LONGITUDINAL

PLANE

Lecture 5
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Introduction

- **So far we have:**
  - **Introduced a curvilinear co-ordinate system that follows the reference or central orbit and described the transverse behaviour of the beam with respect to this reference orbit using the ‘hard-edge’ model.**
  - **Derived the transverse motion in the local curvilinear co-ordinate system for magnetic and electrostatic elements and expressed the solutions in the form of transfer matrices.**
  - **Used the matrices to track ions through a lattice.**
  - **Introduced the Twiss parameterisation, which is ubiquitous to lattice design.**
- **The longitudinal plane will be analysed in a similar way:**
  - **The behaviour of the beam will be defined with reference to the same local curvilinear co-ordinates with the addition of a so-called *synchronous ion*.**
  - **The longitudinal behaviour will be introduced into the transfer matrices of the basic lattice elements.**
  - **RF structures, the Transit Time Factor and Phase Stability will be described.**
  - **There will not be time to cover the equivalent Twiss parameterisation in the longitudinal plane nor the derivation of transfer matrices for cavities and RFQs.**

Terminolgy

- **The beam will be described with reference to a *synchronous ion* that follows a particular *space-time trajectory*. The 'space' trajectory is the central orbit of the transverse motion and the 'time' trajectory is defined by initial conditions.**

- **When crossing a cavity, the synchronous ion receives a kick in momentum ($\Delta_s p$). Non-synchronous ions receive slightly different kicks ($\Delta_s p + \Delta p$).**

- **The motion of the non-synchronous ions is then expressed in terms of how much they lead or lag (Δs) the synchronous ion in their flight through the lattice and by how much they deviate from the synchronous ion in momentum ($\Delta p/p$).**

- **Δs-$\Delta p/p$ defines the *longitudinal phase space*.**

- **A large number of ions concentrated around a synchronous ion are referred to as *bunch*.**

- **Without longitudinal focusing, a bunch of ions will progressively spread out.**

- **A focusing region in longitudinal phase-space around the synchronous ion is known as an *RF bucket*.**

Terminology continued

- A ***stationary RF bucket*** is one that does not alter the momentum of the synchronous ion ($\Delta_s p = 0$), but does modify the momenta of the non-synchronous ions ($\Delta p \neq 0$).
- An ***accelerating bucket*** applies a positive momentum kick to the synchronous ion ($\Delta_s p > 0$).
- RF cavities are usually (but not always) configured to bring non-synchronous ions closer to the synchronous ion.
- In transfer lines, this is called ***longitudinal focusing***.
- In a ring, it is called ***phase stability***.

NOTE:

'Δ_s' refers to the synchronous ion.

'Δ' refers to the difference with respect to the synchronous ion.

'δ' refers to energy exchanges in electrostatic fields.

'd' and '∂' used for mathematical differentials.

Normally, no distinctions are made in the literature.

Leading and Lagging

- **The variable Δs has two components, given by differentiating $s = vt$.**

$$s = vt$$

$$\Delta s = v\Delta t + t\Delta v$$

$$\Delta s = \Delta s_{\text{path length}} + \Delta s_{\text{velocity}} \quad (1)$$

- **The first term is the *geometric difference in path length*, given by the velocity of the reference ion multiplied by the extra time taken by the given ion to traverse the element. The second term is the distance due to the *difference in velocity* between the given ion and the synchronous ion applied for the time needed for the reference ion to traverse the element.**

- **If the change in path length compensates the effect of the velocity difference (i.e. $\Delta s_{\text{path length}} = - \Delta s_{\text{velocity}}$), so that $\Delta s = 0$, the transit time is the same for ions of all momenta and the lattice is known as an *isochronous lattice*.**

Δs for drift spaces and similar elements

- **Drift spaces, quadrupoles, multipoles and solenoids are considered to have the same geometric path length to first order for all momenta, so $\Delta s_{\text{path length}}$ in (1) is zero in these cases.**

- **The second term in (1) is derived from the basic relativistic expression, $p_0 = m_0 \gamma \beta c$ by differentiation to give,**

$$\frac{\Delta p}{p_0} = \gamma^2 \frac{\Delta \beta}{\beta}$$

which gives,

$$\Delta v = \frac{v}{\gamma^2} \frac{\Delta p}{p_0}$$

so that,

$$\boxed{\Delta s_{\text{velocity}} = t \Delta v = \frac{\ell}{\gamma^2} \frac{\Delta p}{p_0} \qquad (2)}$$

- **You will frequently see this term in 6×6 transfer matrices (This is the easiest term to derive so watch for examination questions) .**

Longitudinal terms for transfer matrices of non-bending elements

- **For drift spaces, quadrupoles (magnetic & electrostatic), skew quadrupoles, solenoids and similar 'straight' elements.**

Transverse coupling off-axis sub-matrices not treated in these lectures

$$\left[\begin{array}{cc:cc:cc} m_{11} & m_{12} & m_{13} & m_{14} & 0 & 0 \\ m_{21} & m_{22} & m_{23} & m_{24} & 0 & 0 \\ \hdashline m_{31} & m_{32} & m_{33} & m_{34} & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & \ell/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} x \\ x' \\ \hdashline y \\ y' \\ \hdashline \Delta s \\ \Delta p/p \end{array}\right]$$

Uncoupled transverse sub-matrices, see earlier lectures

Longitudinal sub-matrix

The complete treatment of bends

- The complete treatment of bends to obtain all the matrix elements for the transfer matrix is above ‘introductory level’, especially for the case of electrostatic bends.

- However, this work is included in an Annex for completeness and future reference.

Fields in RF devices

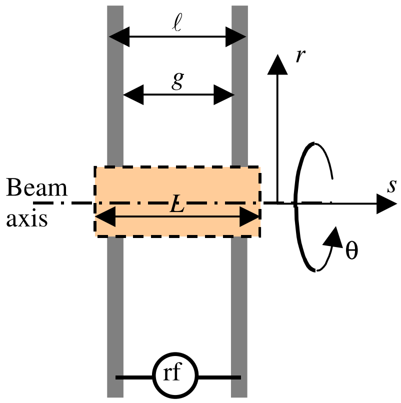
- RF devices require a little more understanding than the uniform blocks of field used in the 'hard-edge' model.

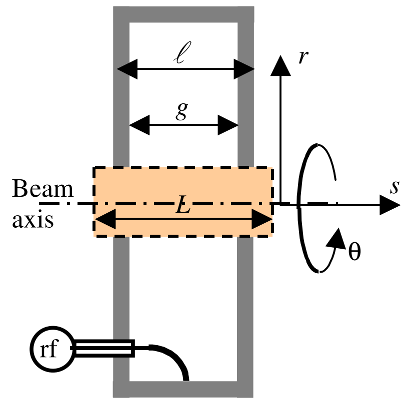
- This section is devoted to ***accelerating structures of the standing-wave type*** with rotational symmetry excited by a TM010 mode. In the TM010 mode, only the E_r, E_s and B_Θ components are non-zero.

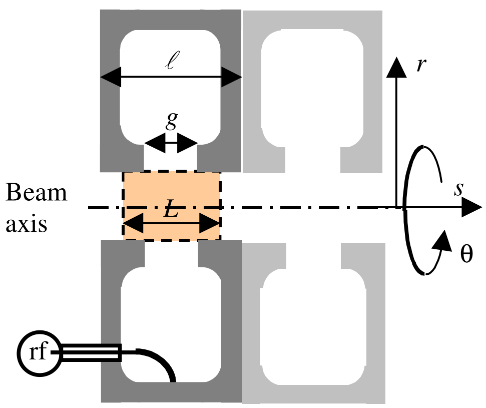
- Whether the standing-wave structure is called a gap, a cavity, or a tank with drift tubes depends on the external geometry, see next slide.

- The basic modules can be used individually or in a periodic array operating in the so-called ***π-mode*** in which the fields of adjacent cells are π out of phase, or the ***2π-mode*** for drift tubes in a tank.

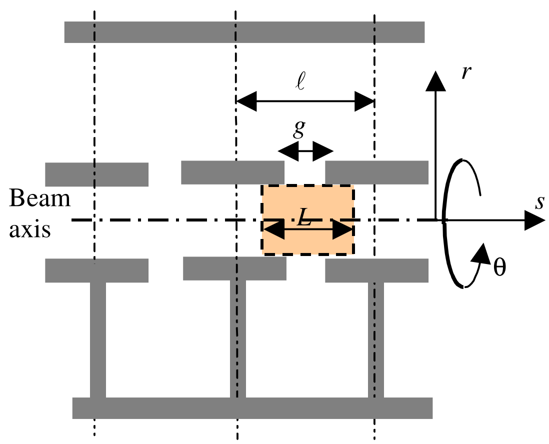
RF standing-wave structures







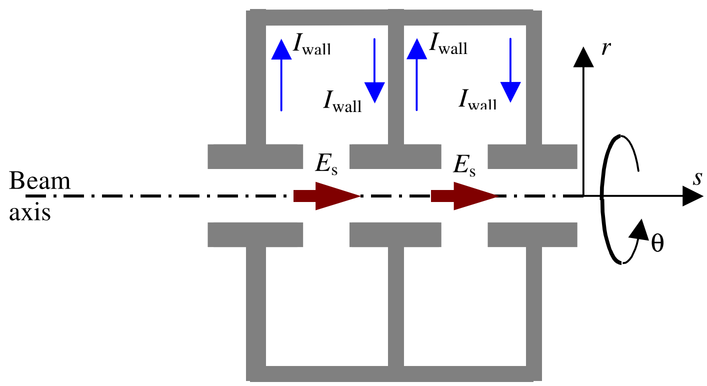
Cavities with noses



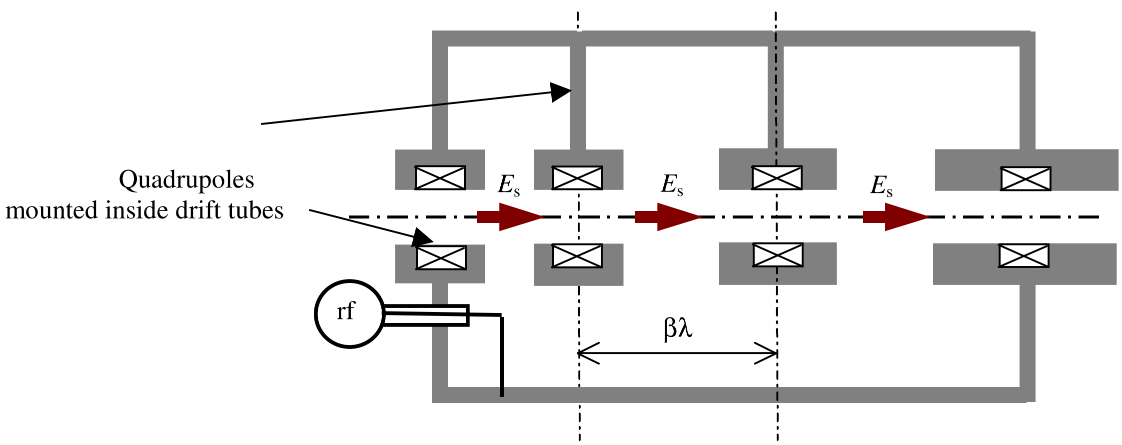
Drift tubes in a tank

The beige colour shows the 'useful' RF field region.

Alvarez linac (non-relativistic)



❖ **Start with a series of cavities with 'noses' or drift tubes and excite all cavities in phase (2π mode). Note wall currents cancel.**



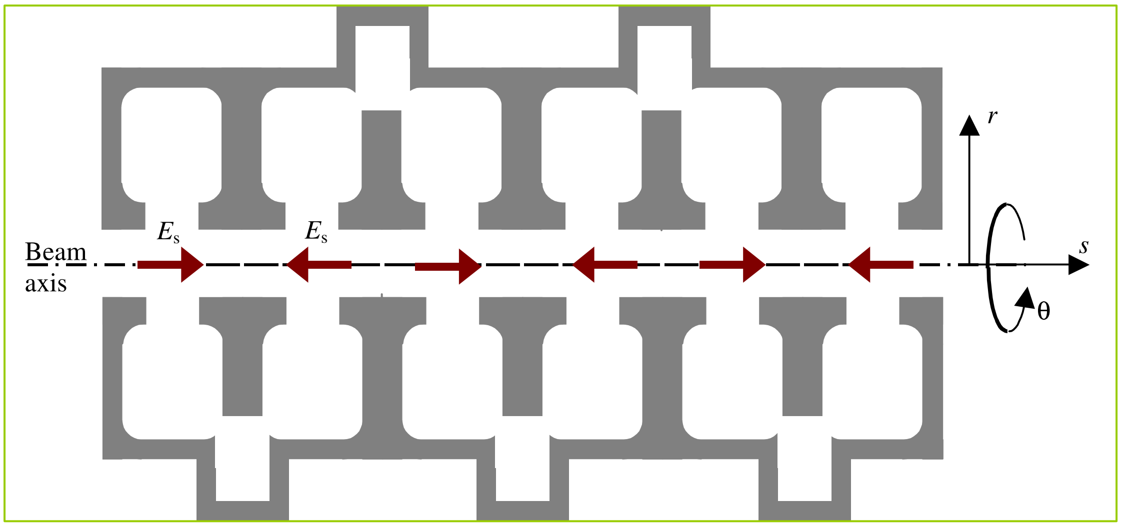
❖ **As wall currents cancel, remove walls except for a support column for the drift tubes.**

❖ **Now adjust the drift tube lengths for the velocity.**

❖ **Note there are quadrupoles lodged inside the drift tubes for additional focusing.**

❖ **You have an Alvarez.**

Coupled-cavity linac (relativistic)



- **The beam velocity is virtually that of light, so the cavities are identical.**
- **The cavities are coupled to be excited in the π-mode. This saves having an RF source for each cavity and synchronising them.**

All different, but all the same

- **The field on the axis always has the form,**

$$E_s(r,s,t) = E_s(r,s)\sin(\omega t + \phi_p)$$

where ω is the angular frequency of the standing wave and ϕ_p is the RF phase at $t = 0$.

- **It is too complicated and unnecessary to follow the full derivation, but it can be shown that the linearised fields are,**

$$E_s(s,t) = E_0 \sum_n A_n \cos\left(\frac{n\pi}{L}s\right)\sin\phi$$

$$E_r(r,s,t) = E_0 \sum_n A_n \frac{n\pi}{2L} r \sin\left(\frac{n\pi}{L}s\right)\sin\phi$$

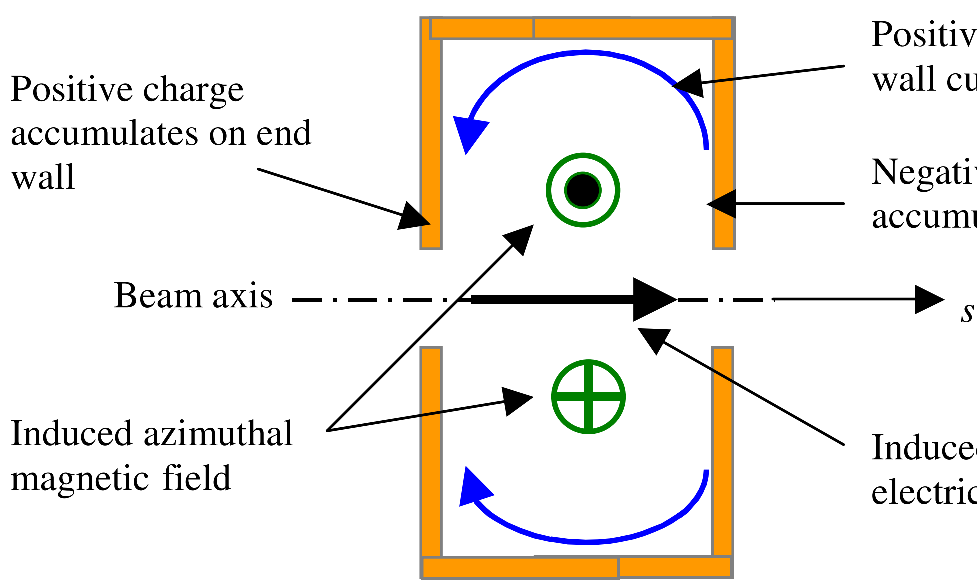
$$B_\theta(r,s,t) = E_0 \sum_n A_n \frac{\pi}{c\lambda} r \cos\left(\frac{n\pi}{L}s\right)\cos\phi \qquad (3)$$

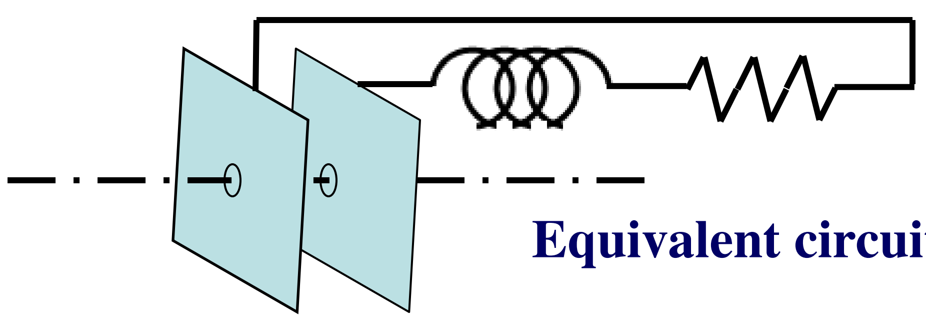
where n is odd, the amplitudes A_n depend on the mechanical shape of the cavity, E_0 is the average electric field across the 'useful' region at the time of peak field, L is the length of the active region, λ is the free-space wavelength at the RF frequency ω and $\phi = \omega t+\phi_p$.

Qualitative action of a cavity

- **Wall currents flow back and forth between the end plates that store the charge.**
- **The current flow supports an azimuthal magnetic field.**
- **The charge accumulation on the end plates drives an electric field that acts on the beam.**
- **To relate the azimuthal magnetic field to the induced axial electric field use Faraday's law.**

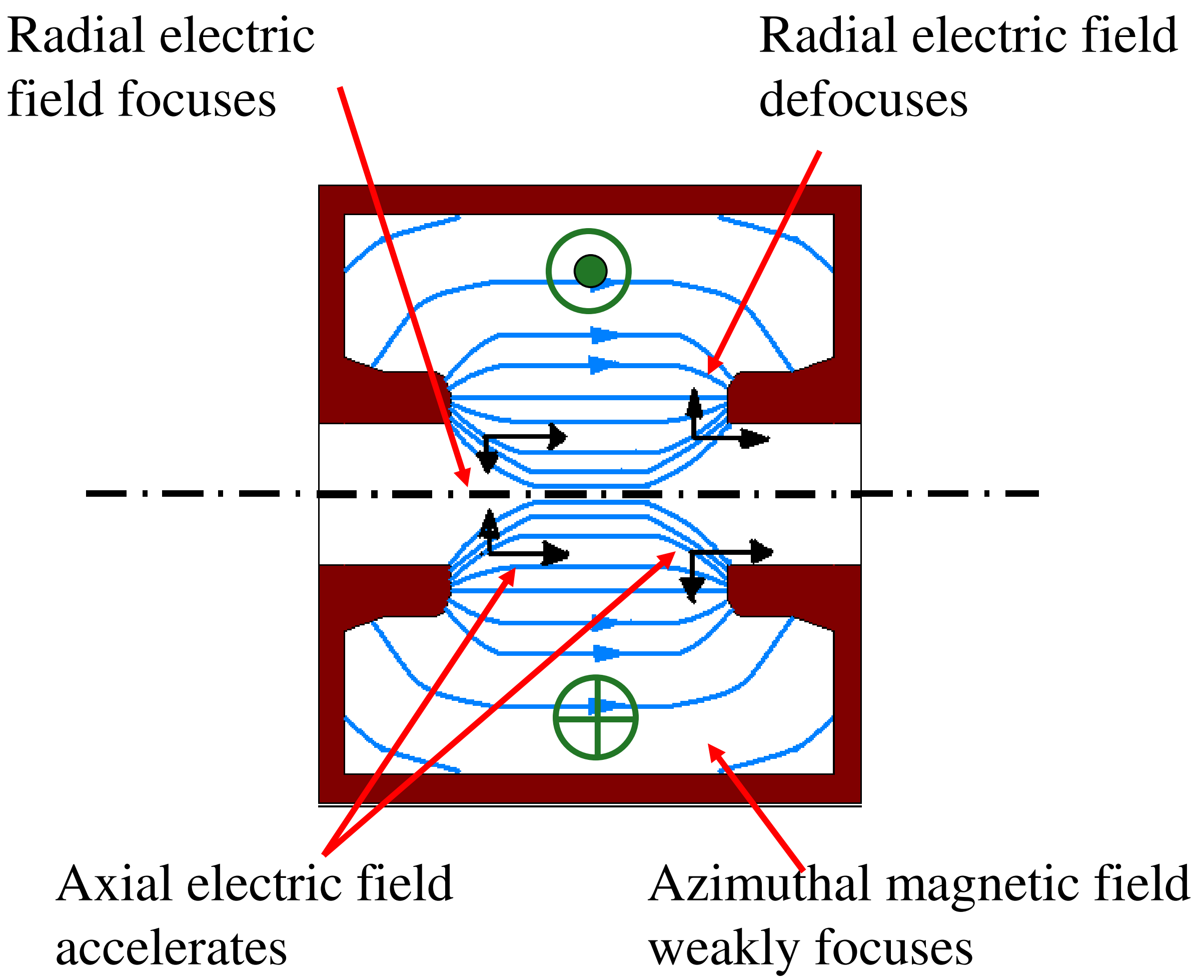
$$\nabla \times E = -\partial B/\partial t$$





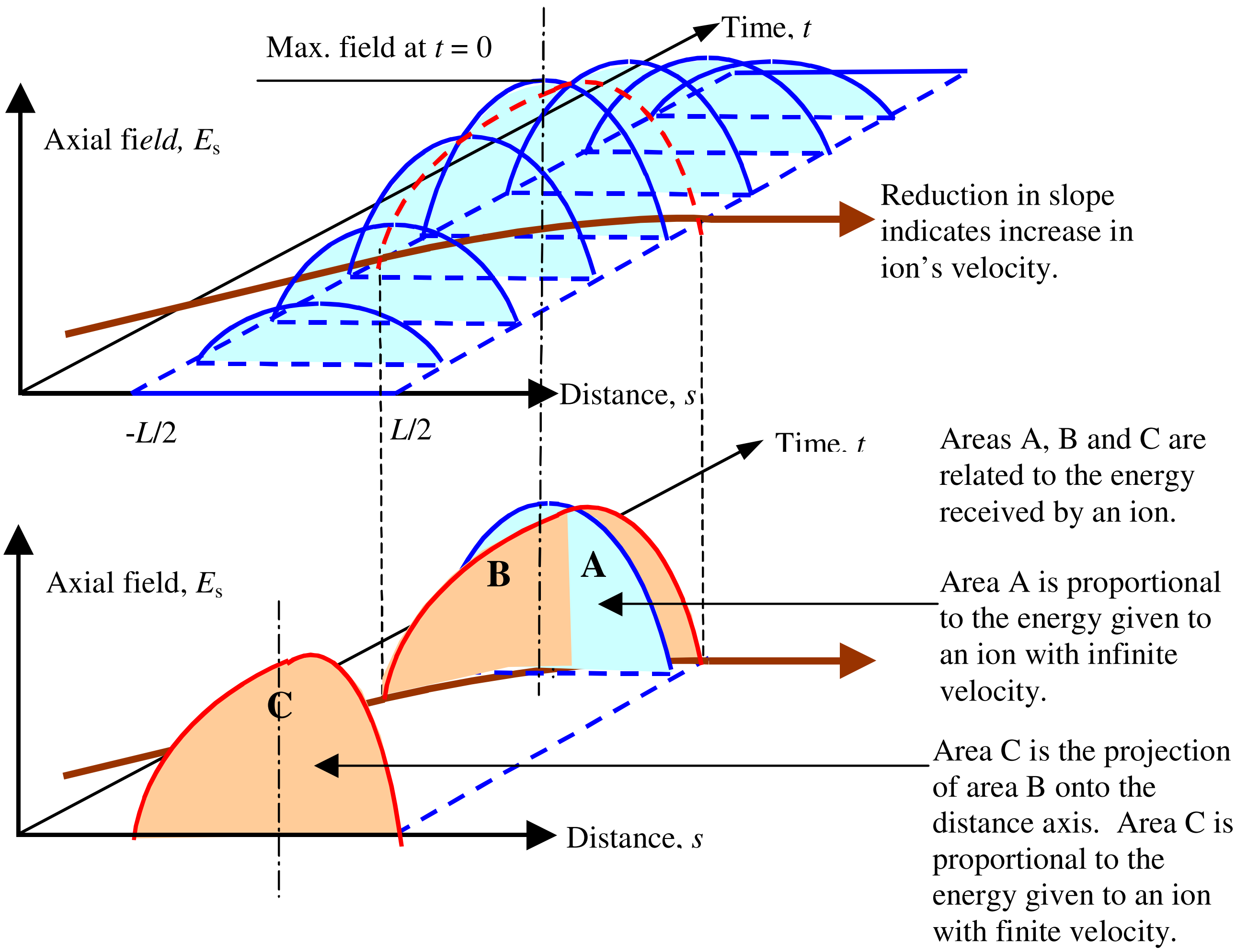
Equivalent circuit

Qualitative action of the cavity fields



- **The ‘noses’ shield the incoming ion from the axial field.**
- **As the ion enters the gap, the axial field rises with a cosine-like form.**
- **The radial focusing at the entry slightly exceeds the defocusing at the exit because the ion has a higher energy and is stiffer.**

Crossing a cavity



- **The blue half waves show the axial cavity field as it changes sinusoidally with time.**
- **The brown curve shows the ion's trajectory in space and time.**
- **The beige areas are defined by the position of the ion in space and time and the corresponding axial field.**
- **Beige area 'C' is the projection on the *s*-axis of the axial field 'seen' by the ion. The area of 'C' is the energy given to the ion.**

Transit time factor

- The exact solution for the transit of an ion in an RF cavity is complicated.

- The problem is partially avoided by defining something called the ***Transit Time Factor, T***.

- *T* is defined as the ratio of the maximum integral of the axial electric field that can be 'seen' by a ion traversing an RF cavity with velocity $v_s(s)$ to the maximum integral that can be 'seen' by a particle traversing with infinite velocity.

- To obtain the maximum integral the ion must enter shortly before the peak field is reached and exit shortly after.

- The general energy gain, $\Delta_s E$, is then defined as,

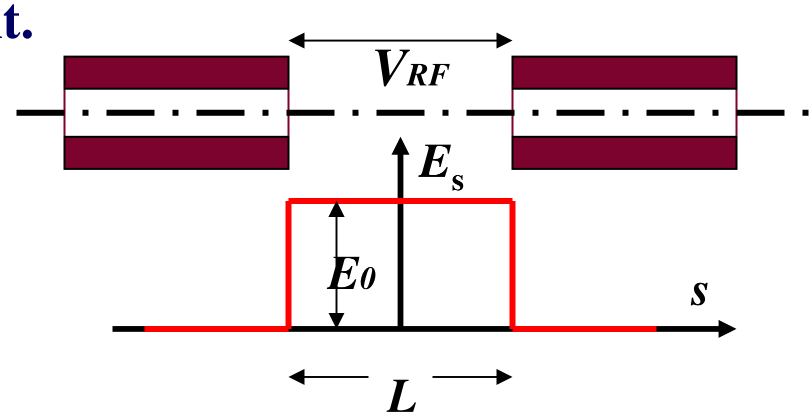
$$\boxed{\Delta E_s = qE_0 LT \sin\phi_p = qV_{RF} T \sin\phi_p \qquad (4)}$$

where E_0 is the average of the field distribution across the gap at peak field, L is the 'active' gap length, V_{RF} is the peak voltage and ϕ_p is the phase of the cavity field as the ion crosses the centre point (due to different origins you will also see $\cos\phi_p$).

- All the problems are now hidden in *T*, which can be estimated and a numerical calculation can be put off until really necessary.

First approximation for T

- Let the field in the gap have an amplitude $E_0 = V_{RF}/L$. Assume the field is constant with respect to distance s and is zero in the drift tubes. Assume the velocity is constant.



- The accelerating field is then, $E(t) = (V_{RF}/L)\cos(\omega t)$.
- For an ion passing the centre of the gap at $t = 0$ and with an average velocity v_0, its position is $s = v_0 t$ and its energy gain will be,

$$\Delta E_v = q\int_{-L/2}^{+L/2}\frac{V_{\mathrm{RF}}}{L}\cos(\omega_{\mathrm{RF}}t)\,\mathrm{d}s = q\frac{V_{\mathrm{RF}}}{L}\int_{-L/2}^{+L/2}\cos\left(\frac{\omega_{\mathrm{RF}}}{v_0}s\right)\mathrm{d}s$$

$$\Delta E_v = qV_{\mathrm{RF}}\frac{\sin(\theta/2)}{(\theta/2)} \quad \text{where } \theta = \frac{\omega_{\mathrm{RF}}L}{v_0}, \text{ Transit Angle}$$

- The maximum energy that can be extracted by an ion with infinite velocity is qV_{RF} so that,

$$\boxed{\text{Transit time, } T = \frac{\Delta E_v}{\Delta E_\infty} = \frac{\sin(\theta/2)}{(\theta/2)} \qquad (5)}$$

Second approximation for T

- **The field distribution with distance in the gap is close to a cosine [equation (3) 1st harmonic only], so we can improve our approximation,**

$$\Delta E_v = q\frac{V_{\mathrm{RF}}}{L}A_1\int_{-L/2}^{+L/2}\cos\left(\frac{\pi}{L}s\right)\cos(\omega_{\mathrm{RF}}t)\,\mathrm{d}s$$

$$\Delta E_v = \frac{qV_{\mathrm{RF}}}{2L}A_1\int_{-L/2}^{+L/2}\cos\left(\frac{\pi}{L}s+\frac{\omega_{\mathrm{RF}}}{v_0}s\right)+\cos\left(\frac{\pi}{L}s-\frac{\omega_{\mathrm{RF}}}{v_0}s\right)\mathrm{d}s$$

I leave you to do the mathematics,

$$\Delta E_v = \frac{\pi}{2}qV_{\mathrm{RF}}A_1\left(\frac{\cos(\theta/2)}{(\pi/2)^2-(\theta/2)^2}\right) \quad \text{where} \quad \theta = \frac{\omega_{\mathrm{RF}}L}{v_0}$$

- **The ion crossing with infinite velocity receives,**

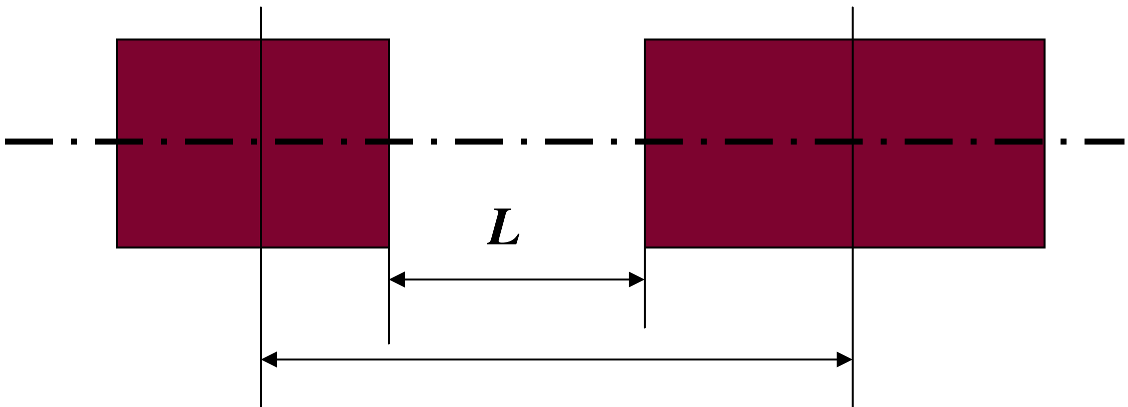
$$\Delta E_\infty = q\frac{V_{\mathrm{RF}}}{L}A_1\int_{-L/2}^{+L/2}\cos\left(\frac{\pi}{L}s\right)\mathrm{d}s = \frac{2qV_{\mathrm{RF}}}{\pi}$$

- **So finally,**

$$\boxed{\text{Transit time}, T = \frac{\Delta E_v}{\Delta E_\infty} = \frac{\cos(\theta/2)}{1-(\theta/\pi)^2} \qquad (6)}$$

Transit time factor continued

- **Consider an Alvarez structure operating in the 2π-mode and let the gap length be 0.4 of the cell length.**



Cell length = 2.5L = v/f_{RF} = $\beta c(2\pi/\omega_{RF})$

- **The Transit Angle is,**

$$\theta = \frac{\omega_{RF} L}{v_0} = \frac{\omega_{RF}}{\beta c}\frac{\beta c\, 2\pi}{2.5\,\omega_{RF}} = 0.8\pi$$

- **First approximation gives, *T* = 0.757**
- **Second approximation gives, *T* = 0.858**

[Note that the first and second approximation have singular points at θ = 0 and θ = π, respectively]

Linacs and Rings

- If you are interested in linacs, then the fields, Transit Time Factor etc. will be important to you.
- If you are interested in rings, then it is likely that you will be able to take a very simplified view of the cavities.
- In linacs, the RF period will be a few times the gap transit time.
- In a ring, the RF period will be related to the revolution period by a factor called the harmonic number, h,

$$h = \frac{\text{Revolution period}}{\text{RF period}} \qquad (7)$$

- In most cases, the time to cross the gap in a ring will be very small compared to the RF period and the ion will be fully relativistic, so that the Transit Time Factor will be close to unity.
- In this case, the energy gained by the beam will be,

$$\Delta_{\mathrm{s}} E = q V_{\mathrm{RF}} \sin \phi_{\mathrm{s}} \qquad (8)$$

[ϕ_s refers to the synchronous ion]

RF acceleration

- **From Lecture (2), equation (4) for bending in a dipole (omit sign),**

$$qv_0 B = \frac{m{v_0}^2}{\rho_0} \quad \Rightarrow \quad p_0 = q\rho_0 B$$

so that,
$$\frac{\mathrm{d}p}{\mathrm{d}t} = q\rho_0 \frac{\mathrm{d}B}{\mathrm{d}t}.$$

- **Re-writing for one turn,**

$$\Delta p_{\mathrm{Turn}} = q\rho_0 \dot{B}\, T_{\mathrm{Rev}}$$

- **Let *R* be the average machine radius, then,**

$$\Delta p_{\mathrm{Turn}} = q\rho_0 \dot{B}\, \frac{2\pi R}{\beta c}$$

- **Now, $\Delta E = \beta c \Delta p$, so,**

$$\Delta_{\mathrm{s}} E_{\mathrm{Turn}} = 2\pi q R \rho_0 \dot{B}$$

- **but we already have for *N* cavities,**

$$\Delta_{\mathrm{s}} E_{\mathrm{Turn}} = qV_{\mathrm{RF}} N \sin\phi_{\mathrm{s}}$$

- **Finally,**

$$\boxed{V_{\mathrm{RF}} N \sin\phi_s = 2\pi\rho_0 R\, \dot{B} \qquad (9)}$$

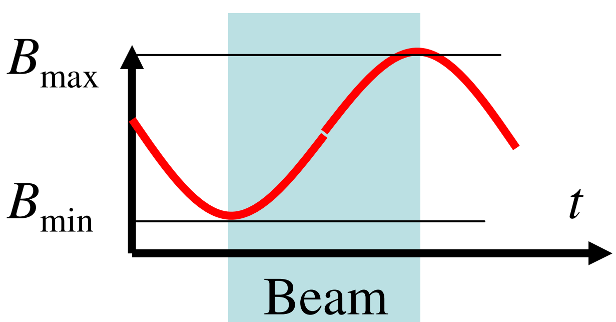
RF acceleration continued

- **The harmonic number, *h* sets the number of RF oscillations in one revolution. There will be one stable RF bucket per RF oscillation, i.e. *h* buckets and correspondingly up to *h* bunches in the machine.**

$$f_{\mathrm{Rev}} = \frac{1}{T_{\mathrm{Rev}}} = \frac{\beta c}{2\pi R}$$

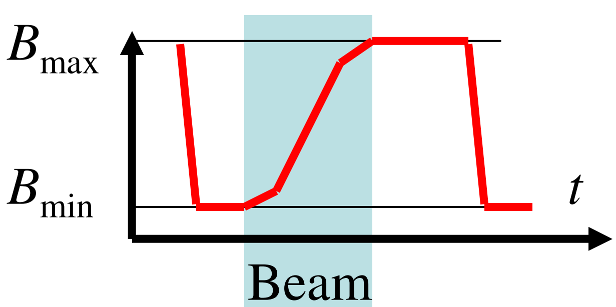
$$f_{\mathrm{RF}} = \frac{ch}{2\pi R}\sqrt{1-\frac{1}{\left(1+T/E_0\right)^2}}$$

- **The magnetic field ramp is the 'driving' parameter behind the RF programmes for f_{RF}, V_{RF} and ϕ_s.**
- **Fast cycling machines have resonant power supplies.**



Maximum dB/dt at mid-cycle

- **Slow cycling machines are 'ramp and hold'**



dB/dt mostly constant with 'round in' and 'round-out' curves

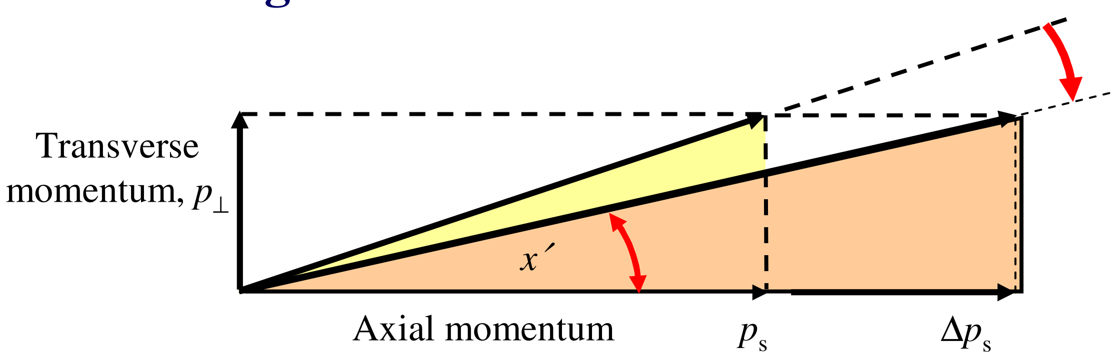
Adiabatic damping

- **A beam is a cluster of singular points in 6-D phase space (x, x', y, y', Δs, $\Delta p/p$). It is assumed that the three component phase spaces (x, x'), (y, y') and (Δs, $\Delta p/p$) are independent, i.e. there are no initial correlations and the motion is uncoupled. Most beams contain typically 10^8 to 10^{14} ions and it is natural to forget the point-like structure and refer to areas in phase space.**

Geometric emittance, $$\varepsilon_z = \iint_{\text{Beam}} \mathrm{d}z\,\mathrm{d}z' \quad (10)$$

where (z, z') is any one of the component phase spaces.

- **It is well known that the phase-space area, or emittance, of a beam shrinks as it is accelerated, giving rise to what is known as *adiabatic damping*. The physics of this effect can be illustrated by a simple vector diagram.**



Adiabatic damping continued

- **It can be shown that the phase-space quantity that remains invariant with acceleration is the**

Normalised emittance, $\varepsilon_{\mathrm{n,z}} = \beta\gamma\,\varepsilon_{\mathrm{z}} \quad (11)$

- **The derivation of the transfer matrix for an RF cavity makes astute use of the above.**

1) The cavity is treated as a thin lens, or a series of thin lenses. Each lens has zero thickness, so it is flanked by drift spaces on either side that add up to the length of the active gap.

2) Since a 'thin' lens has zero thickness the transverse positions of the ions (*x, y*) cannot change, *so the full effect of the adiabatic damping is in the angles (remember this trick)*,

$$z'_{\mathrm{exit}} = (\beta\gamma)_{\mathrm{entry}}(\beta\gamma)^{-1}_{\mathrm{exit}}\, z'_{\mathrm{entry}}$$

3) The Δs term is transferred through the thin lens unchanged.

4) The $\Delta p/p$ term is changed according to the ion's value of Δs.

5) The drift spaces flanking the thin lens are treated in the normal way.

[The full derivation is rather complicated.]

Transition energy

- **The angular revolution frequency is,**

$$\Omega_{\mathrm{Re}\,v} = \frac{2\pi v}{C}$$

which gives,

$$\frac{\Delta\Omega_{\mathrm{Re}\,v}}{\Omega_{\mathrm{Re}\,v}} = \frac{\Delta v}{v} - \frac{\Delta C}{C}. \qquad \text{(A)}$$

- **From relativity,**

$$\frac{\Delta v}{v} = \frac{1}{\gamma^2}\frac{\Delta p}{p}. \qquad \text{(B)}$$

- **Define α as the *momentum compaction function*,**

$$\frac{\Delta C}{C} = \alpha\frac{\Delta p}{p} \qquad \text{(C)}$$

- **So that by substituting (B) and (C) into (A),**

$$\boxed{\frac{\Delta\Omega_{\mathrm{Re}\,v}}{\Omega_{\mathrm{Re}\,v}} = \frac{\Delta p}{p}\left(\frac{1}{\gamma^2} - \alpha\right) \qquad (12)}$$

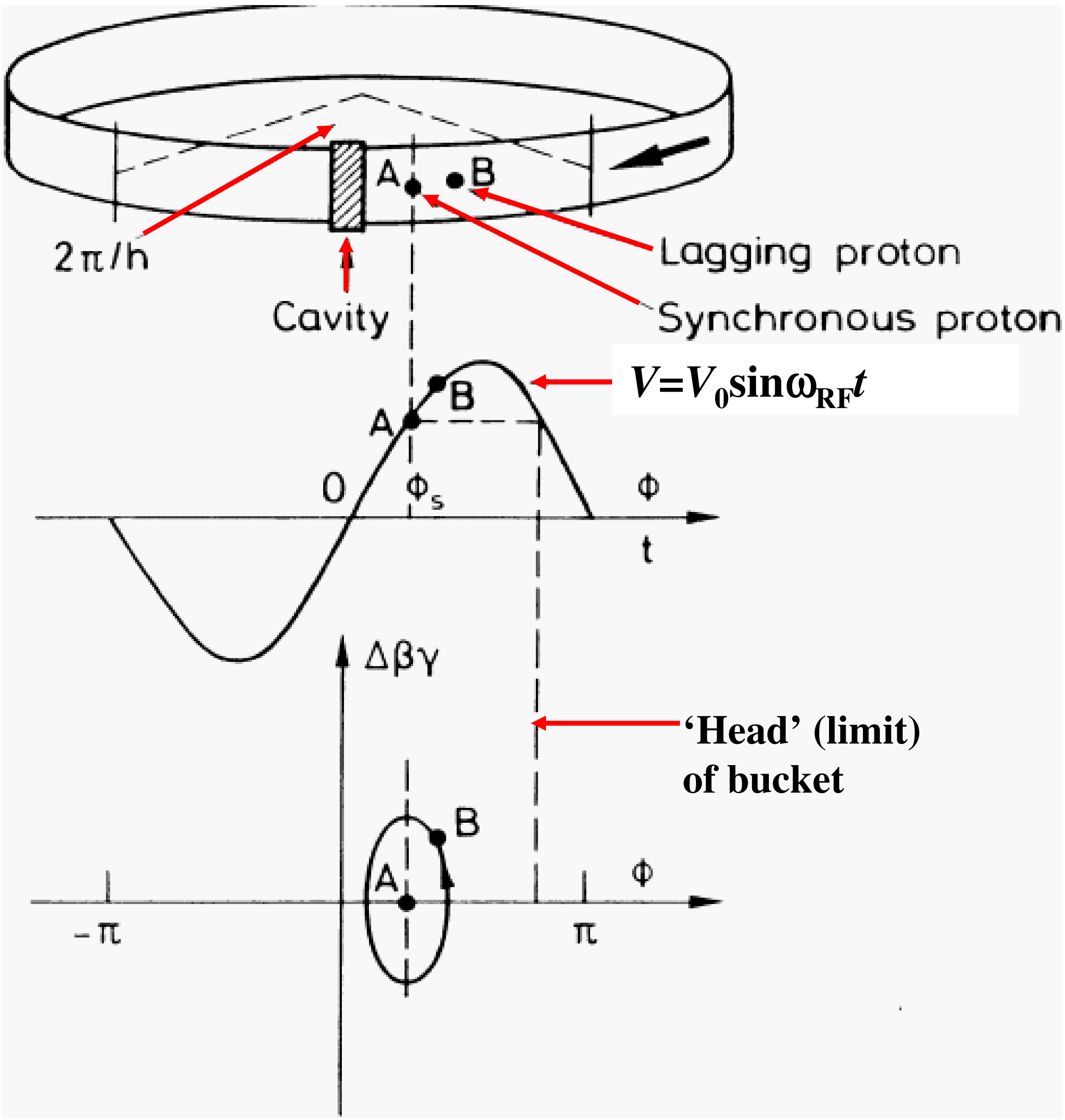
α depends only on the lattice.

- **The *transition energy* is defined as**

$$\boxed{\gamma_{\mathrm{tr}} = \alpha^{-1/2} \qquad (13)}$$

- **Below transition $\frac{\Delta\Omega_{\mathrm{Re}\,v}}{\Omega_{\mathrm{Re}\,v}}$ is positive and above it is negative.**

Phase stability

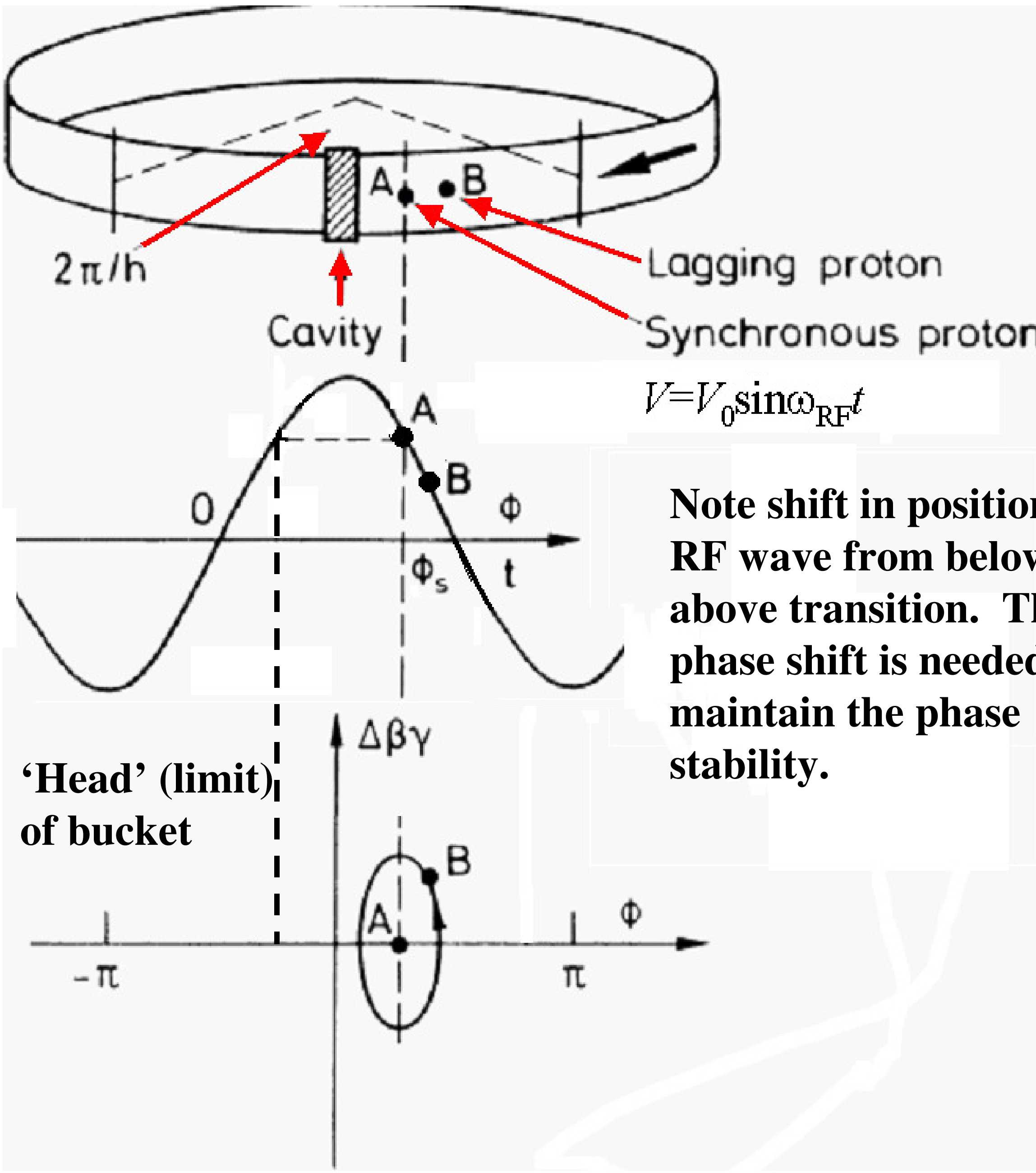


Below transition

This case is intuitive as Δv dominates.

- **Lag behind -get more energy - catch up.**
- **Get ahead - get less energy - fall back.**

Phase stability continued



Note shift in position of RF wave from below to above transition. This phase shift is needed to maintain the phase stability.

Above transition

This case is not intuitive as Δ*C* dominates.

- **Lag behind –get less energy - catch up.**
- **Get ahead - get more energy - fall back.**

Summary

- We defined the synchronous ion and introduced the idea of longitudinal phase space being described by the deviations (Δs, $\Delta p/p$) from the synchronous ion.

- We looked at the derivation of the Δs lag/lead term for inclusion in transfer matrices of general lattice elements.

- We examined standing-wave RF devices and their fields, but only looked qualitatively at how they work.

- We defined the Transit Time Factor.

- We looked at acceleration, Transition Energy and phase stability in rings.

- We did not cover Twiss parameters in the longitudinal plane, nor did we tackle longitudinal matching.

- We hinted at how to calculate a complete transfer matrix for a cavity.

Annex

Δs for dipoles

- In dipoles, the variation in bending with momentum makes the paths followed by ions of different momenta different in length.

- To find this difference, divide the equilibrium orbit into elementary sections of length $d\ell = \rho_0\, d\theta$. To 1st-order, the elementary length of a trajectory crossing this angular slice would be $d\ell^* = (\rho_0 + x)\, d\theta$, where x is the radial position of the trajectory. The full length of the trajectory is then found by integrating over θ.

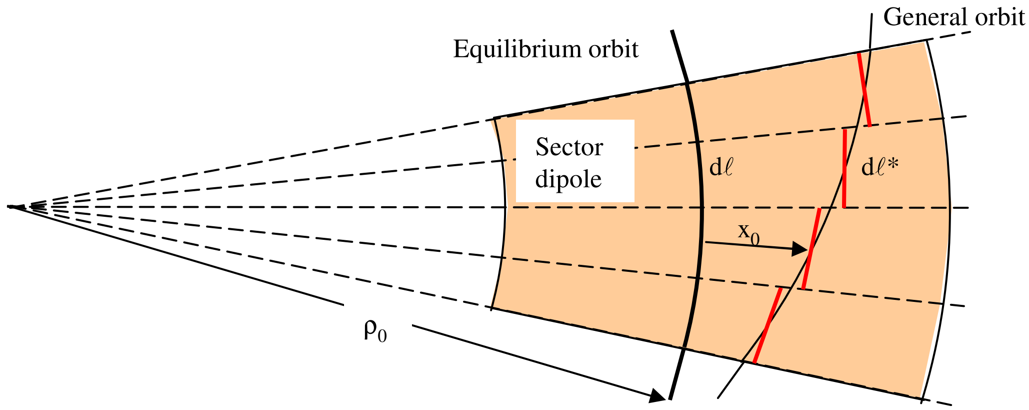
- The change in length of each elementary section of trajectory due to its slope x' is 2nd-order and is neglected.

$$\ell = \int_0^{\ell} \mathrm{d}s$$

$$\ell^* = \int_0^{\alpha} [\rho_0 + x(s)]\mathrm{d}\phi \qquad (1)$$

$$\ell^* = \int_0^{\ell} \frac{[\rho_0 + x(s)]}{\rho_0} \mathrm{d}s$$

Δs for dipoles continued



❖ **Thus, the geometric path difference is given by,**

$$\Delta s_{\text{path length}} = \ell - \ell^{*} = -\frac{1}{\rho_0}\int_0^{\ell} x(s)\mathrm{d}s \qquad (2)$$

❖ **This reduces the problem to the integration of the expression for the radial position $x(s)$ of the orbit in the plane of bending as derived in Lecture 2 and given in the Formula Book.**

❖ **The final expression is also given in the Formula Book.**

❖ **The detailed mathematics is not important, but remember the method, because you will have a hard time finding this explanation in the literature.**

Δs for electrostatic bends

- **As mentioned in Lecture 3, there is a fundamental difference between electric and magnetic bends.**
- **When traversing a magnetic field, the ion's energy is rigorously constant, whereas in an electric field the ion can absorb or release energy to as it moves away from the central orbit.**
- **The energy exchanged with an electric field affects the basic equation for the lead or lag, by introducing a second contribution, δ*v*, to the velocity. Thus for electrostatic elements, equation (1) in the main lecture is more exactly written as,**

$$\Delta s = v\Delta t + t\Delta v + t\delta v \qquad (3)$$

Remember that Δ*v* arises from the momentum deviation Δ*p*/*p* of the incoming beam and that this term is constant.

Δs for electrostatic bends continued

- **The first terms in (3) are calculated as before.**
- **Referring back to Lecture 3, the radial position and the velocity were related for local energy changes by,**

$$\frac{x}{\rho_0} = -\frac{1}{\left(1-\beta^2\right)}\frac{\delta v}{v}$$

which gives,

$$\delta v = -\frac{x}{\rho_0}\left(1-\beta^2\right)v$$

- **Since δv is changing, it is necessary to integrate,**

$$t\delta v = \int_0^t \delta v \, \mathrm{d}t = -\frac{1}{\rho_0}\int_0^\ell \left(1-\beta^2\right)x(s)\,\mathrm{d}s \qquad (4)$$

- **Equations (2) and (4) can be combined so that,**

$$\boxed{\Delta s_{\text{path length}} + t\delta v = -\frac{1}{\rho_0}\int_0^\ell x(s)\left(2-\beta^2\right)\mathrm{d}s \qquad (5)}$$

Thus the final result is that of the magnetic bend multiplied by $(2-\beta^2)$.

- **Do not try to remember the detailed mathematics, but remember the method, because it is not well documented in the literature. The matrices are given on the next slides for reference.**

Longitudinal terms for transfer matrices of horizontal bends

- **For magnetic dipoles & electrostatic bends,**

$$\begin{bmatrix} m_{11} & m_{12} & 0 & 0 & 0 & m_{16} \\ m_{21} & m_{22} & 0 & 0 & 0 & m_{26} \\ 0 & 0 & m_{33} & m_{34} & 0 & 0 \\ 0 & 0 & m_{43} & m_{44} & 0 & 0 \\ m_{51} & m_{52} & 0 & 0 & 1 & m_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \\ \Delta s \\ \Delta p / p \end{bmatrix}$$

- **For $K_x > 0$,** ***for magnetic bends remove $(2-\beta^2)$***

$$m_{51} = \frac{-\left(2-\beta^2\right)}{\rho_x} \frac{\sin\left(\sqrt{|K_x|}\ell\right)}{\sqrt{|K_x|}}$$

$$m_{52} = \frac{-\left(2-\beta^2\right)}{\rho_x} \frac{\left(1-\cos\left(\sqrt{|K_x|}\ell\right)\right)}{|K_x|}$$

$$m_{56} = \frac{\ell}{\gamma^2} - \frac{\left(2-\beta^2\right)}{\rho_x} \frac{\left(\sqrt{|K_x|}\ell - \sin\left(\sqrt{|K_x|}\ell\right)\right)}{\rho_x |K_x|^{3/2}}$$

This looks complicated, but

$m_{16} = -m_{52}$

$m_{26} = -m_{51}$

Longitudinal terms continued

- **For $K_x < 0$,** ***for magnetic bends remove (2-β^2)***

$$m_{51} = \frac{-\left(2-\beta^2\right)}{\rho_x}\frac{\sinh\left(\sqrt{|K_x|}\ell\right)}{\sqrt{|K_x|}}$$

$$m_{52} = \frac{-\left(2-\beta^2\right)}{\rho_x}\frac{\left(\cosh\left(\sqrt{|K_x|}\ell\right)-1\right)}{|K_x|}$$

$$m_{56} = \frac{\ell}{\gamma^2} - \frac{\left(2-\beta^2\right)}{\rho_x}\frac{\left(\sqrt{|K_x|}\ell - \sinh\left(\sqrt{|K_x|}\ell\right)\right)}{\rho_x |K_x|^{3/2}}$$

This looks complicated, but

$m_{16} = -m_{52}$

$m_{26} = -m_{51}$

Longitudinal terms for matrices of vertical bends

- **For magnetic dipoles & electrostatic bends,**

$$\begin{bmatrix} m_{11} & m_{12} & 0 & 0 & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{33} & m_{34} & 0 & m_{36} \\ 0 & 0 & m_{43} & m_{44} & 0 & m_{46} \\ 0 & 0 & m_{53} & m_{54} & 1 & m_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \\ \Delta s \\ \Delta p / p \end{bmatrix}$$

- **For $K_y > 0$,** ***for magnetic bends remove (2-β^2)***

$$m_{53} = \frac{-(2-\beta^2)}{\rho_y} \frac{\sin\left(\sqrt{|K_y|}\ell\right)}{\sqrt{|K_y|}}$$

$$m_{54} = \frac{-(2-\beta^2)}{\rho_y} \frac{\left(1-\cos\left(\sqrt{|K_y|}\ell\right)\right)}{|K_y|}$$

$$m_{56} = \frac{\ell}{\gamma^2} + \frac{(2-\beta^2)}{\rho_y} \frac{\left(\sqrt{|K_y|}\ell - \sin\left(\sqrt{|K_y|}\ell\right)\right)}{\rho_y |K_y|^{3/2}}$$

This looks complicated, but

$m_{36} = -m_{54}$

$m_{46} = -m_{53}$

Longitudinal terms continued

- **For $K_y < 0$,** ***for magnetic bends remove (2-β^2)***

$$m_{53} = \frac{-\left(2-\beta^2\right)}{\rho_y}\frac{\sinh\left(\sqrt{|K_y|}\ell\right)}{\sqrt{|K_y|}}$$

$$m_{54} = \frac{-\left(2-\beta^2\right)}{\rho_y}\frac{\left(\cosh\left(\sqrt{|K_y|}\ell\right)-1\right)}{|K_y|}$$

$$m_{56} = \frac{\ell}{\gamma^2} + \frac{\left(2-\beta^2\right)}{\rho_y}\frac{\left(\sqrt{|K_y|}\ell - \sinh\left(\sqrt{|K_y|}\ell\right)\right)}{\rho_y|K_y|^{3/2}}$$

This looks complicated, but

$m_{36} = -m_{54}$

$m_{46} = -m_{53}$