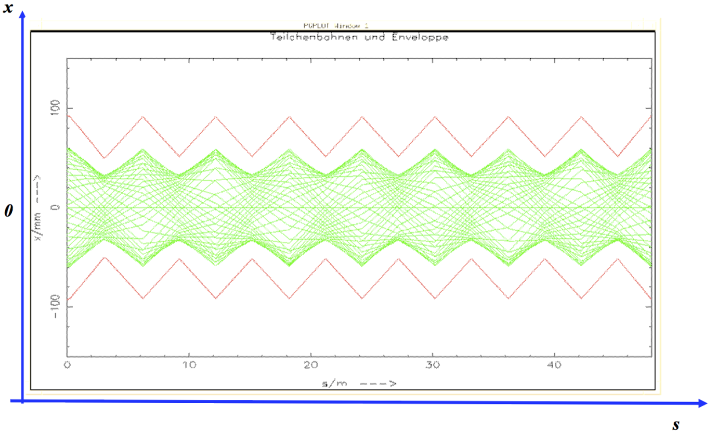


Relevant for beam stability studies is : the non-integer part

Envelope

Question: what will happen, if the particle performs a second turn ?

- ... or a third one or ... 10^{10} turns ...



Summary

beam rigidity: $B\rho = \frac{p}{q}$

bending strength of a dipole: $\frac{1}{\rho}\ [m^{-1}] = \frac{0.2998 \cdot B_{\mathbf{0}}\ [\mathrm{T}]}{p\ [\mathrm{GeV/c}]}$

focusing strength of a quadruple: $k\ [m^{-2}] = \frac{0.2998 \cdot g}{p\ [\mathrm{GeV/c}]}$

focal length of a quadrupole: $f = \frac{1}{k \cdot L_{\mathbf{Q}}}$

equation of motion: $x'' + Kx = \frac{1}{\rho}\frac{\Delta p}{p}$

transfer matrix of a foc. quad: $x_{s_2} = M \cdot x_{s_1}$

$$M_{\mathrm{QF}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix}$$

$$M_{\mathrm{QD}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix} \qquad M_{\mathrm{D}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Bibliography

1. Edmund Wilson: Introduction to Particle Accelerators Oxford Press, 2001
2. Klaus Wille: Physics of Particle Accelerators and Synchrotron Radiation Facilities, Teubner, Stuttgart 1992
3. Peter Schmüser: Basic Course on Accelerator Optics, CERN Acc. School: 5th general acc. phys. course CERN 94-01
4. Bernhard Holzer: Lattice Design, CERN Acc. School: Interm.Acc.phys course, `http://cas.web.cern.ch/cas/ZEUTHEN/lectures-zeuthen.htm`
5. M.S. Livingston, J.P. Blewett: Particle Accelerators, Mc Graw-Hill, New York,1962
6. The CERN Accelerator School (CAS) Proceedings
7. Mathew Sands: The Physics of e+ e- Storage Rings, SLAC report 121, 1970
8. D. Edwards, M. Syphers: An Introduction to the Physics of Particle Accelerators, SSC Lab 1990