MAGNET DESIGN

Foundations of Electromagnetic Fields, Analytical and Numerical Field Computation, Design and Optimization in Magnet Technology

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The Nürnberg Funnel

No “cooking recipes”

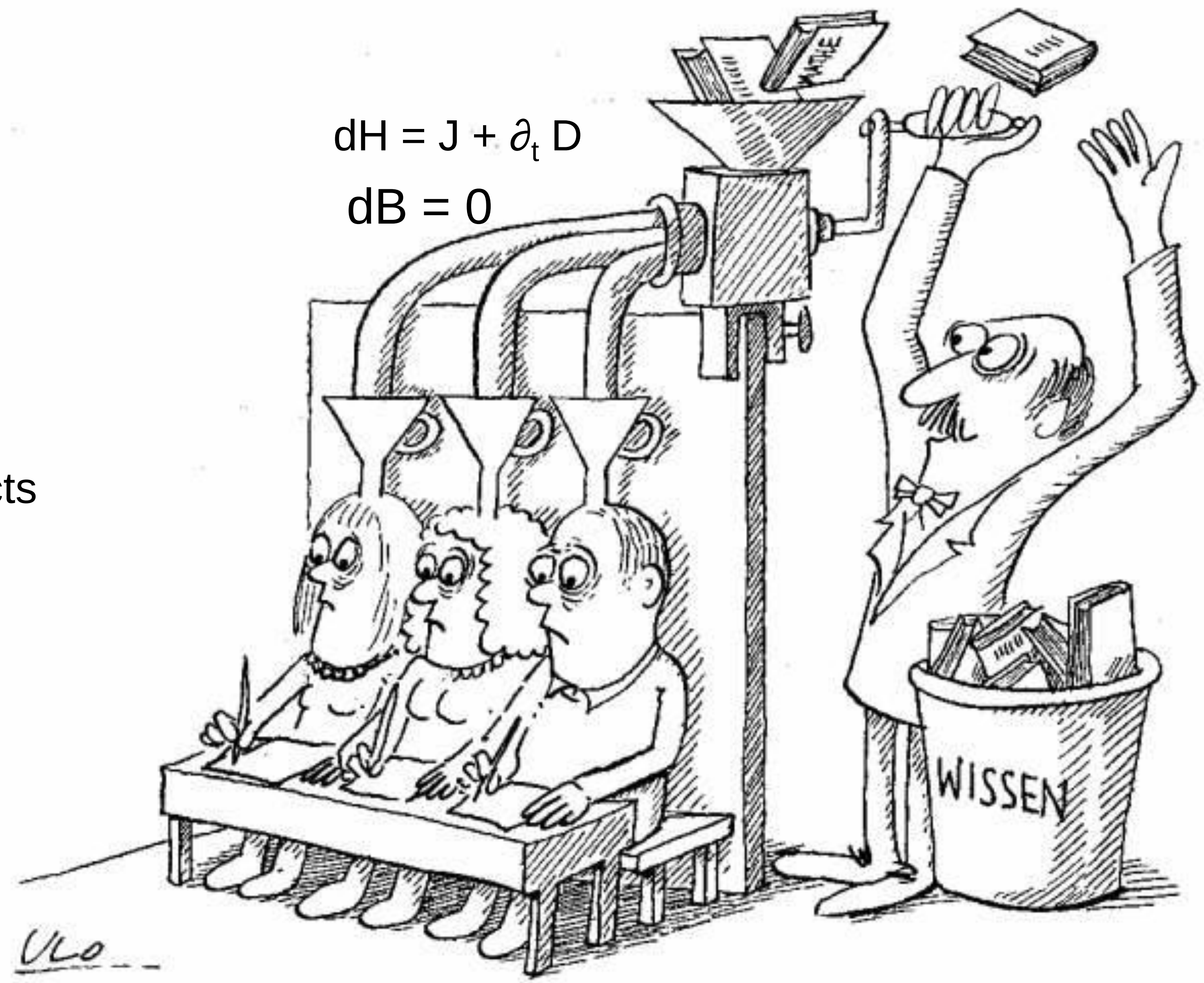
Less powerpoint

More interaction

If needed: Less subjects

$$dH = J + \partial_t D$$

$$dB = 0$$



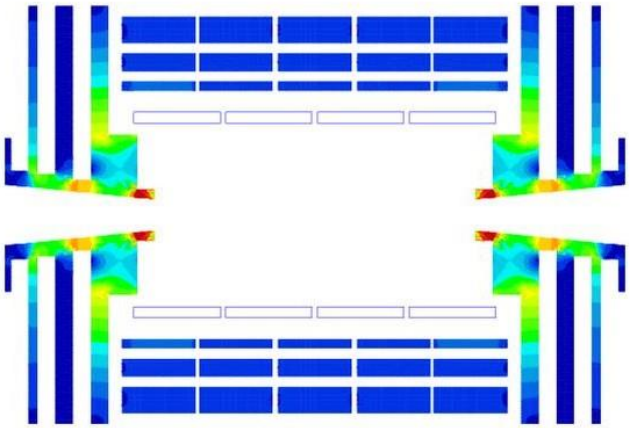
Course Outline

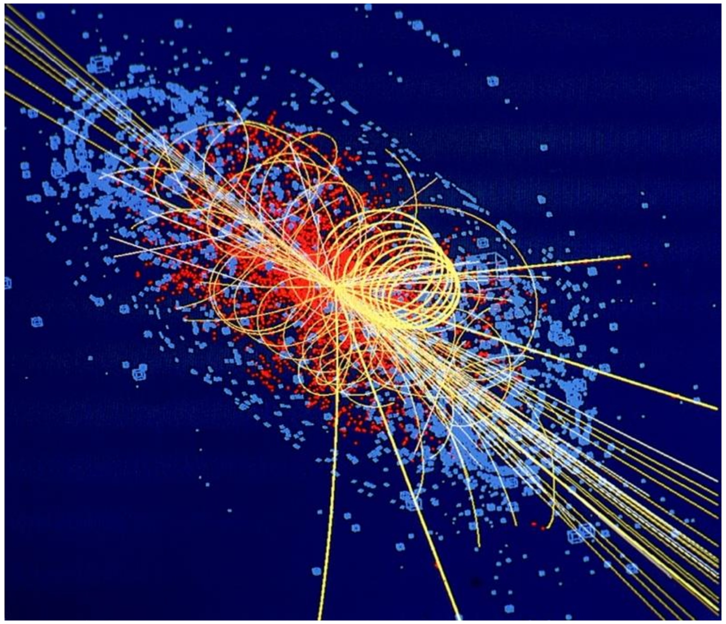
- Introduction and overview
- Mappings, Real functions, Linear algebra
- Vector analysis, Harmonic fields, Complex analysis
- Foundations of electromagnetic fields, the Maxwell equations

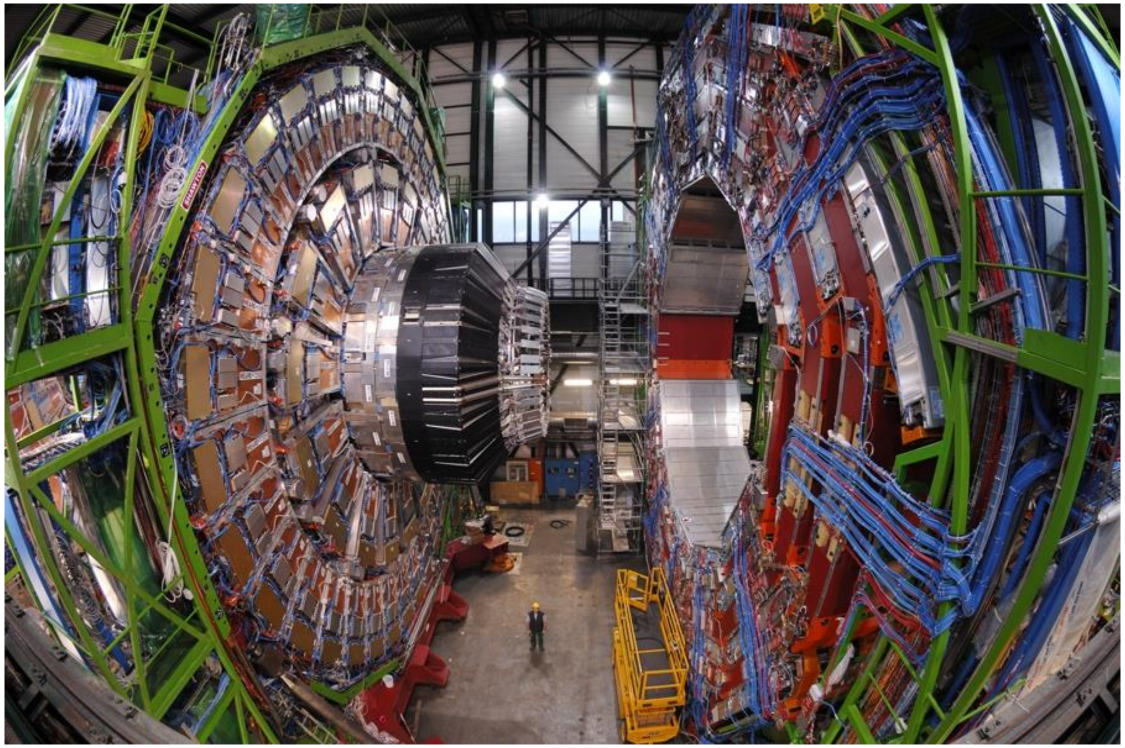
- Normal conducting magnets
- Magnetic field measurements
- Field of line-currents, Biot-Savart
- Coil fields of superconducting magnets, field harmonics

- Principles of numerical field computation
- Numerical field computation for accelerator magnets
- Post-processing: Inductances, persistent, interstrand coupling currents
- Quench simulation
- Mathematical Optimization techniques

CMS (Class 1 Magnets)



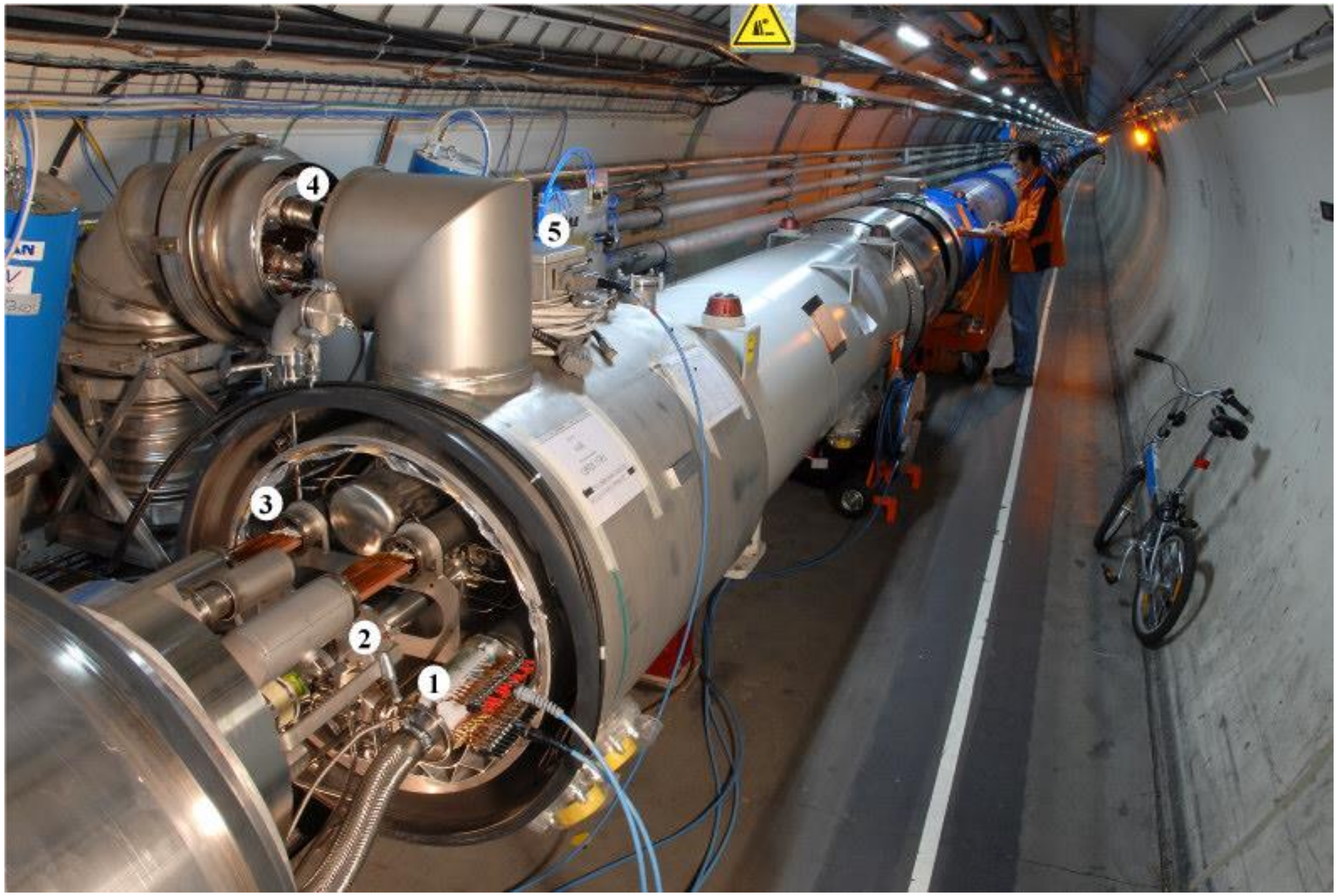




$$S = R(1 - \cos\frac{\alpha}{2}) \approx \frac{R\alpha^2}{8} = \frac{QBL^2}{8p}$$

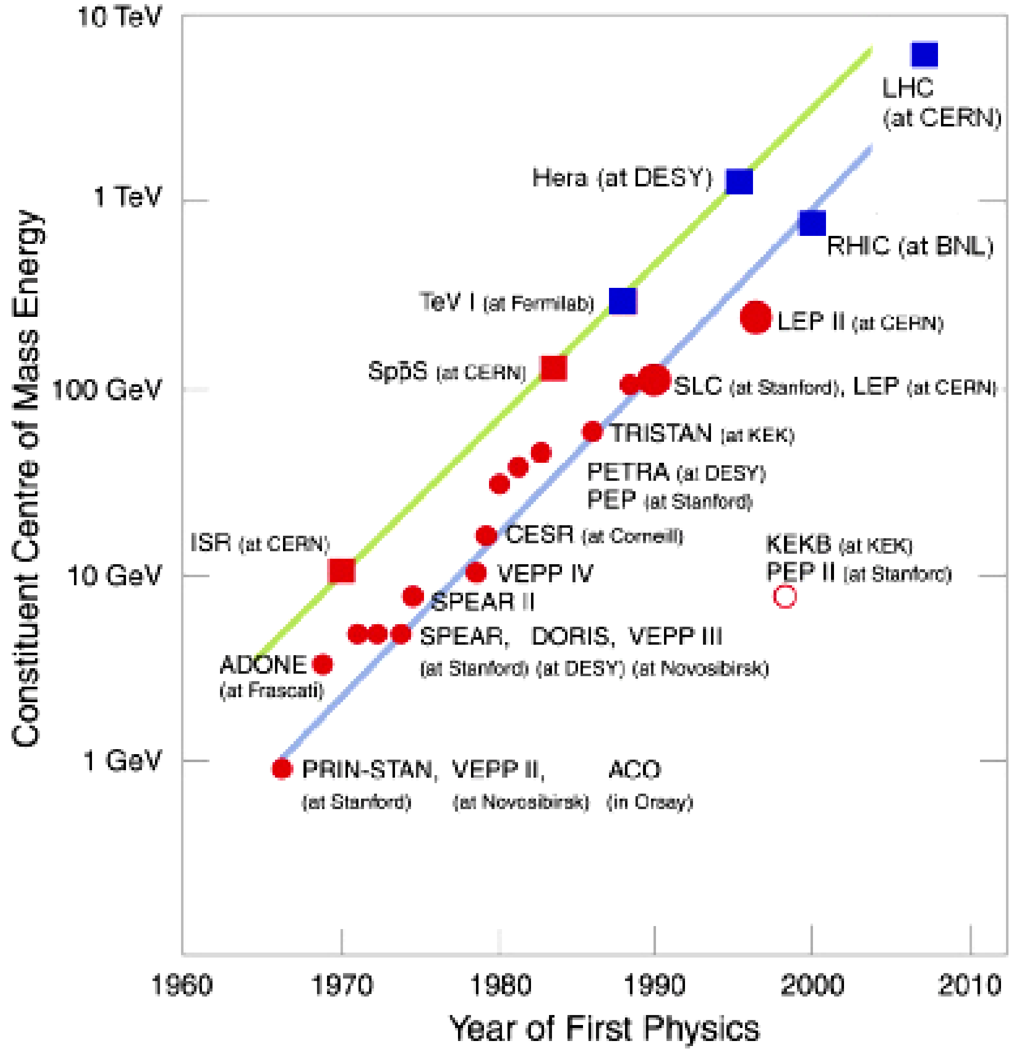
String of LHC Magnets in the Tunnel (Class 2 Magnets)

$$\{p\}_{\mathrm{GeV/c}} \approx 0.3\{Q\}_e\{R\}_{\mathrm{m}}\{B_0\}_{\mathrm{T}}$$



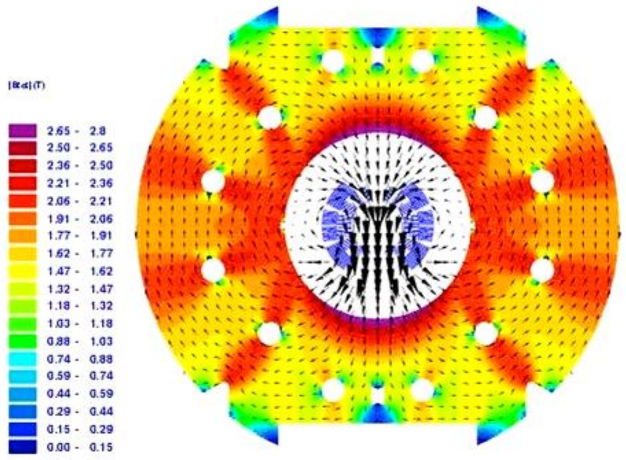
High field and high current density

Livingston Plot

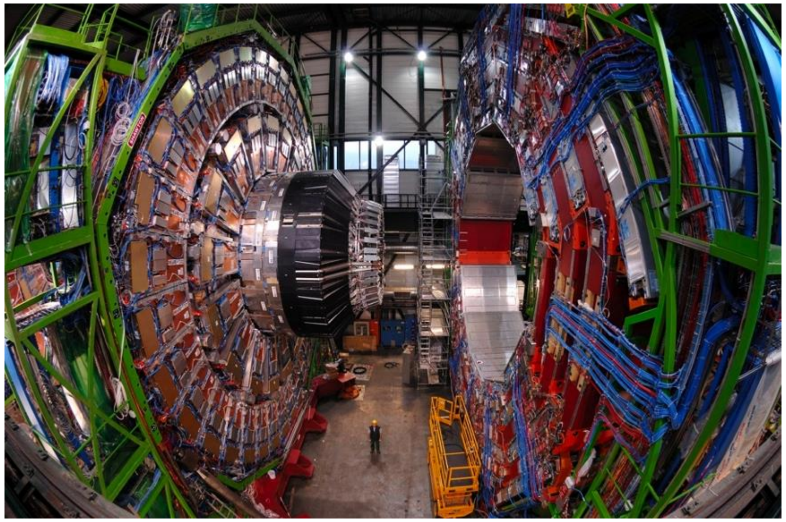


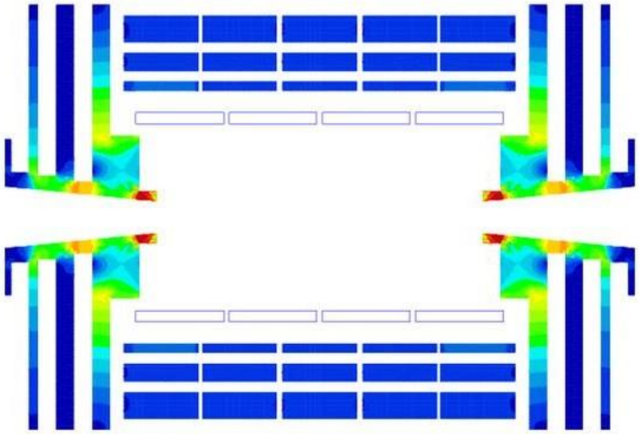
Blue: Accelerator project using SC technology

Coil Dominated Magnets



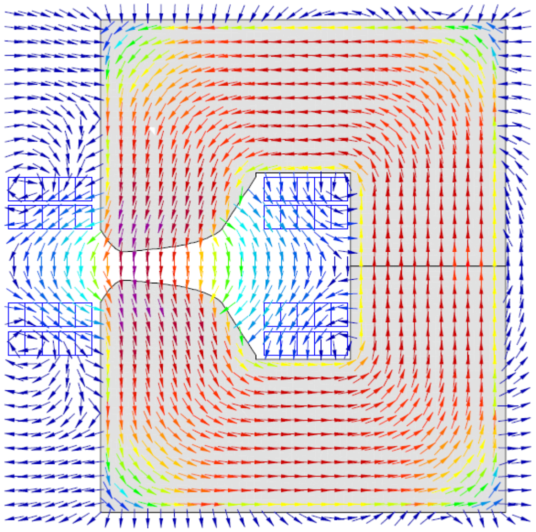
$B = 8.33$ T $\quad B_s = 7.77$ T

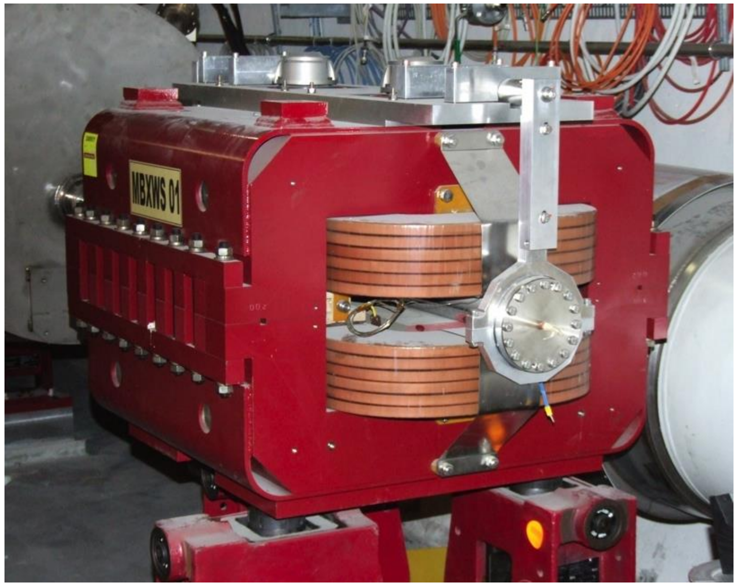


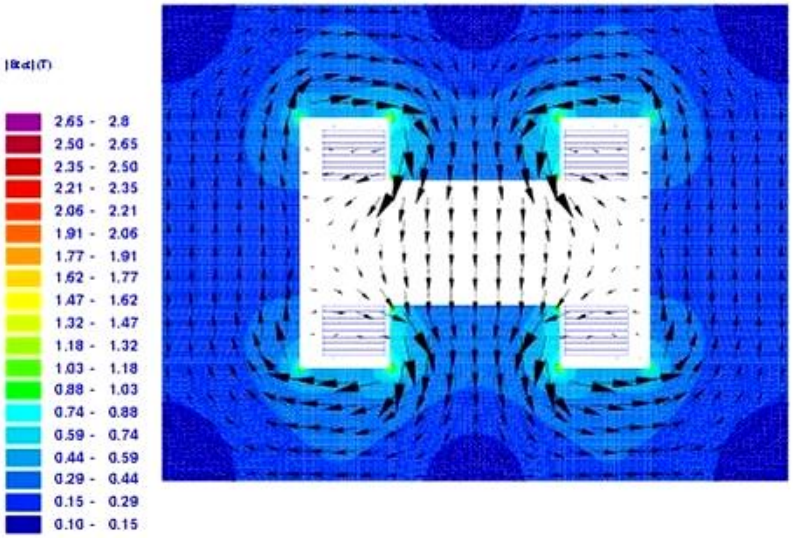


$B = 4$ T $\quad B_s = 3.69$ T

Iron Dominated Magnets

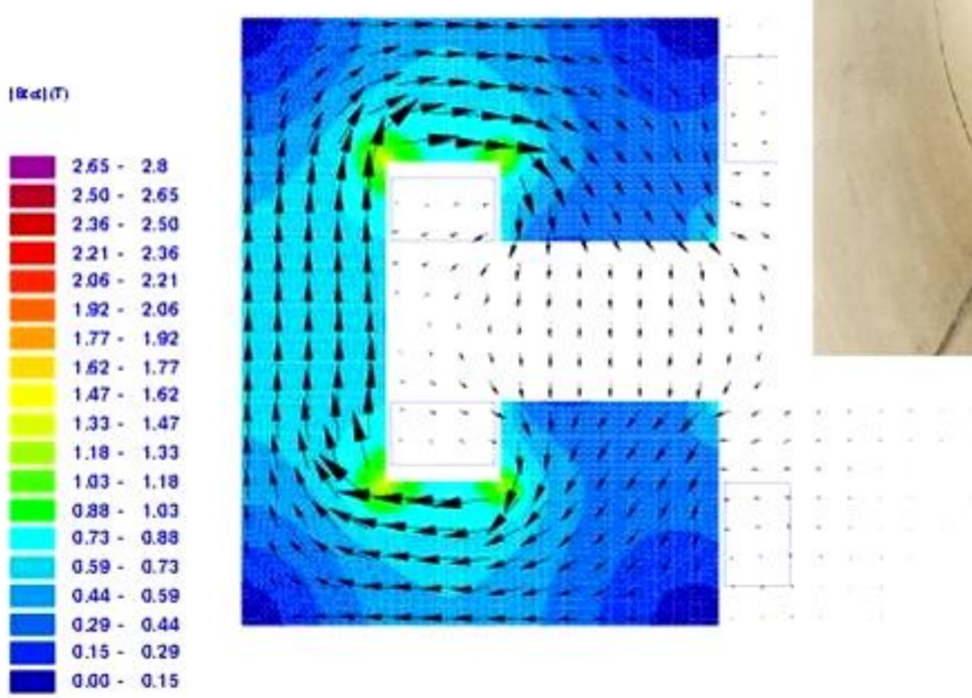






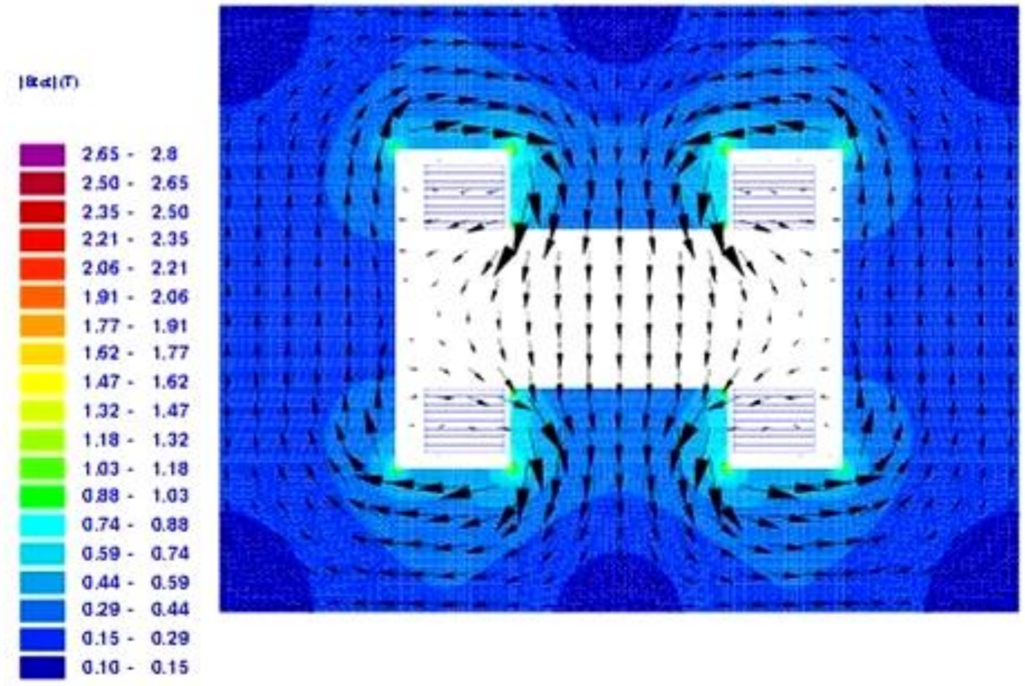
$N \cdot I = 24000$ A $\quad B_1 = 0{,}3$ T $\quad B_s = 0{,}065$ T $\quad$ Fill.fac. 0.98

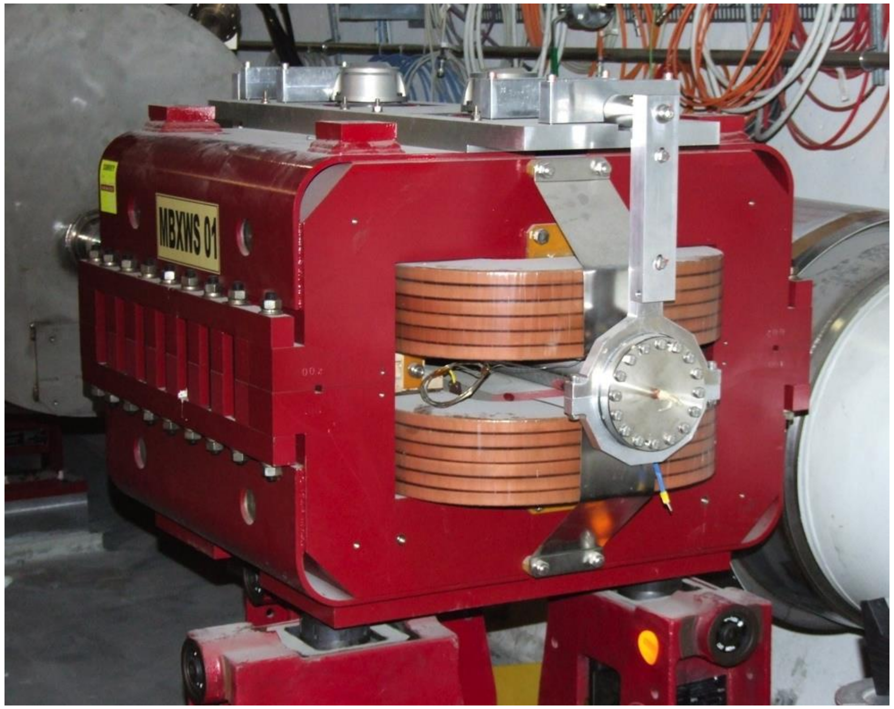
LEP Dipole



$N \cdot I = 4480$ A $B_1 = 0.13$ T $B_S = 0.042$ T Fill.fac. 0.27

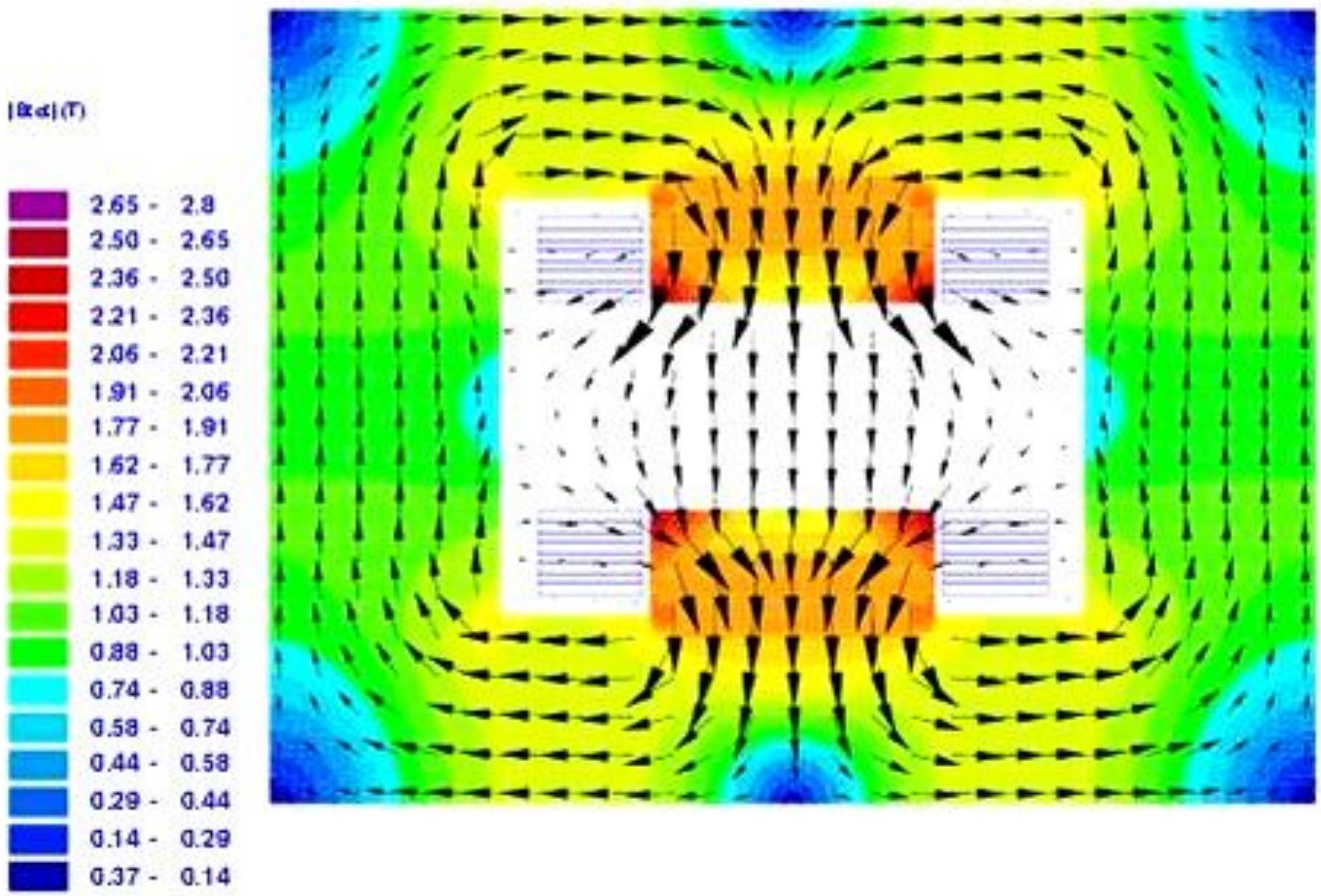
H Magnet (LHC transfer line)





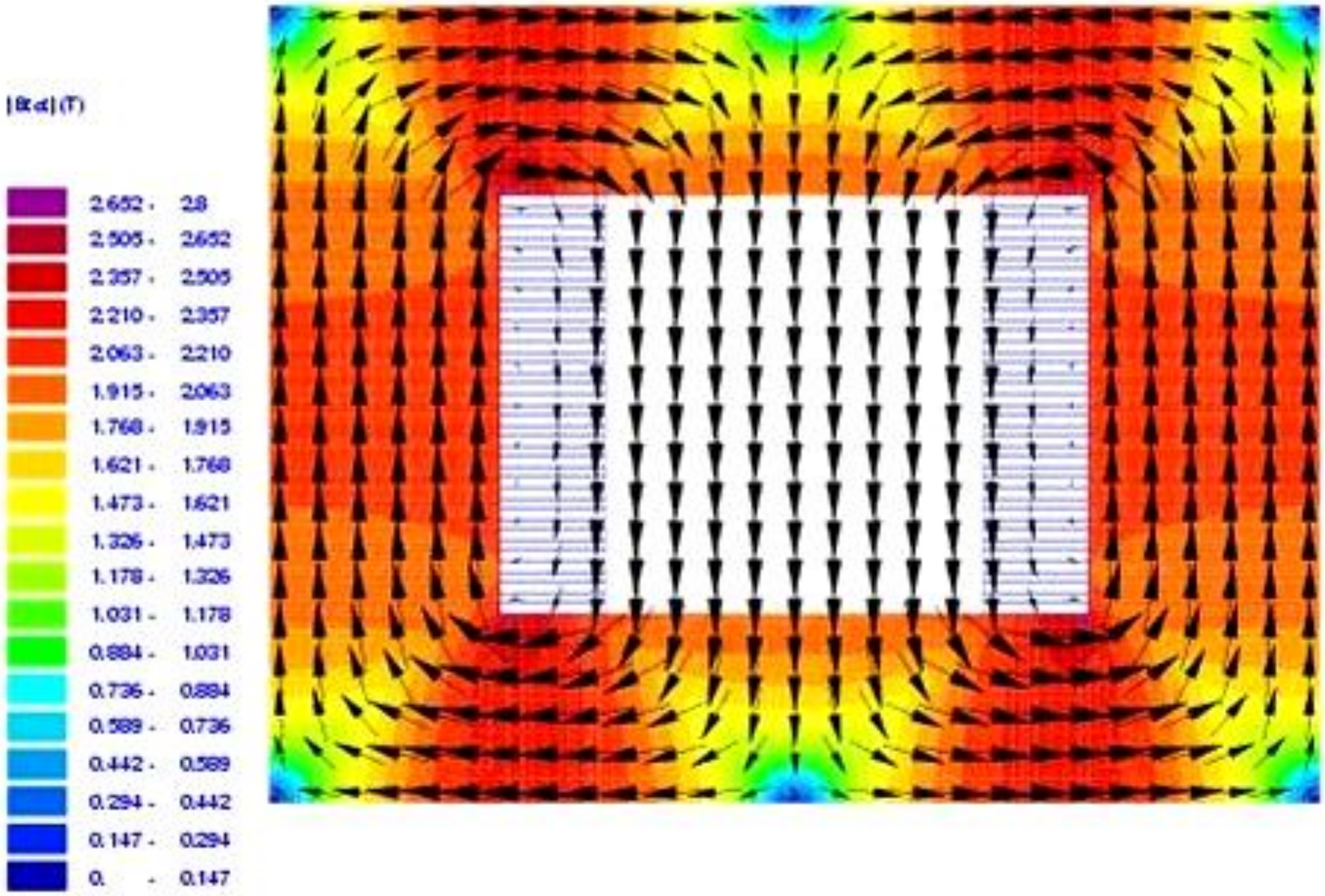
$N \cdot I = 24000$ A $\quad B_1 = 0.3$ T $\quad B_S = 0.065$ T $\quad$ Fill.fac. 0.98

Super-Ferric H Magnet



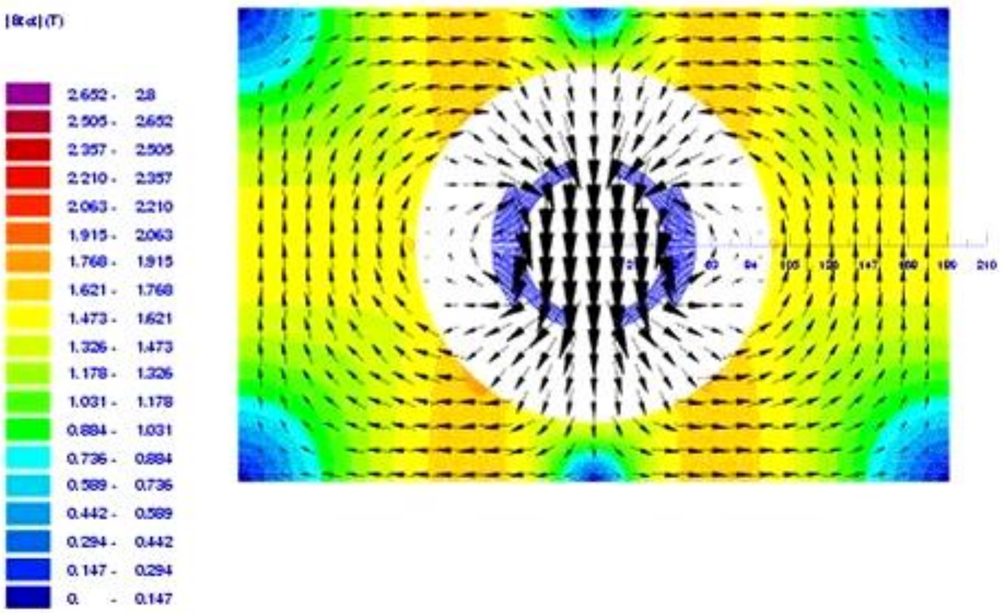
$N \cdot I = 96000$ **A** $\qquad B_1 = 1.18$ **T** $\qquad B_S = 0.26$ **T**

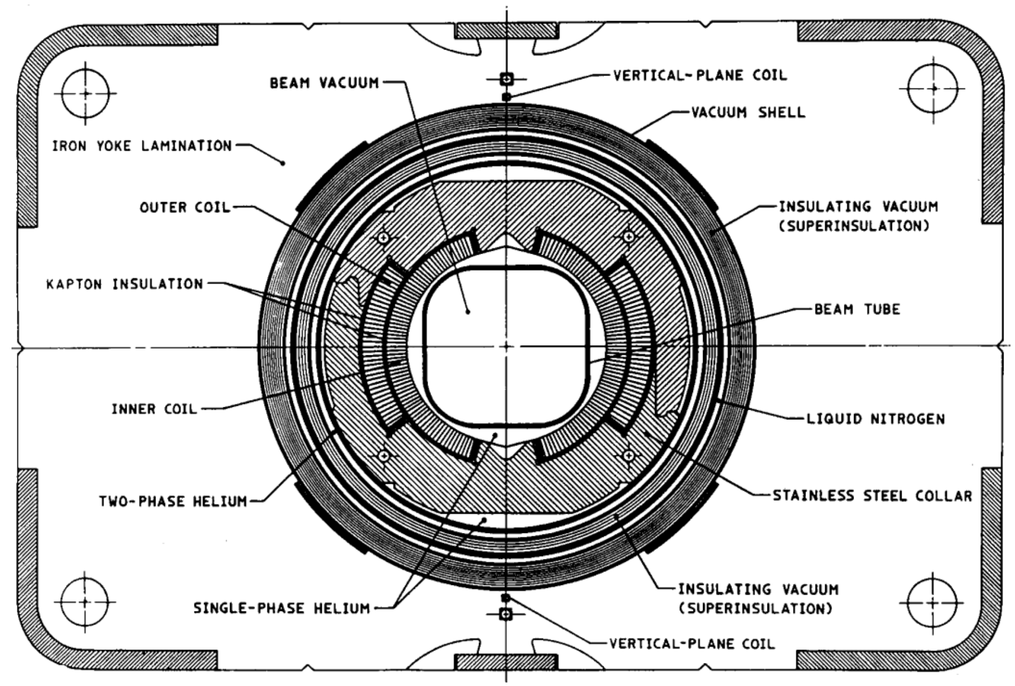
Window Frame Magnet



$N \cdot I = 360000 \text{ A}$ $\qquad B_1 = 2.08 \text{ T}$ $\qquad B_S = 1.04 \text{ T}$

Cos θ (Warm iron yoke) - Tevatron Dipole

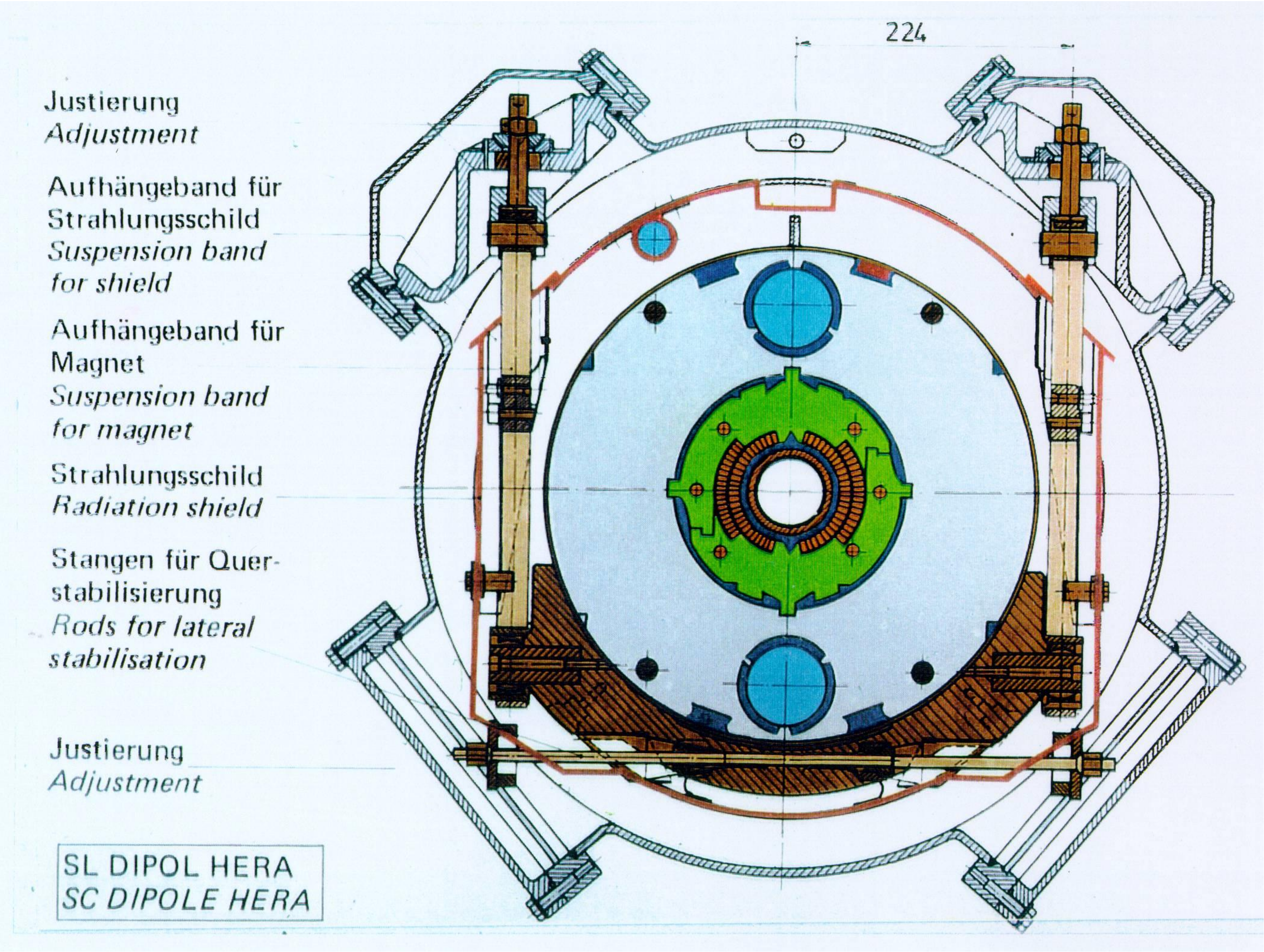




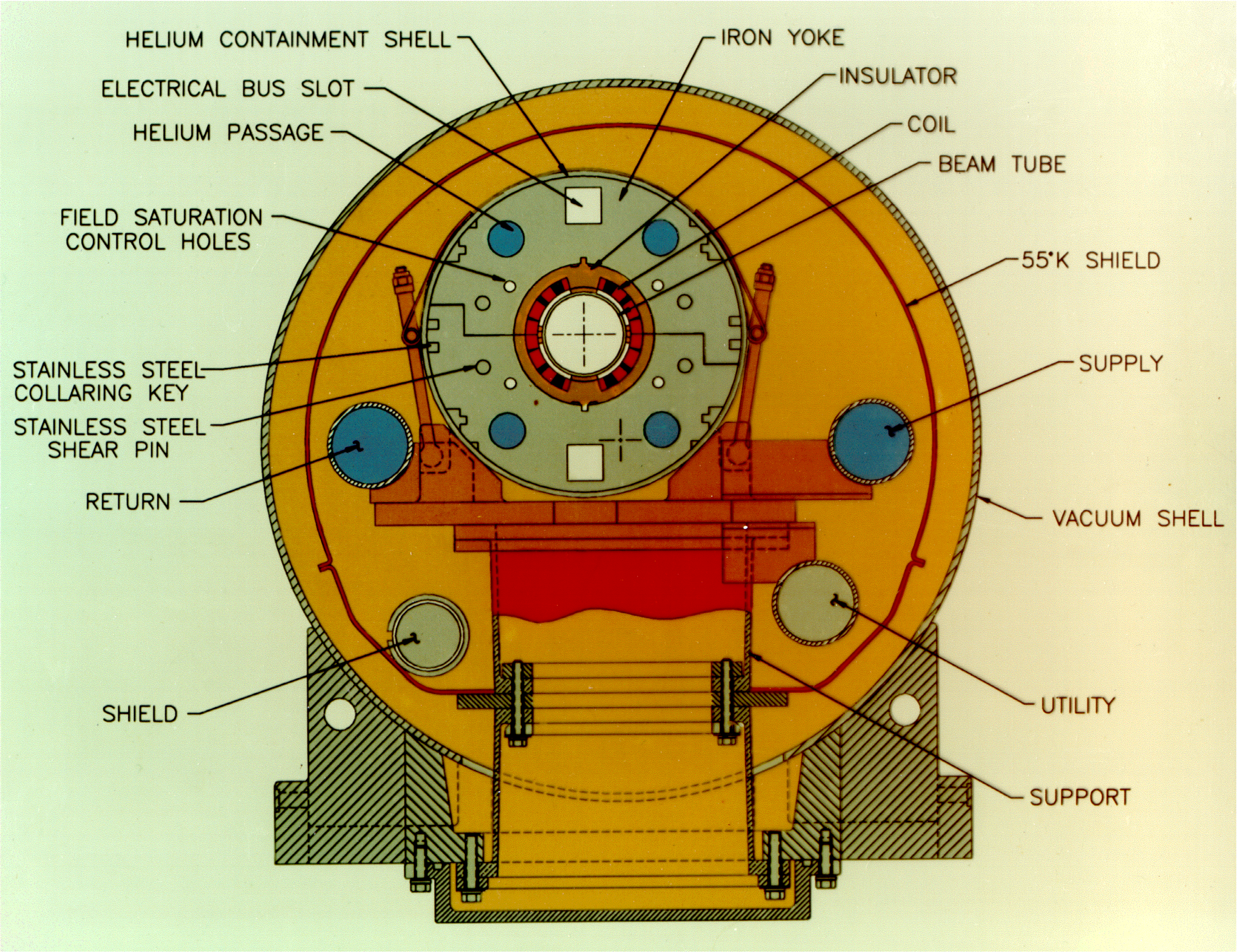
$N \cdot I = 471000$ A $\quad B_1 = 4.16$ T $\quad B_S = 3.39$ T

Notice the lower field in the iron yoke compared to the window frame

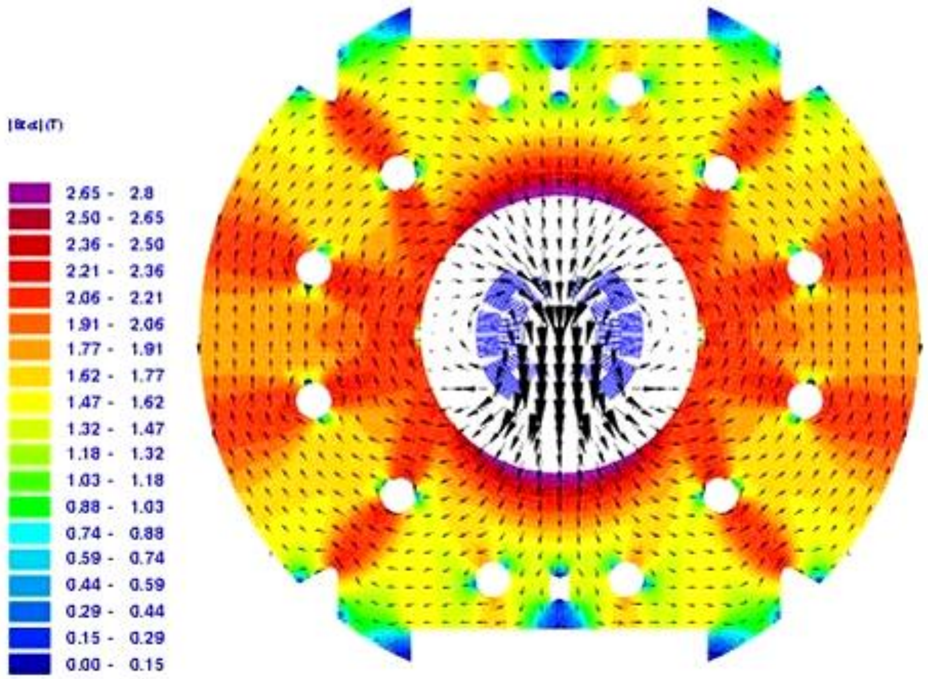
HERA Dipole in its Cryostat



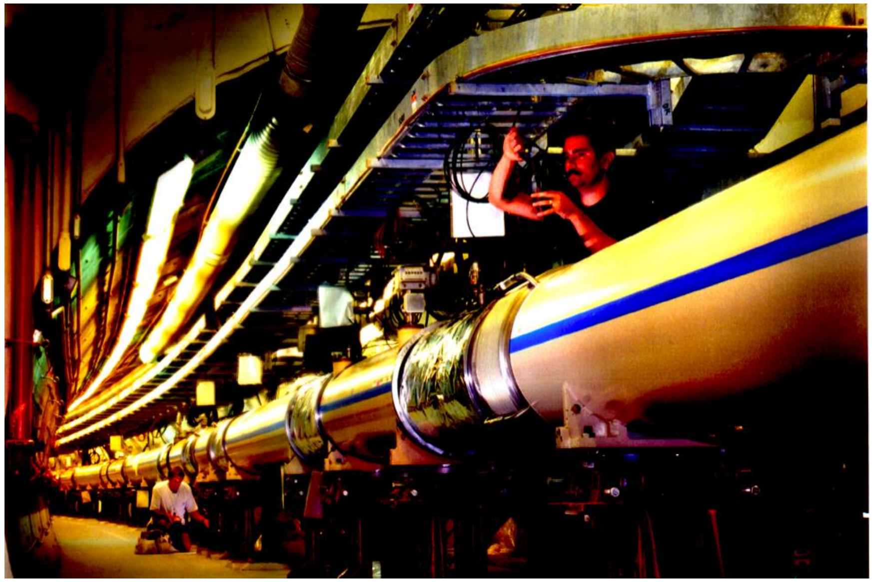
RHIC Dipole in its Cryostat



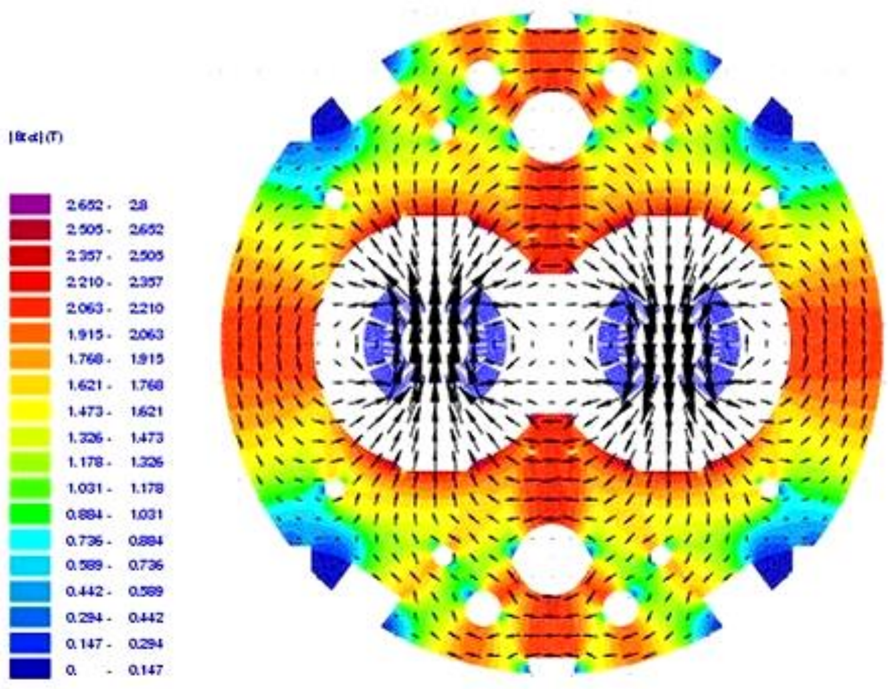
LHC Coil Test Facility for LHC (Based on HERA/RHIC Magnet Technology)



$N \cdot I = 960000$ A $\qquad B_1 = 8.33$ T $\qquad B_s = 7.77$ T



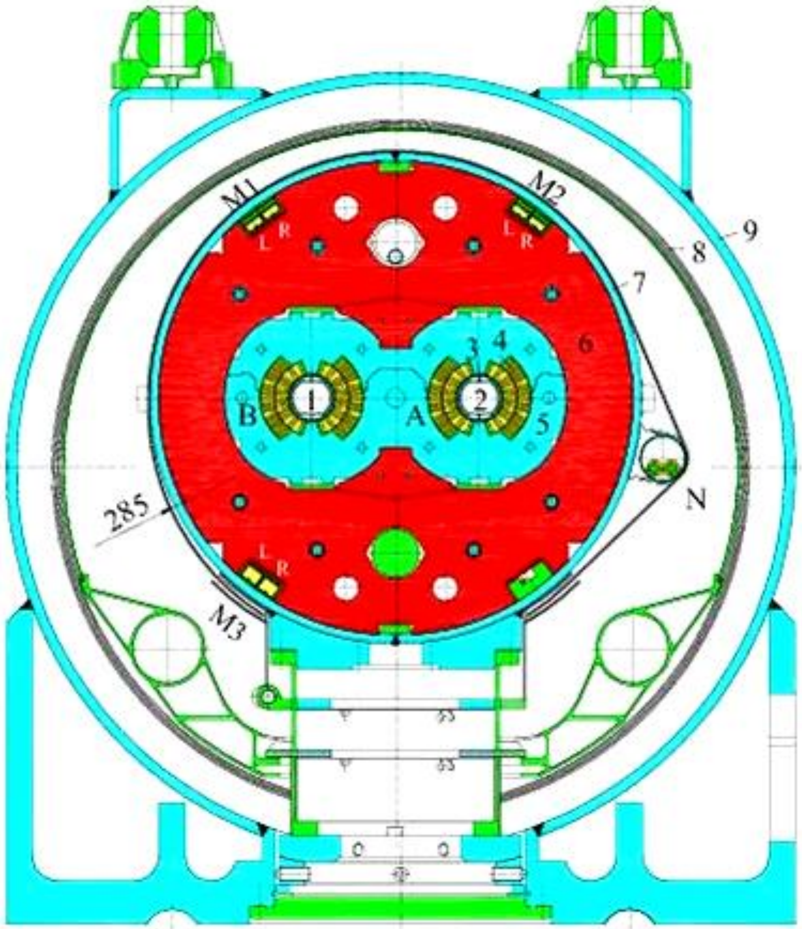
LHC Two-in-one Dipole



$N \cdot I = 2x944000$ A $\quad B_1 = 8.32$ T $\quad B_S = 7.44$ T

Storage of cold-masses

Cross-section of Cryodipole



Cryostat integration at CERN

Conventional and Superconducting Magnets

➔ Conventional magnets

- Important ohmic losses require water cooling
- Field is defined by the iron pole shape (max 1.5 T)
- Easy electrical and beam-vacuum interconnections
- Voltage drop over one coil of the MBW magnets = 22 V

➔ Superconducting magnets

- Field is defined by the coil layout
- Maximum field limited to 10 T (NbTi), 12 T (Nb_3Sn)
- Enormous electromagnetic forces (400 tons/m in MB for LHC)
- Quench protection system required
- Cryogenic installation (1.8 K)
- Electrical interconnections in cryo-lines
- Voltage drop on LHC magnet string (154 MB) 155 V

Comparison (EM-Design)

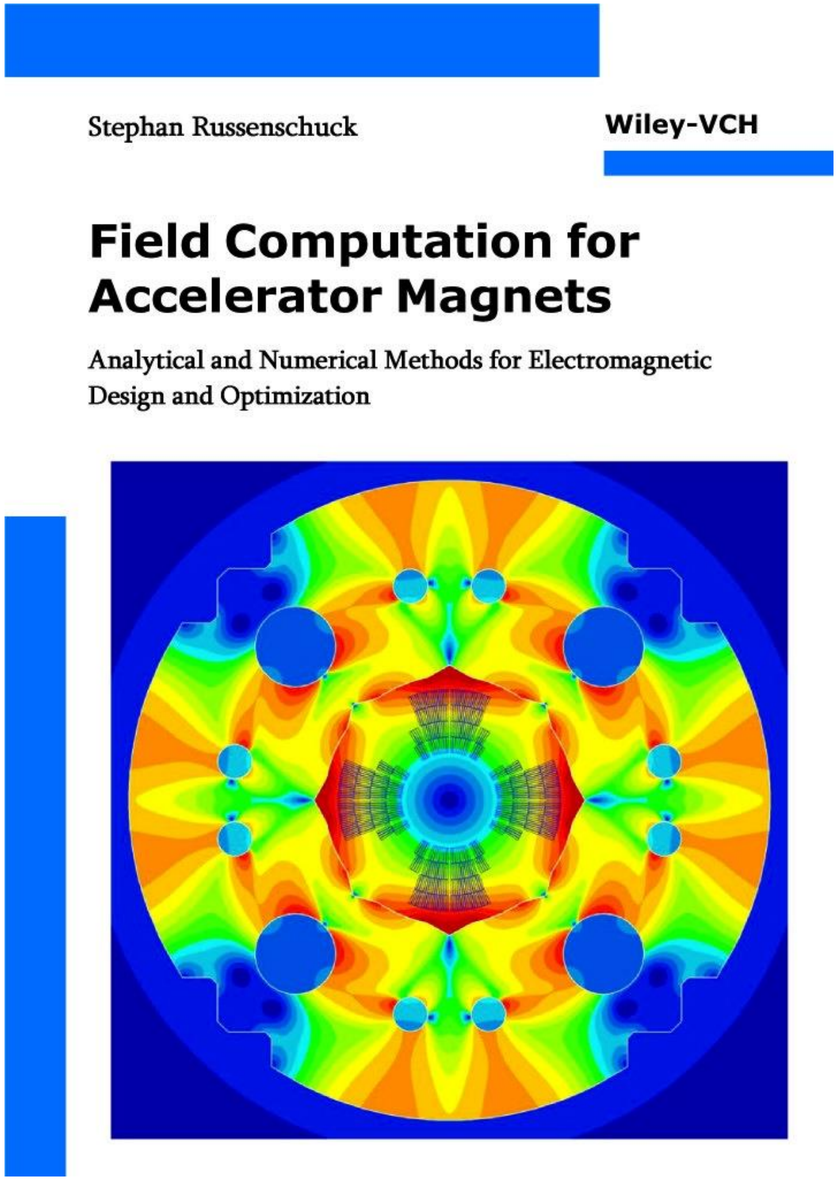
- **Conventional magnets**
  - Ideal pole shape known from potential theory
  - One-dimensional (analytical) field computation for main field
  - Commercial FEM software can be used as a black box (hysteresis modeling)
- **Superconducting magnets**
  - Decoupling of coil and yoke optimization
  - Accuracy of the field solution
  - Modeling of the coils
  - Filament magnetization
  - Quench simulations

A Multiphysics Problem

- Beam physics
- Material science: Superconducting cable, Steel, Insulation
- Mechanics and large-scale mechanical engineering
- Vacuum technology
- Cryogenics (Superfluid helium)
- Metrology and alignment
- Field measurements
- Electrical engineering (Power supplies, leads, buswork, quench detection and magnet protection
- Analytical and numerical field computation

Analytical and Numerical Field Computation

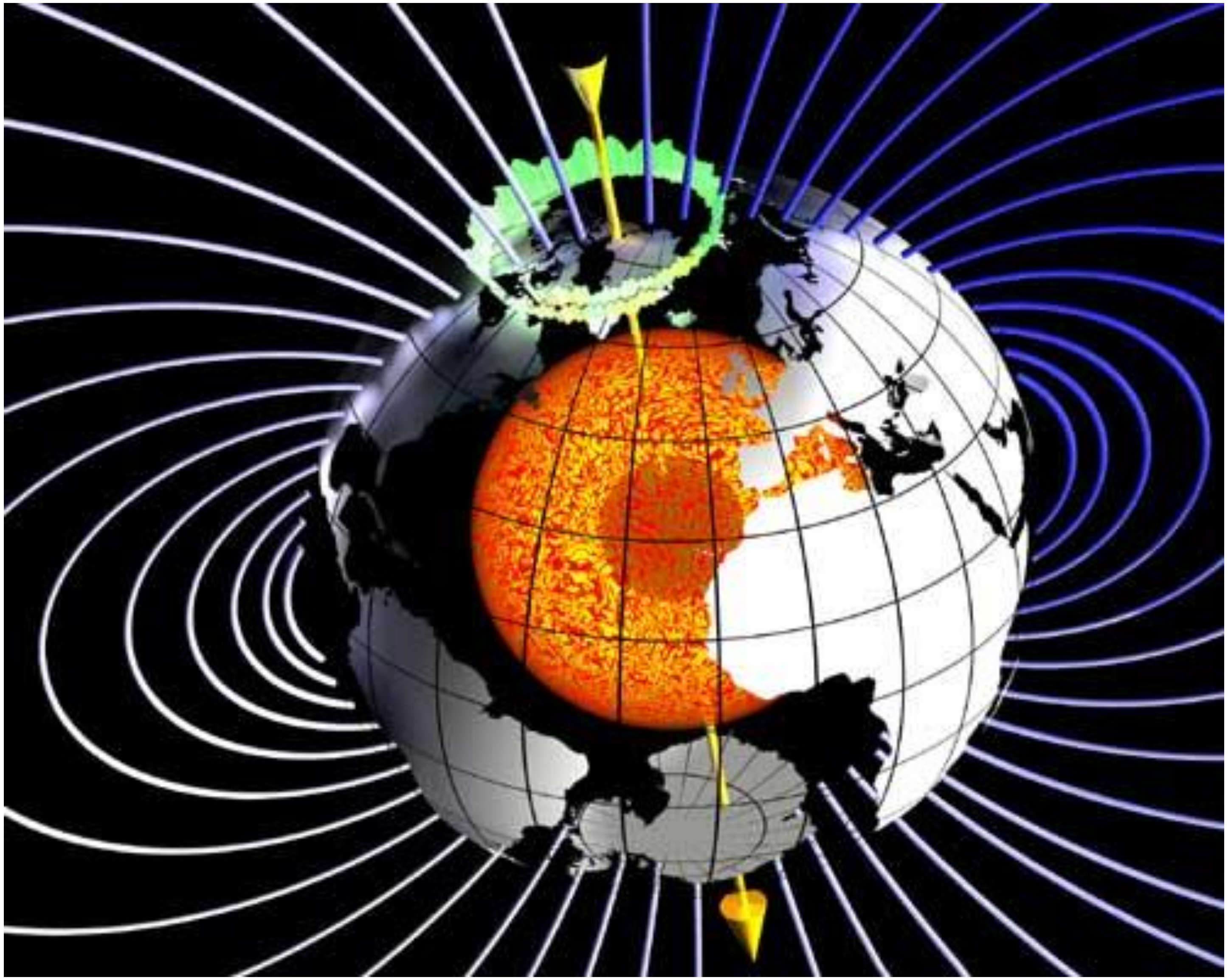
- Linear algebra
- Vector analysis
- Harmonic fields
- Green's functions and the method of images
- Complex analysis
- Differential geometry
- Numerical field computation
- Hysteresis modeling
- Coupled (thermo, magnetic, electric) systems
- Mathematical optimization



A Short Self-Test

- What is a vector? An arrow, a tuple of numbers, a quantity having direction and magnitude, a solution of a linear equation system, a contravariant tensor?
- What is more fundamental, **B** or **H**?
- What **is** a (linear) field
- Is there a difference between coefficients, components, and coordinates?
- We know how to add vectors represented as arrows by means of the parallelogram law. Can we add a force vector at the tip of the position vector?
- What is between the field lines
- In Maxwell's equation we find **J** · d**a** and **B** · d**a**. Is there a difference between these vectorial surface elements.
- What is a "convention recepteur".

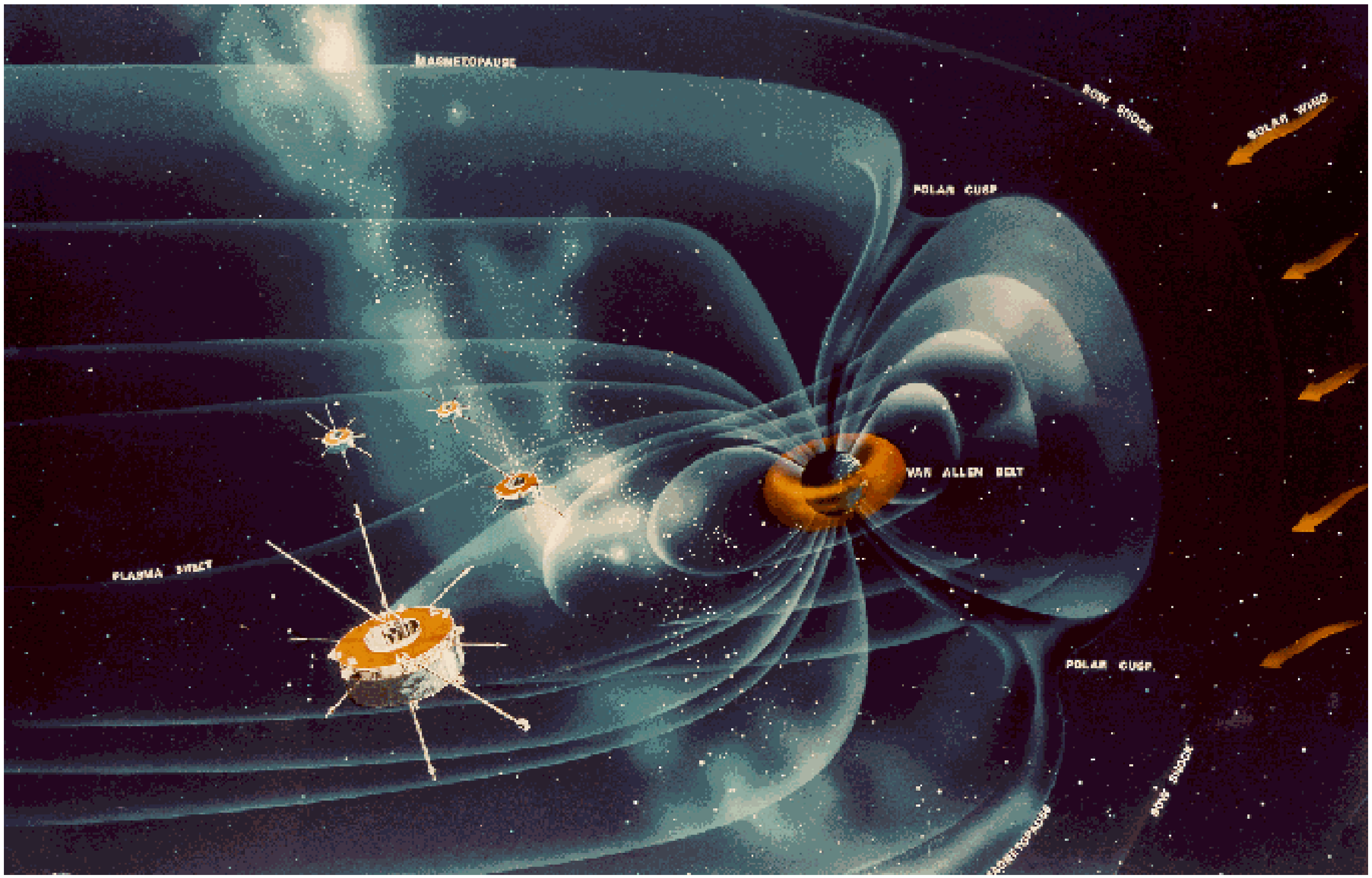
Flux Tubes of Mother Earth (or what is a magnetic field)



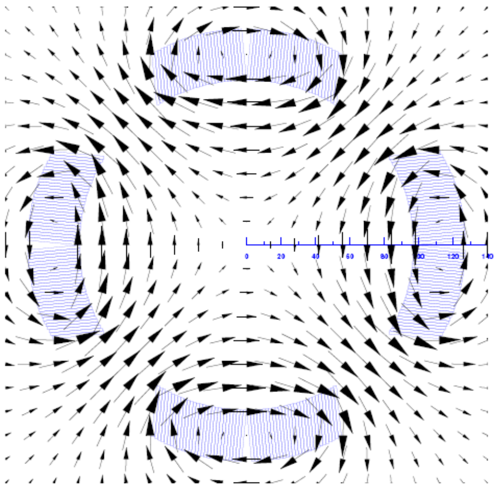
Earth Magnetic Field (better representation)

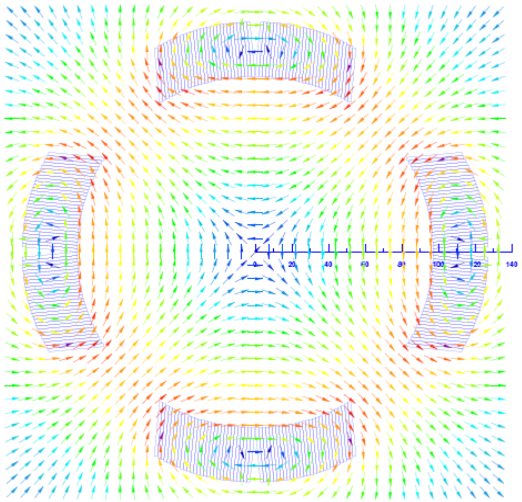
Erdmagnetfeld

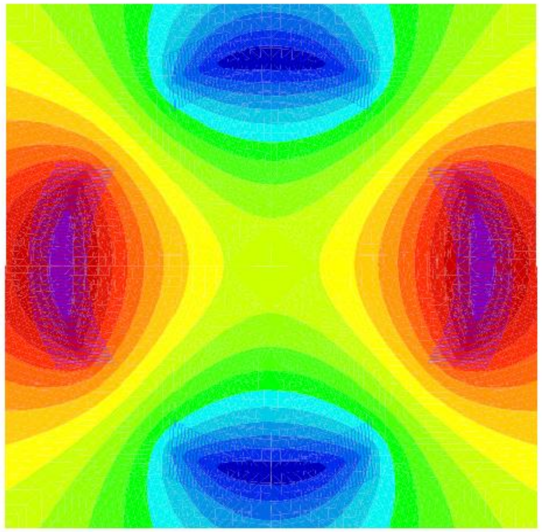
PTB

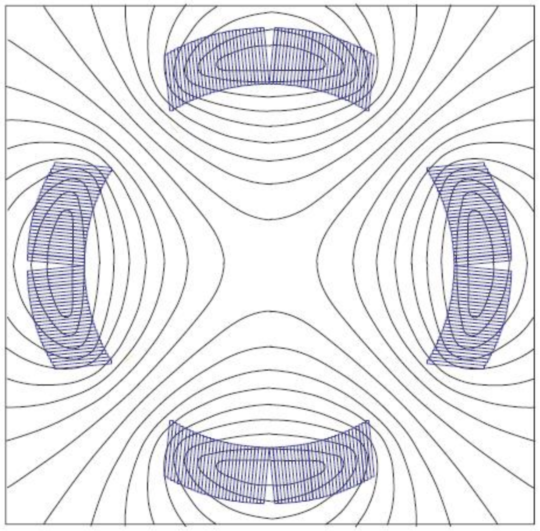


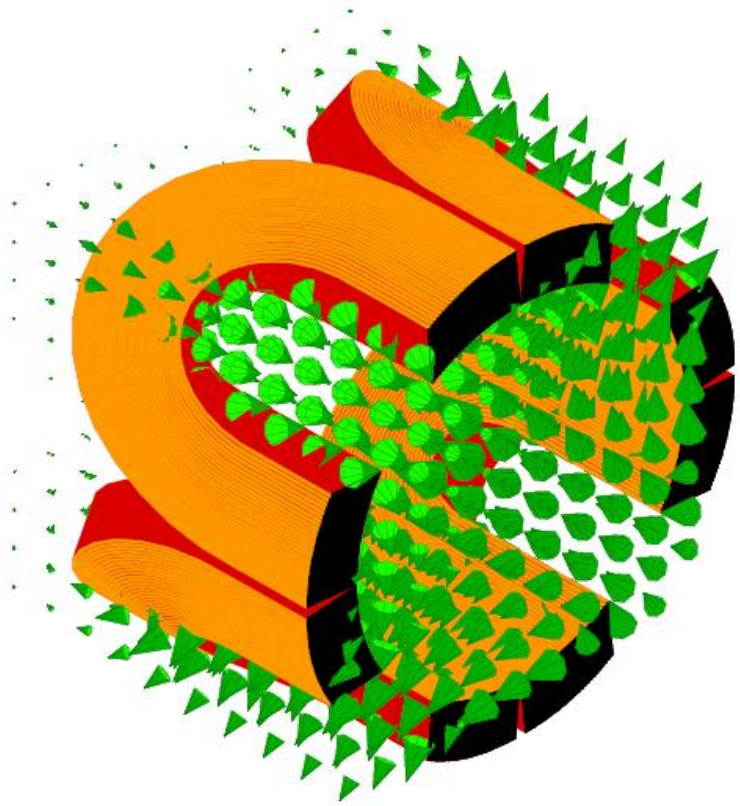
Different Renderings of the Same Vector Field







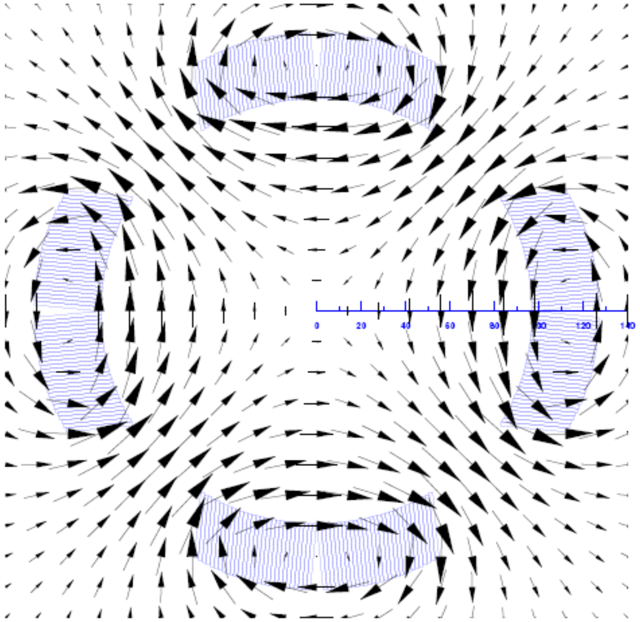




Framework of our Vectorfields E_3

- ➔ E_3 has the structure of the affine point space
- ➔ It carries the vector (linear) space structure of its associated vector space
- ➔ It is equipped with a metric that gives rise to distance and angles

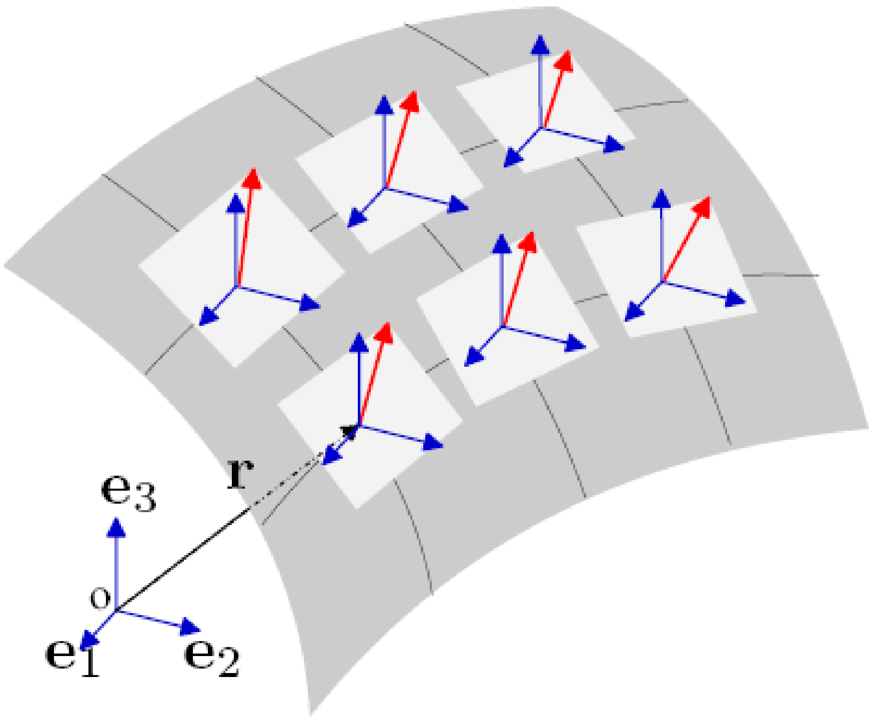
- ➔ If an origin and basis is selected, the projection of the position vector on the basis yields the coordinates (in $\mathbb{R}^3$)
- ➔ The canonical basis can be made to a basis field by translation
- ➔ The components of the field at some point are then the projection on this basis field



Vector and Scalar Fields

$$\mathbf{a} : \Omega \to \mathbb{R}^3 : \mathbf{r} \mapsto \mathbf{a}(\mathbf{r}) : \mathbf{a}(\mathbf{r}) = (a^1(\mathbf{r}), a^2(\mathbf{r}), a^3(\mathbf{r}))$$

$$\Omega \subset \mathbb{R}^3$$



$$\phi : \Omega \to \mathbb{R} : \phi \mapsto \phi(\mathbf{r})$$

Different Incarnations of Maxwell's Equations

$$\int_{\partial\mathscr{A}} \mathbf{H}\cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{A}} \mathbf{J}\cdot \mathrm{d}\mathbf{a} + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{A}} \mathbf{D}\cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{A}} \mathbf{E}\cdot \mathrm{d}\mathbf{r} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{A}} \mathbf{B}\cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{V}} \mathbf{B}\cdot \mathrm{d}\mathbf{a} = 0,$$

$$\int_{\partial\mathscr{V}} \mathbf{D}\cdot \mathrm{d}\mathbf{a} = \int_{\mathscr{V}} \rho \,\mathrm{d}V.$$

$$V_{\mathrm{m}}(\partial\mathscr{A}) = I(\mathscr{A}) + \frac{\mathrm{d}}{\mathrm{d}t}\Psi(\mathscr{A}),$$

$$U(\partial\mathscr{A}) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathscr{A}),$$

$$\Phi(\partial\mathscr{V}) = 0,$$

$$\Psi(\partial\mathscr{V}) = Q(\mathscr{V}).$$

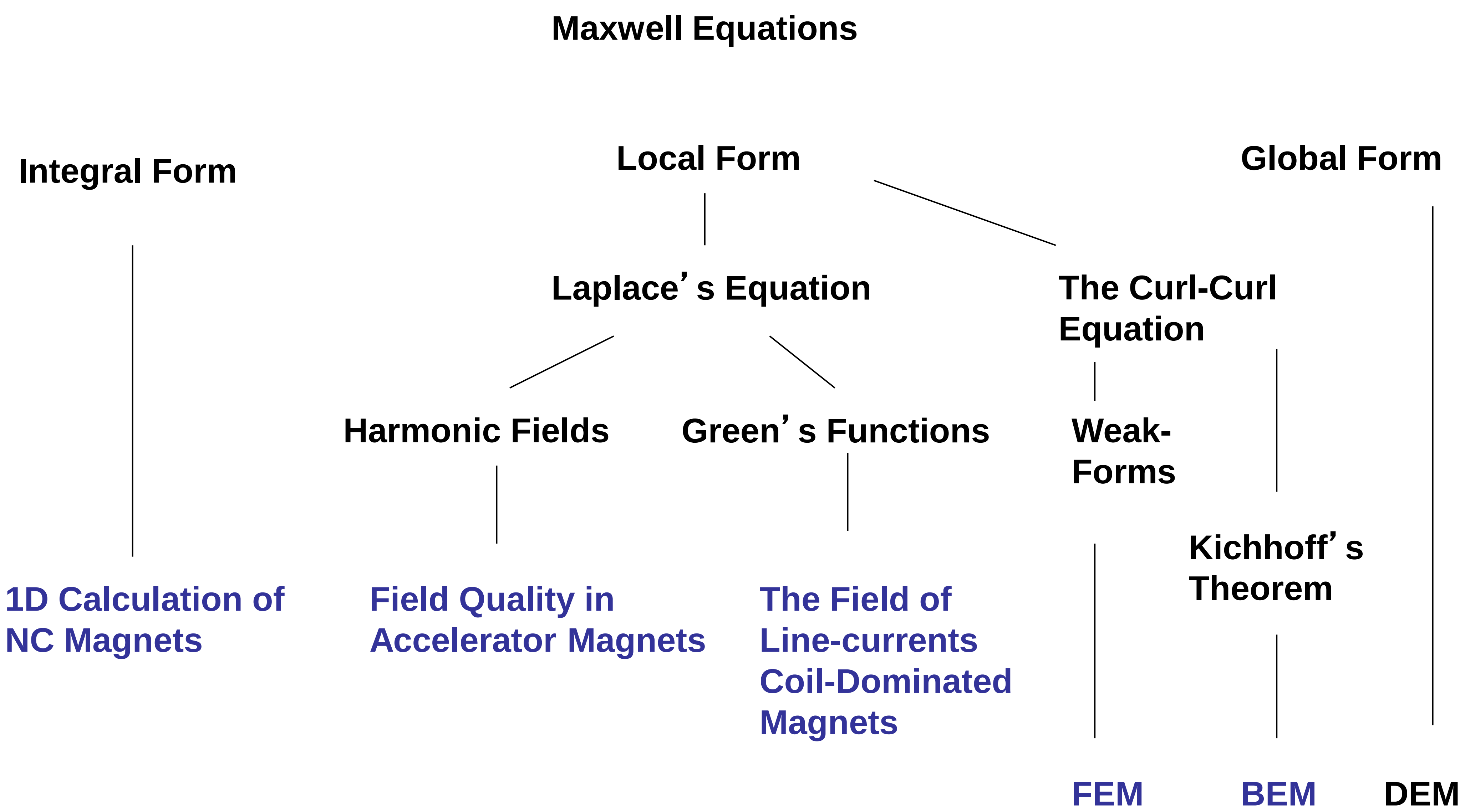
$$\operatorname{curl}\mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}\mathbf{D},$$

$$\operatorname{curl}\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B},$$

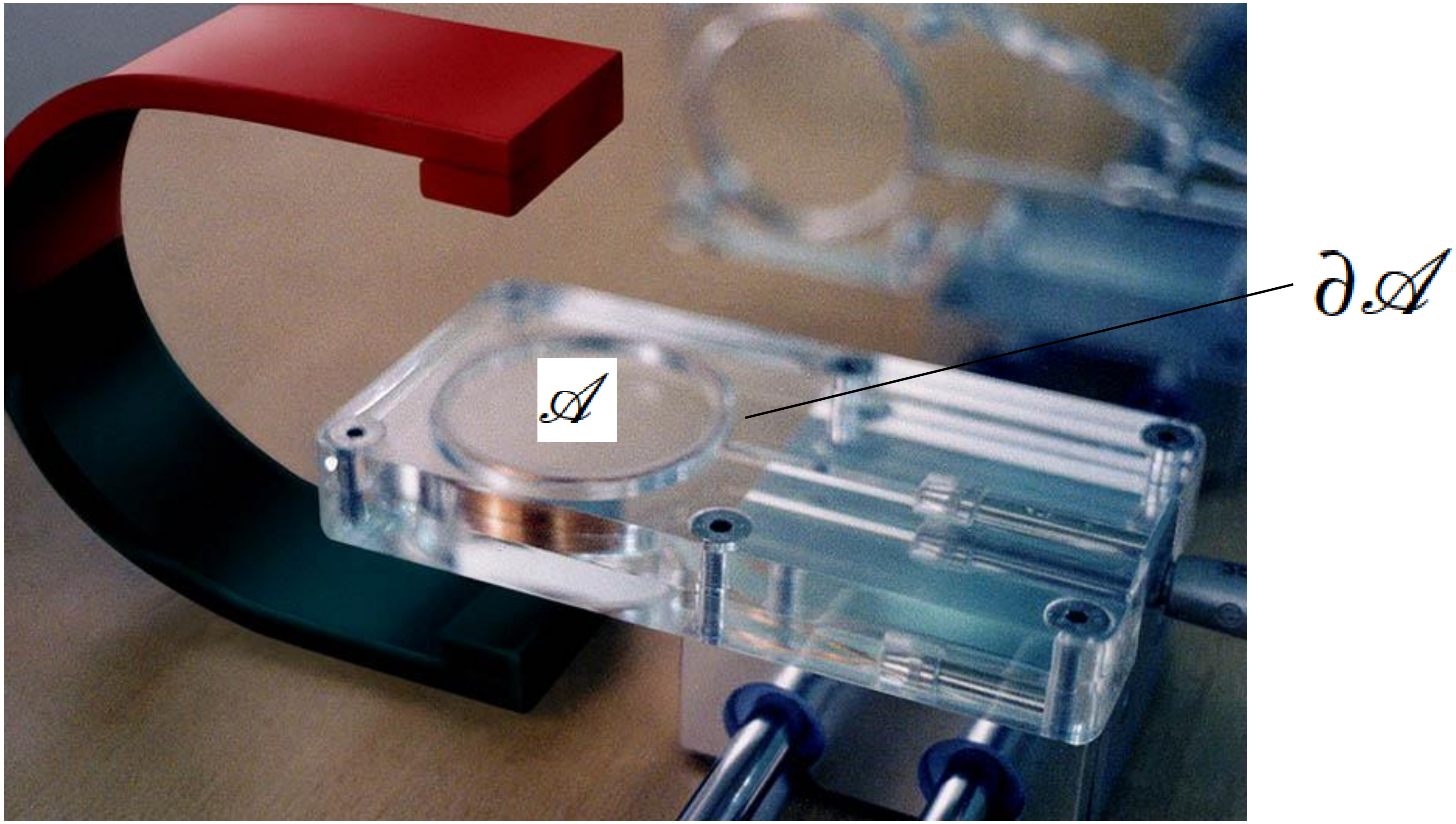
$$\operatorname{div}\,\mathbf{B} = 0,$$

$$\operatorname{div}\,\mathbf{D} = \rho.$$

Mathematical Foundations of Magnet Design



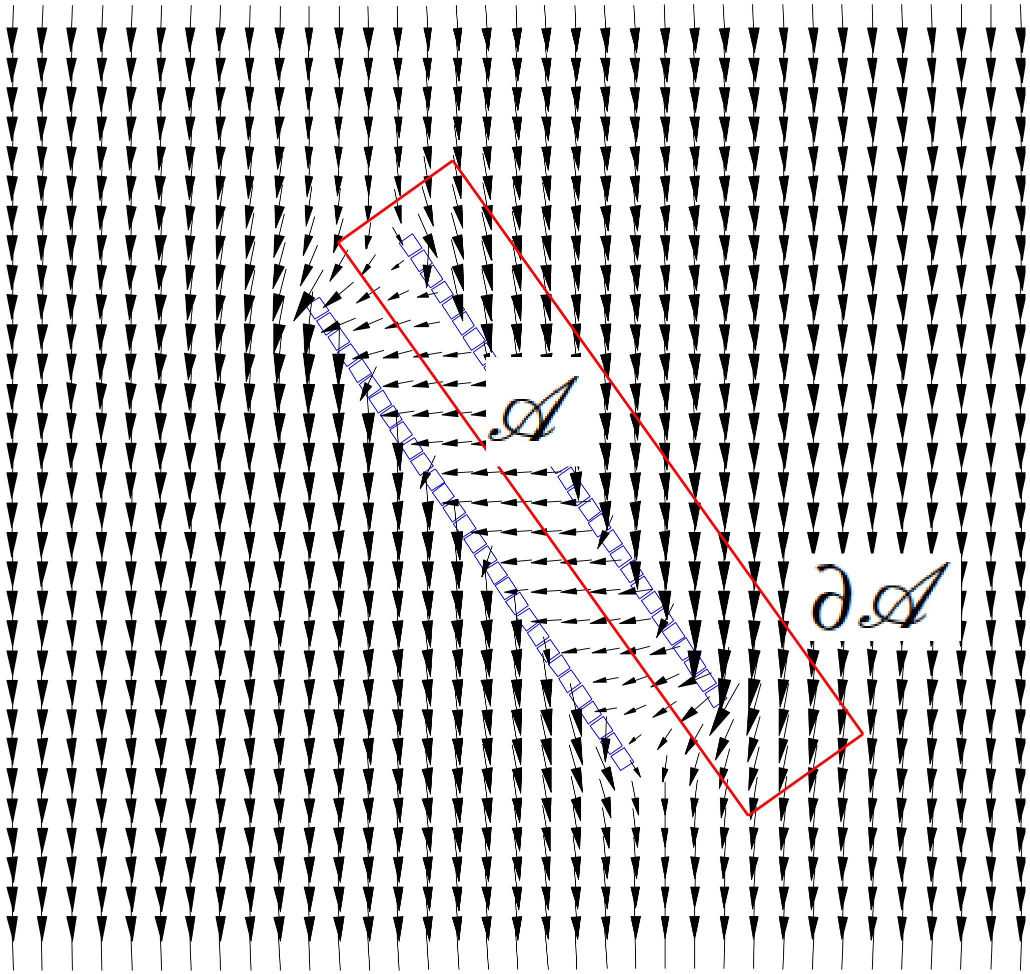
Faraday's Law (Inner Oriented Surface, Voltage along its Rim)



$$U(\partial\mathscr{A}) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathscr{A})$$

The potential to induce a voltage

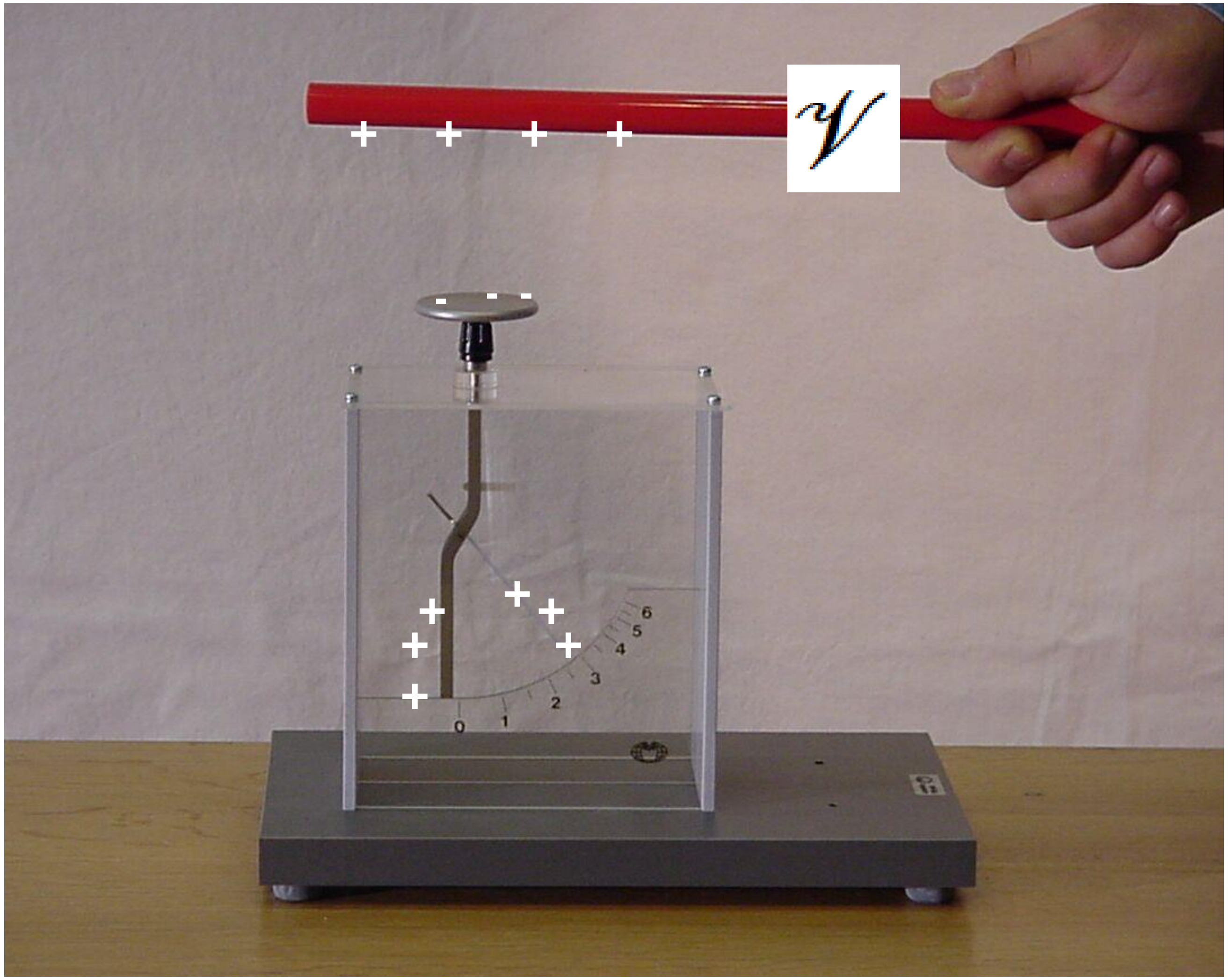
Ampere’s Law (Outer Oriented Surface; Current crossing)



$$V_{\mathrm{m}}(\partial\mathscr{A}) = I(\mathscr{A})$$

The current needed to cancel the longitudinal field component

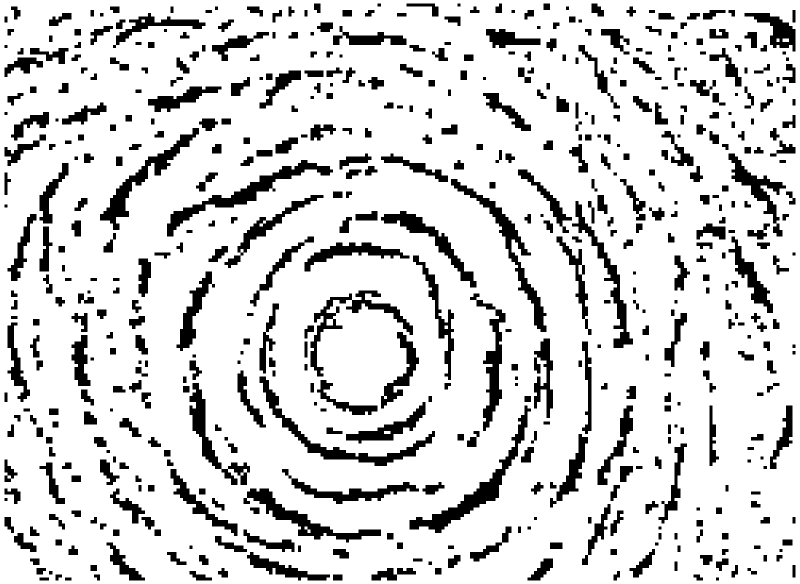
Gauss Law (Outer Oriented Volume; Electric Charge that can be influenced)



$$\Psi(\partial\mathscr{V}) = Q(\mathscr{V})$$

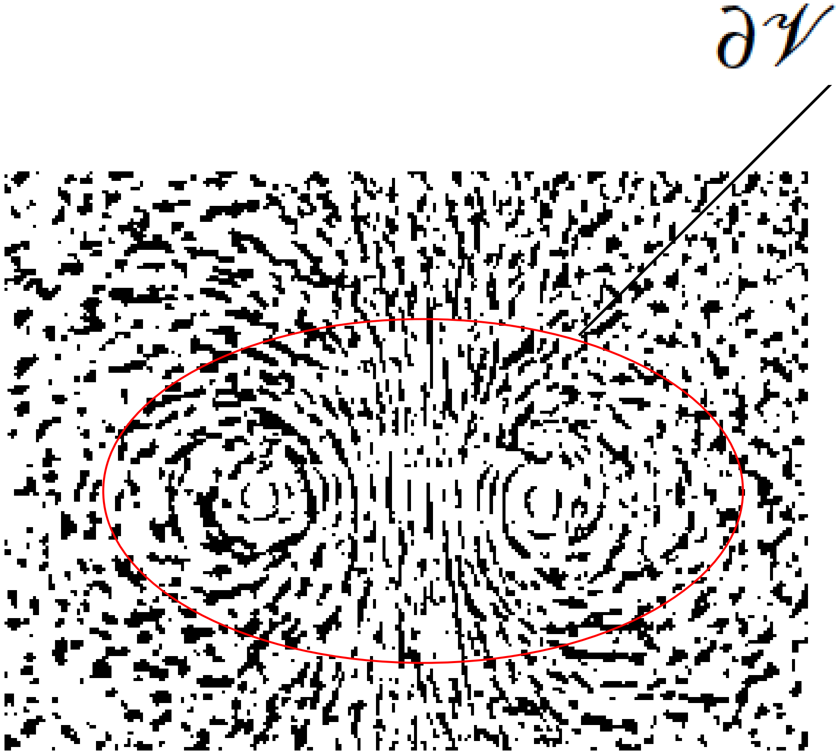
The capacity to induce charge

Magnetic Flux Conservation Law (Inner Oriented Volume)

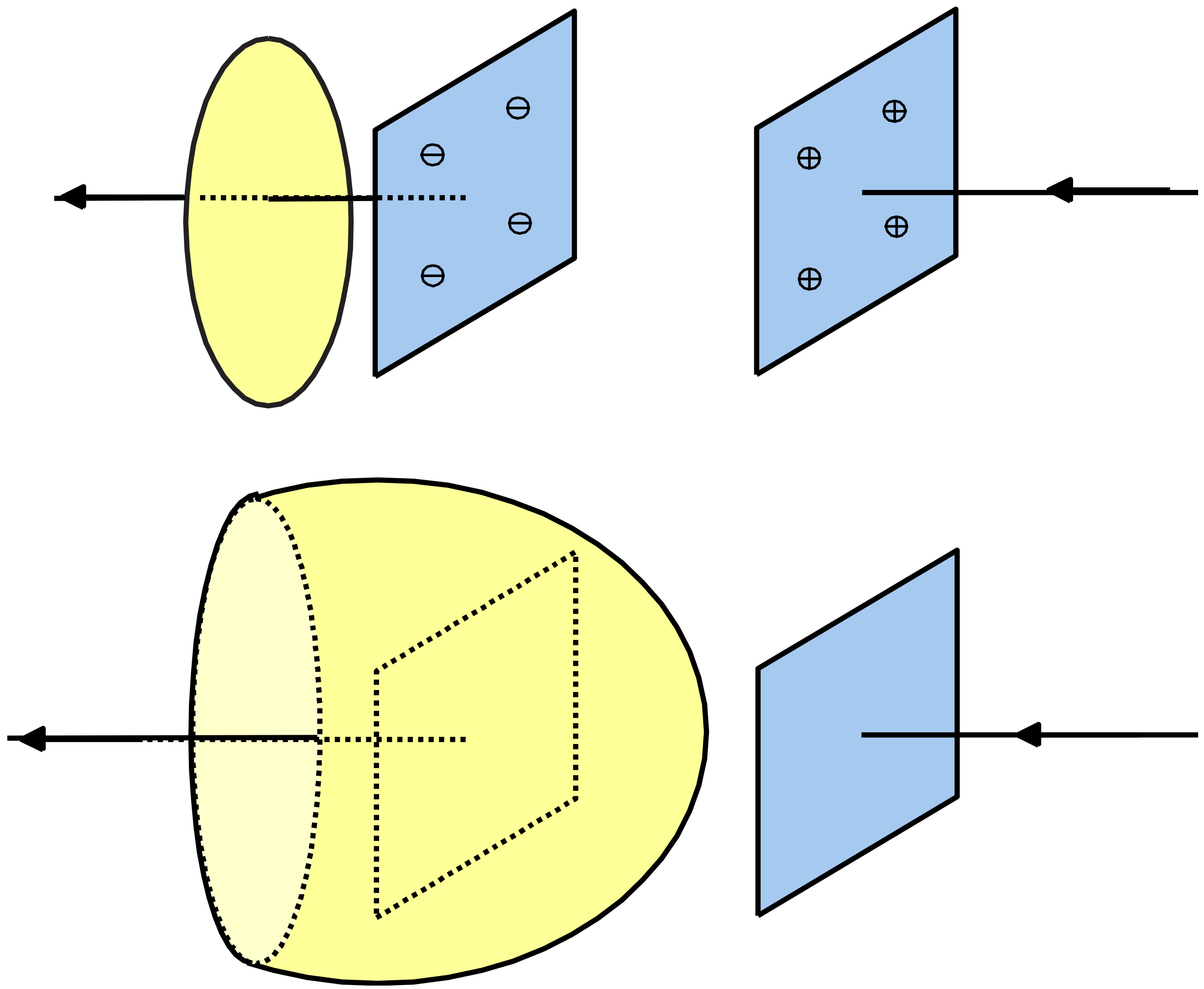


$$\Phi(\partial \mathscr{V}) = 0$$

Conservation of flux



Maxwell' s Extension



Ampere

$$V_m(\partial\mathscr{A}) = I(\mathscr{A}) + \frac{d}{dt}\Psi(\mathscr{A})$$

Rate of change of charge

Maxwell's Equations in Global Form

Ampere	$$V_{\mathrm{m}}(\partial a) = I(a) + \frac{\mathrm{d}}{\mathrm{d}t}\Psi(a)$$
Faraday	$$U(\partial a) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(a)$$
Flux conservation	$$\Phi(\partial V) = 0$$
Gauss	$$\Psi(\partial V) = Q(V)$$

Conservation of charge / Kirchhoff law

$$V_{\mathrm{m}}(\partial(\partial V)) = 0 = I(\partial V) + \frac{\mathrm{d}}{\mathrm{d}t}Q(V)$$

In words: The current exiting a volume is equal to the negative rate of the charge in that volume

Electromagnetic Fields

Global quantity	SI unit	Relation	SI unit	Field
MMF	$1\,\mathrm{A}$	$V_{\mathrm{m}}(\mathscr{S}) = \int_{\mathscr{S}} \mathbf{H} \cdot \mathrm{d}\mathbf{r}$	$1\,\mathrm{A\,m^{-1}}$	Magnetic field
Electric voltage	$1\,\mathrm{V}$	$U(\mathscr{S}) = \int_{\mathscr{S}} \mathbf{E} \cdot \mathrm{d}\mathbf{r}$	$1\,\mathrm{V\,m^{-1}}$	Electric field
Magnetic flux	$1\,\mathrm{V\,s}$	$\Phi(\mathscr{A}) = \int_{\mathscr{A}} \mathbf{B} \cdot \mathrm{d}\mathbf{a}$	$1\,\mathrm{V\,s\,m^{-2}}$	Magnetic flux density
Electric flux	$1\,\mathrm{A\,s}$	$\Psi(\mathscr{A}) = \int_{\mathscr{A}} \mathbf{D} \cdot \mathrm{d}\mathbf{a}$	$1\,\mathrm{A\,s\,m^{-2}}$	Electric flux density
Electric current	$1\,\mathrm{A}$	$I(\mathscr{A}) = \int_{\mathscr{A}} \mathbf{J} \cdot \mathrm{d}\mathbf{a}$	$1\,\mathrm{A\,m^{-2}}$	Electric current density
Electric charge	$1\,\mathrm{A\,s}$	$Q(\mathscr{V}) = \int_{\mathscr{V}} \rho \cdot \mathrm{d}V$	$1\,\mathrm{A\,s\,m^{-3}}$	Electric charge density

Interrupt: The vectorial line and surface elements, and the volume element

Maxwell's Equations in Integral Form

$$\int_{\partial\mathscr{A}} \mathbf{H}\cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{A}} \mathbf{J}\cdot \mathrm{d}\mathbf{a} + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{A}} \mathbf{D}\cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{A}} \mathbf{E}\cdot \mathrm{d}\mathbf{r} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{A}} \mathbf{B}\cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{V}} \mathbf{B}\cdot \mathrm{d}\mathbf{a} = 0,$$

$$\int_{\partial\mathscr{V}} \mathbf{D}\cdot \mathrm{d}\mathbf{a} = \int_{\mathscr{V}} \rho\, \mathrm{d}V.$$

Contains the Kirchhoff laws

$$V_{\mathrm{m}}(\partial\mathscr{A}) = I(\mathscr{A}) + \frac{\mathrm{d}}{\mathrm{d}t}\Psi(\mathscr{A}),$$

$$U(\partial\mathscr{A}) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\mathscr{A}),$$

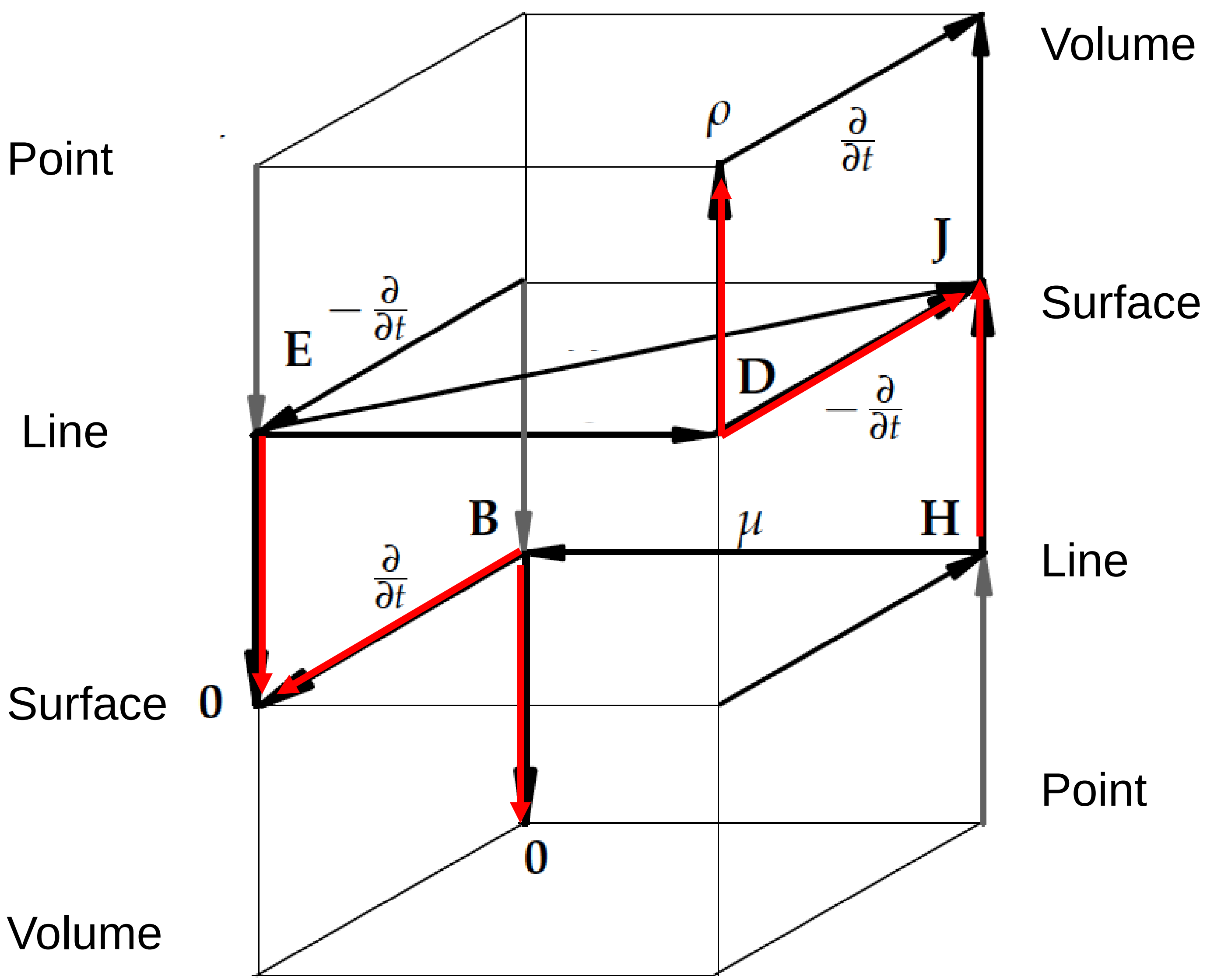
$$\Phi(\partial\mathscr{V}) = 0,$$

$$\Psi(\partial\mathscr{V}) = Q(\mathscr{V}).$$

8 Equations
16 Unknowns

Exercise: Ampere's law in integral form

Maxwell's House



Constitutive Equations

$$\mathbf{B} = \mu \mathbf{H}, \qquad \mathbf{D} = \varepsilon \mathbf{E}, \qquad \mathbf{J} = \varkappa \mathbf{E},$$

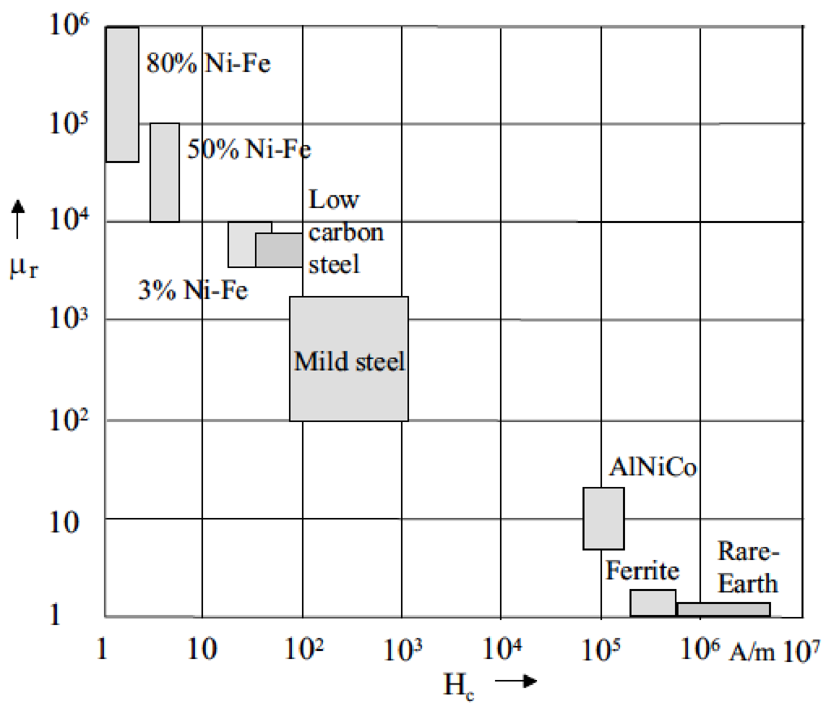
$$[\mu] = 1\,\mathrm{V\,s\,A^{-1}\,m^{-1}} = 1\,\mathrm{H\,m^{-1}},$$
$$[\varepsilon] = 1\,\mathrm{A\,s\,V^{-1}\,m^{-1}},$$
$$[\varkappa] = 1\,\mathrm{A\,V^{-1}m^{-1}} = 1\,\Omega^{-1}\,\mathrm{m^{-1}}.$$

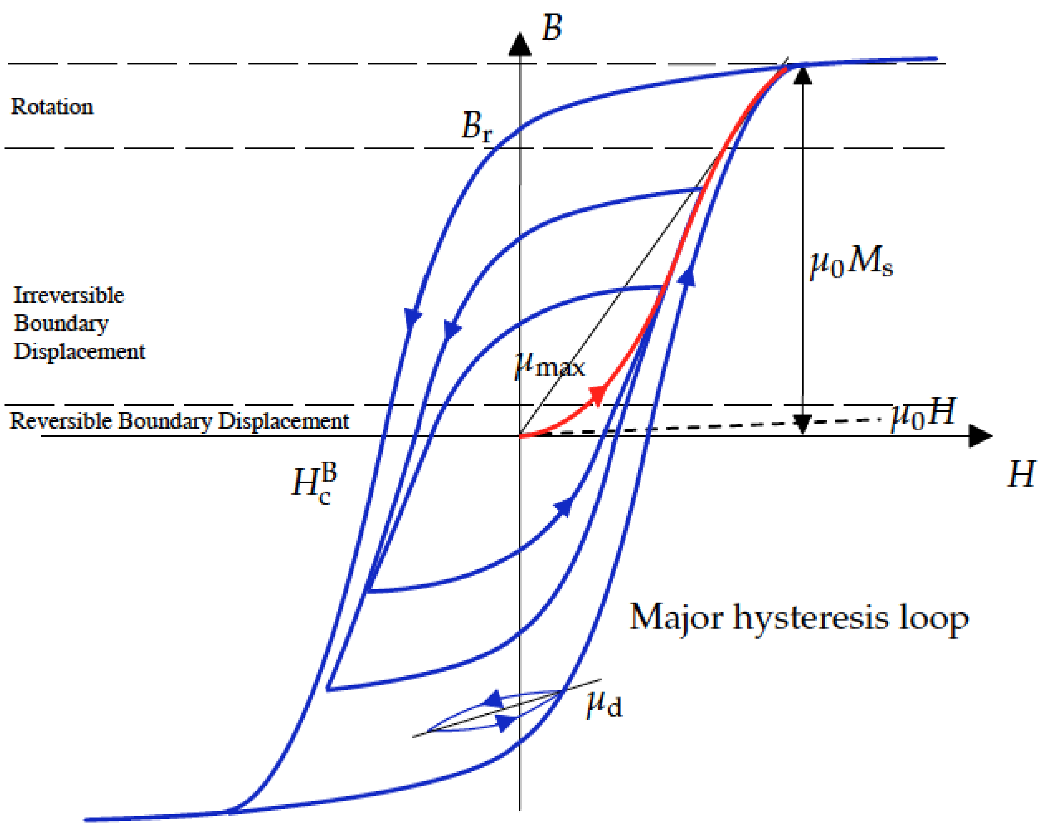
Linear (field independent, homogeneous (position independent), lossless, isotropic (direction independent, stationary

$$\mu = \mu_r \mu_0, \qquad \varepsilon = \varepsilon_r \varepsilon_0,$$

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H\,m^{-1}}, \qquad \varepsilon_0 = 8.8542\ldots \times 10^{-12}\,\mathrm{F\,m^{-1}},$$

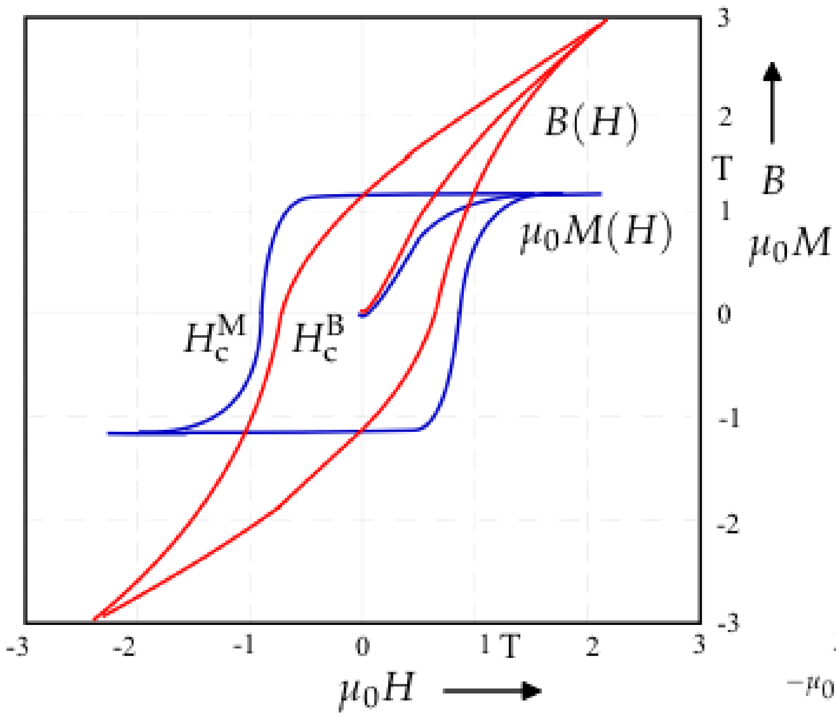
Hysteresis

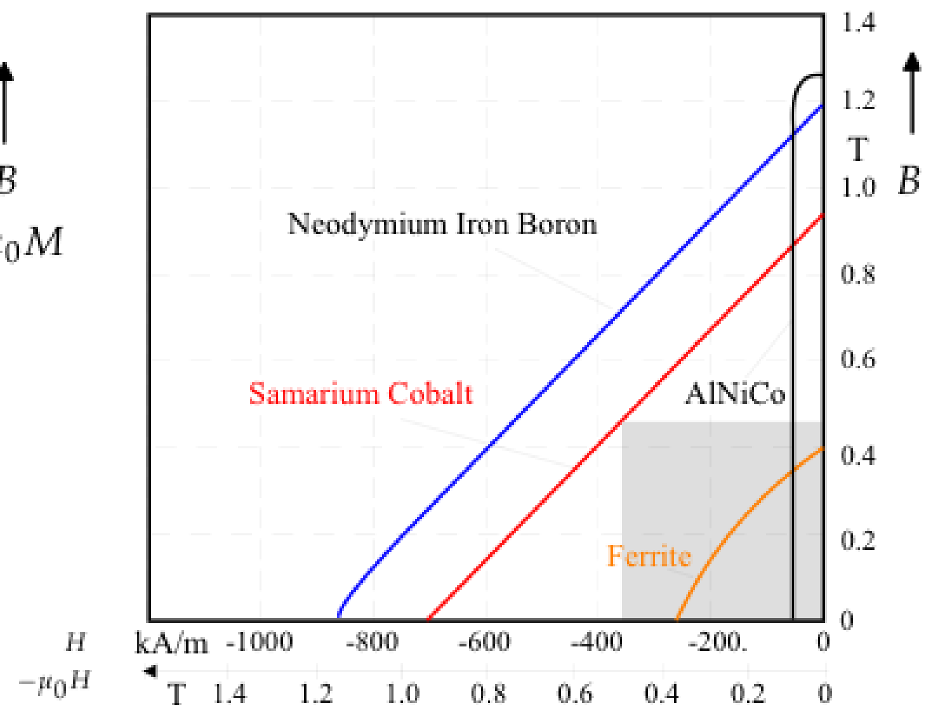




$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{P}_m(\mathbf{H}) = \mu_0(\mathbf{H} + \mathbf{M}(\mathbf{H})),$$

M(H) and B(H) for Permanent Magnets





	B_r	$\mu_0 H_c^M$	H_c^B	$(BH)_{max}$	$(BH)_{max}^{id}$	T_c
	T	T	T	$kJ\,m^{-3}$	$kJ\,m^{-3}$	°C
AlNiCo	1.3	0.06	0.06	50	336	857
Ferrite	0.4	0.4	0.37	30	32	447
$SmCo_5$	0.9	2.5	0.87	160	161	727
Sm_2Co_{17}	1.1	1.3	0.97	220	241	827
NdFeB	1.3	1.5	1.25	320	336	313

B(H) Measurement

$H = NI/2\pi r$ within the specimen, which is

$$\overline{H} = \frac{NI}{2\pi(r_2 - r_1)} \ln\left(\frac{r_2}{r_1}\right) .$$

$$U = \frac{\mathrm{d}}{\mathrm{d}t}\Phi = \frac{\mathrm{d}}{\mathrm{d}t}\overline{B}a,$$

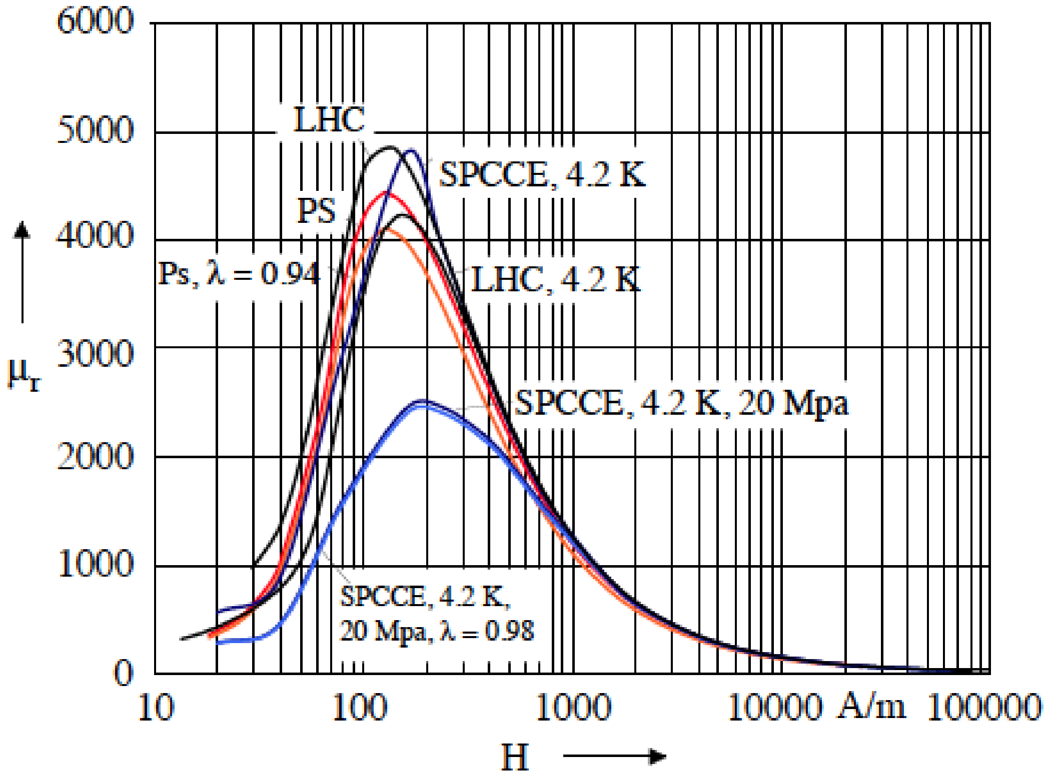
$$\int U \mathrm{d}t = \overline{B}\, a \qquad \mu = \overline{B}/\overline{H}$$

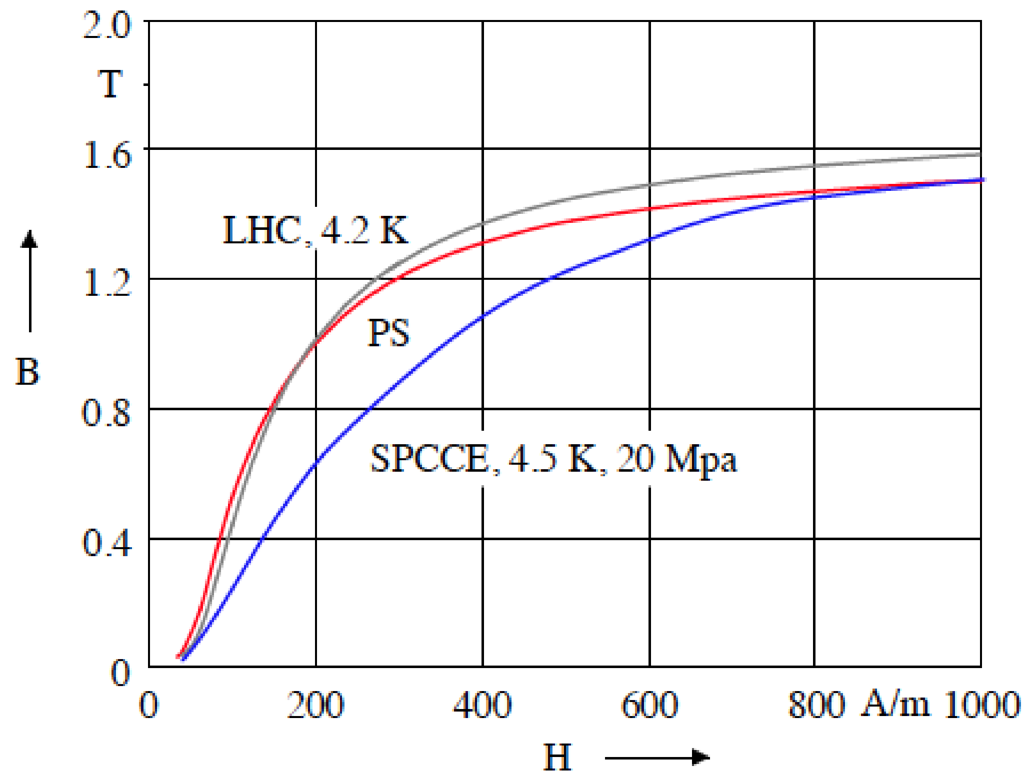
$$\overline{B} = \overline{B}_{\mathrm{meas}} + B_{\mathrm{corr}}.$$

Always check conditions of measurements

Temperature T	Stress	Coercive field H_c^B	Remanence B_r	max μ_r
K	MPa	$\mathrm{A\,m^{-1}}$	T	
300	0	68.4	1.07	5900
77	0	79.6	1.12	5600
4.2	0	85.1	1.06	4800
4.2	20	110	0.67	2460

Nonlinear Iron Magnetization





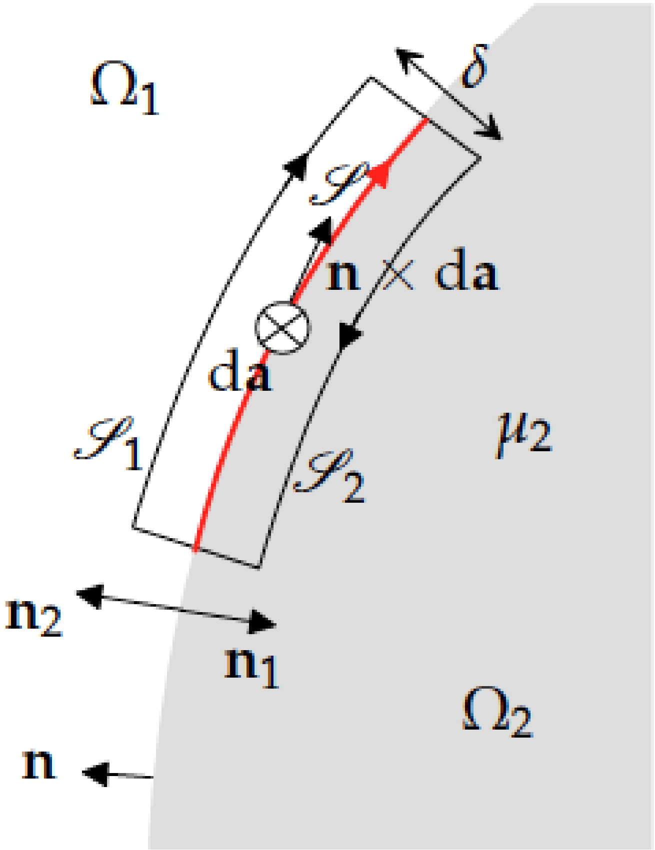
Continuity Conditions (1)

Applying Ampère's law $\int_{\partial\mathscr{A}} \mathbf{H} \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{A}} \mathbf{J} \cdot \mathrm{d}\mathbf{a}$ to the rectangular loop, yields for $\delta \to 0$

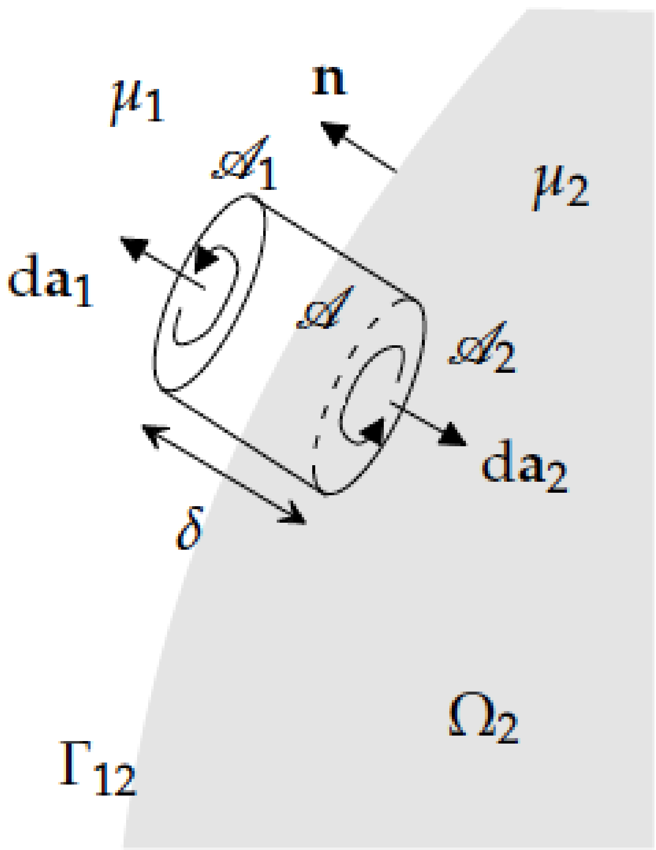
$$\int_{\mathscr{S}_2} \mathbf{H}_2 \cdot \mathrm{d}\mathbf{r} + \int_{\mathscr{S}_1} \mathbf{H}_1 \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{S}} (\mathbf{H}_1 - \mathbf{H}_2) \cdot \mathrm{d}\mathbf{r} = -\int_{\mathscr{S}} (\mathbf{n} \times \boldsymbol{\alpha}) \cdot \mathrm{d}\mathbf{r},$$

where the surface normal vector $\mathbf{n}$ points from Ω_2 to Ω_1

$$H_{t1} = H_{t2} \qquad \equiv \qquad \mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{0}$$



Continuity Conditions (2)



$$\int_{\partial V} \mathbf{B} \cdot \mathbf{da} = 0 \qquad \delta \to 0$$

$$\begin{aligned} \int_a \sigma_{\mathsf{mag}} \mathsf{d}a &= \int_a \mathbf{B}_1 \cdot \mathbf{da}_1 + \mathbf{B}_2 \cdot \mathbf{da}_2 \\ &= \int_a (\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n}_1 \mathsf{d}a \end{aligned}$$

Holds for any surface a if

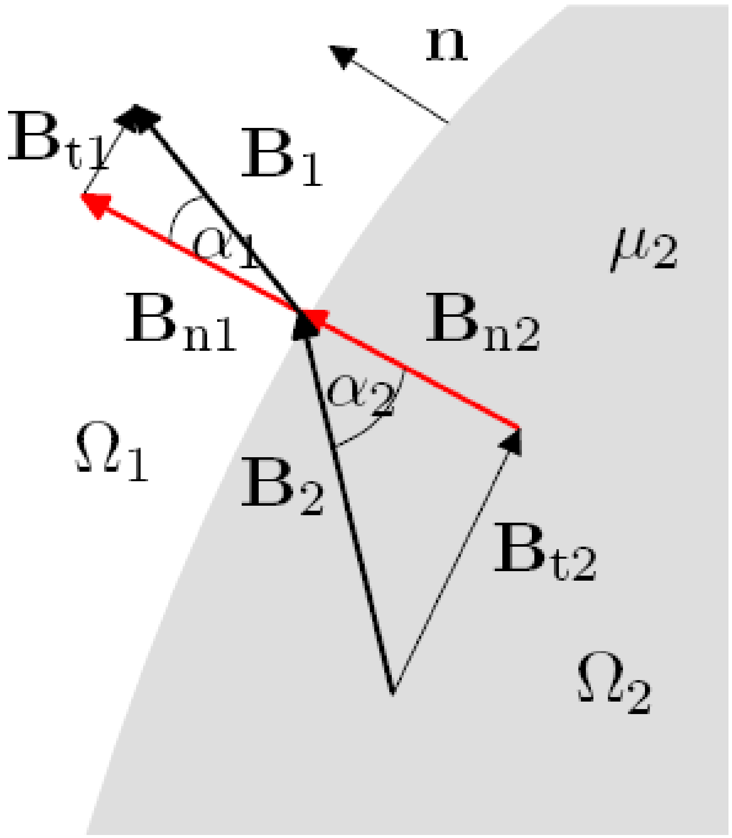
$$\begin{aligned} \sigma_{\mathsf{mag}} &= (\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n} \\ &= [\mathbf{B} \cdot \mathbf{n}]_{12} \end{aligned}$$

$$B_{\mathsf{n}1} = B_{\mathsf{n}2} \quad \equiv \quad (\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n} = 0 \quad \equiv \quad [\mathbf{B} \cdot \mathbf{n}]_{12} = 0$$

Continuity Conditions (3)

No surface currents:

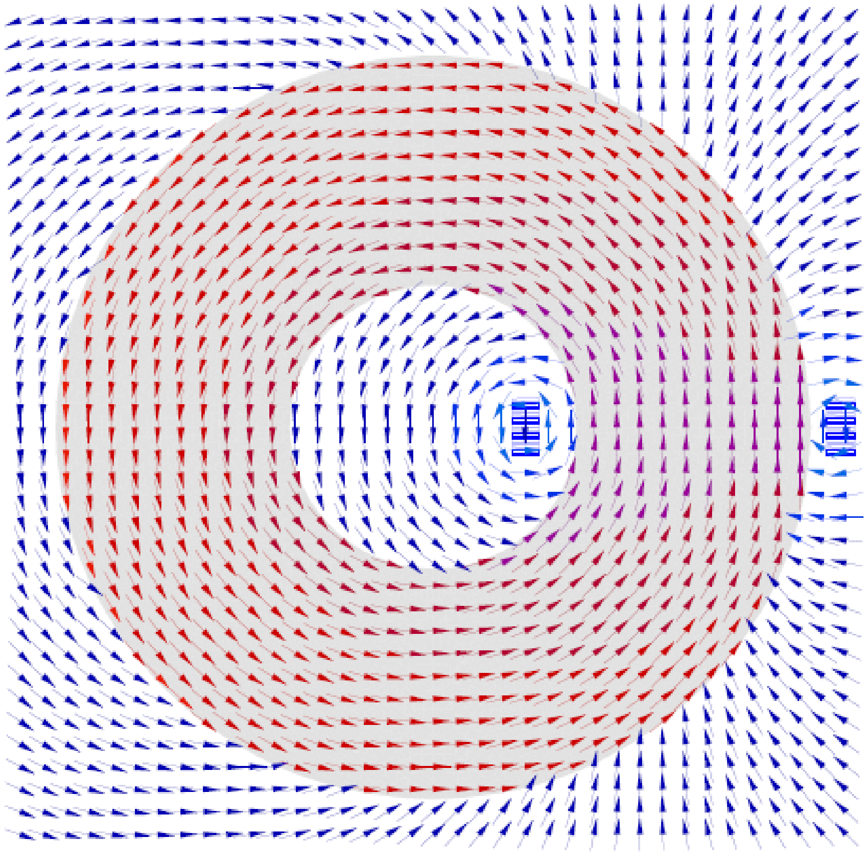
$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\frac{B_{t1}}{B_{n1}}}{\frac{B_{t2}}{B_{n2}}} = \frac{\mu_1 \frac{H_{t1}}{B_{n1}}}{\mu_2 \frac{H_{t2}}{B_{n2}}} = \frac{\mu_1 H_{t1}}{\mu_2 H_{t2}} = \frac{\mu_1}{\mu_2}$$

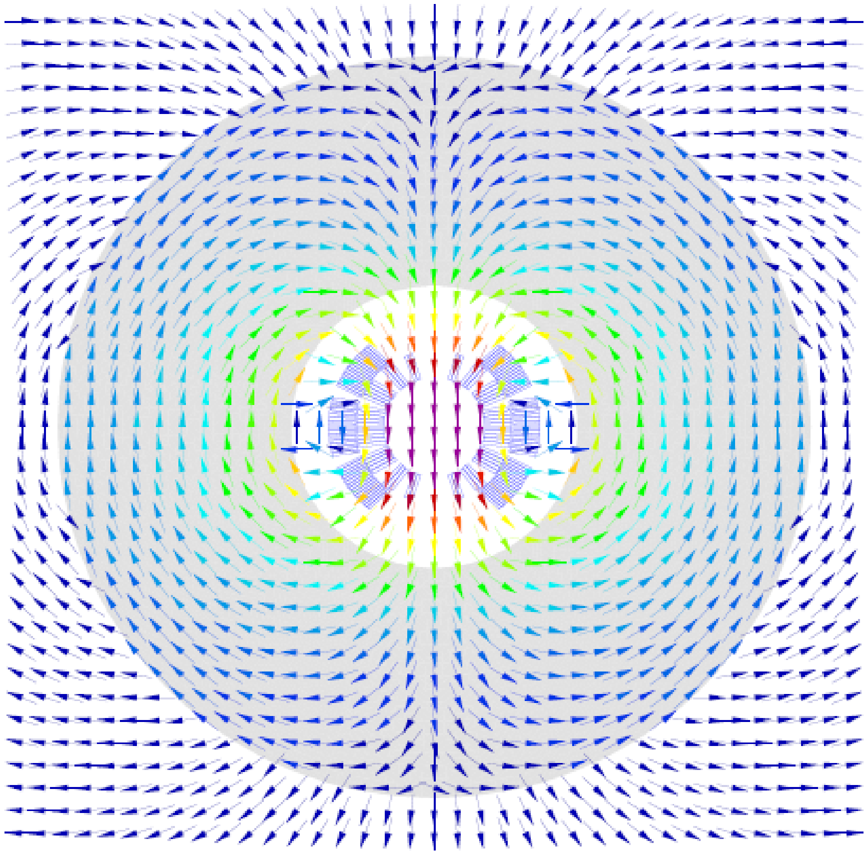


$$\mu_2 \gg \mu_1$$

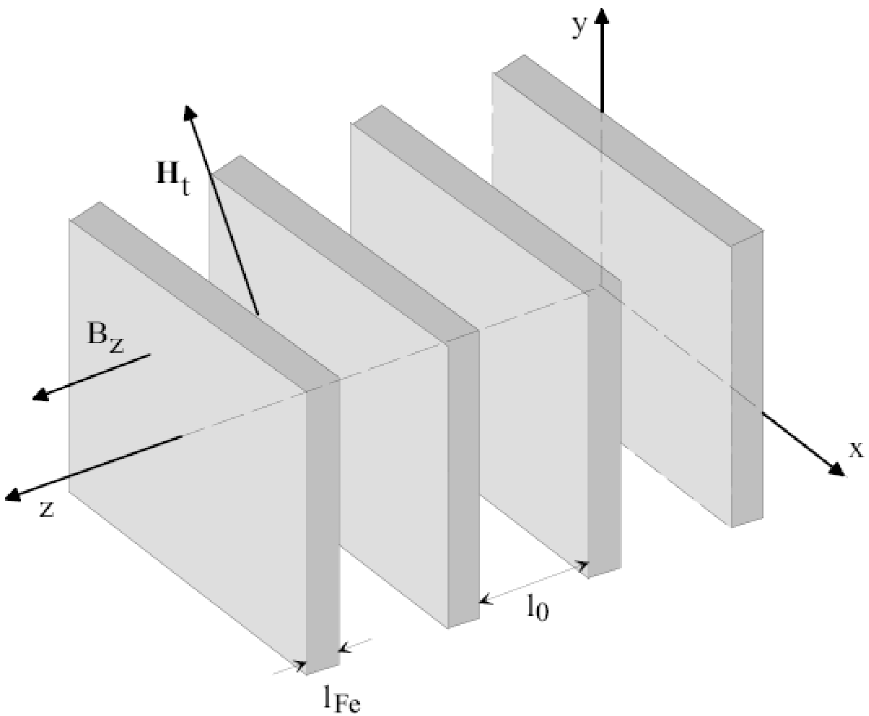
$$\alpha_1 \approx 0, \quad \text{or} \quad \alpha_2 \approx \pi/2,$$

Continuity at Iron Boundaries





Stacking Factor for Yoke Laminations



$$\mathbf{H}_\mathrm{t}^0 = \mathbf{H}_\mathrm{t}^\mathrm{Fe} = \overline{\mathbf{H}}_\mathrm{t}$$

$$\overline{\mathbf{B}}_\mathrm{t} = \frac{1}{l_\mathrm{Fe} + l_0} \left(l_\mathrm{Fe} \mu \overline{\mathbf{H}}_\mathrm{t} + l_0 \mu_0 \overline{\mathbf{H}}_\mathrm{t} \right)$$

$$B_z^0 = B_z^\mathrm{Fe} = \overline{B}_z$$

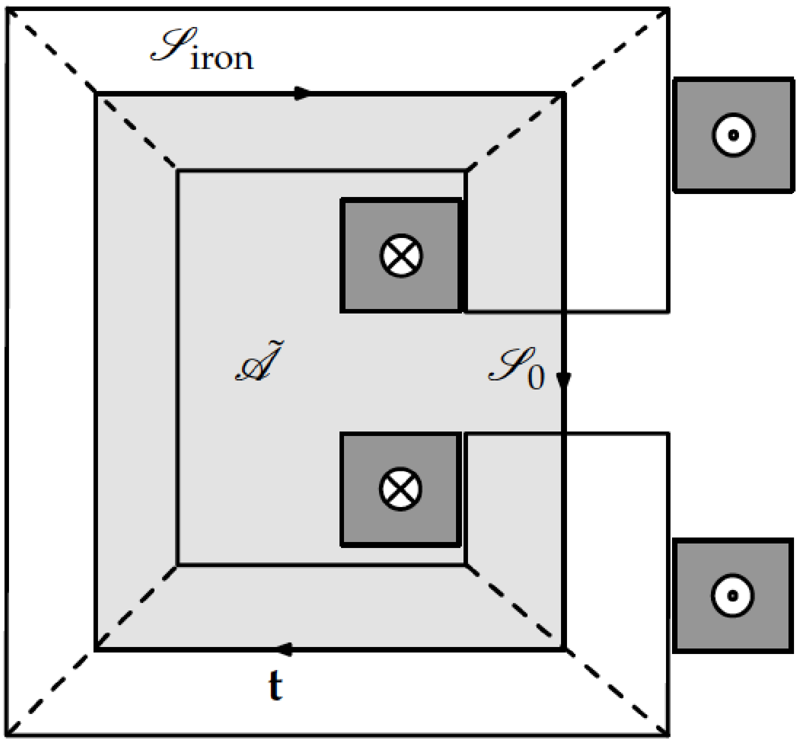
$$\overline{H}_z = \frac{1}{l_\mathrm{Fe} + l_0} \left(l_\mathrm{Fe} \frac{\overline{B}_z}{\mu} + l_0 \frac{\overline{B}_z}{\mu_0} \right)$$

$$\lambda = \frac{l_\mathrm{Fe}}{l_\mathrm{Fe} + l_0}$$

$$\overline{\mu}_\mathrm{t} = \lambda \mu + (1 - \lambda) \mu_0$$

$$\overline{\mu}_z = \left(\frac{\lambda}{\mu} + \frac{1 - \lambda}{\mu_0} \right)^{-1}$$

Main Field in Normal Conducting Dipole



$$\int_{\partial\tilde{\mathscr{A}}} \mathbf{H}\cdot \mathrm{d}\mathbf{r} = \int_{\tilde{\mathscr{A}}} \mathbf{J}\cdot\mathbf{n}\,\mathrm{d}a\,,$$

$$\int_{\mathscr{S}_{\text{iron}}} \mathbf{H}\cdot \mathrm{d}\mathbf{r} + \int_{\mathscr{S}_0} \mathbf{H}\cdot \mathrm{d}\mathbf{r} = \int_{\tilde{\mathscr{A}}_{\text{coil}}} \mathbf{J}\cdot\mathbf{n}\,\mathrm{d}a\,,$$

$$H_{\text{iron}}\, s_{\text{iron}} + H_0\, s_0 = N\,I\,,$$

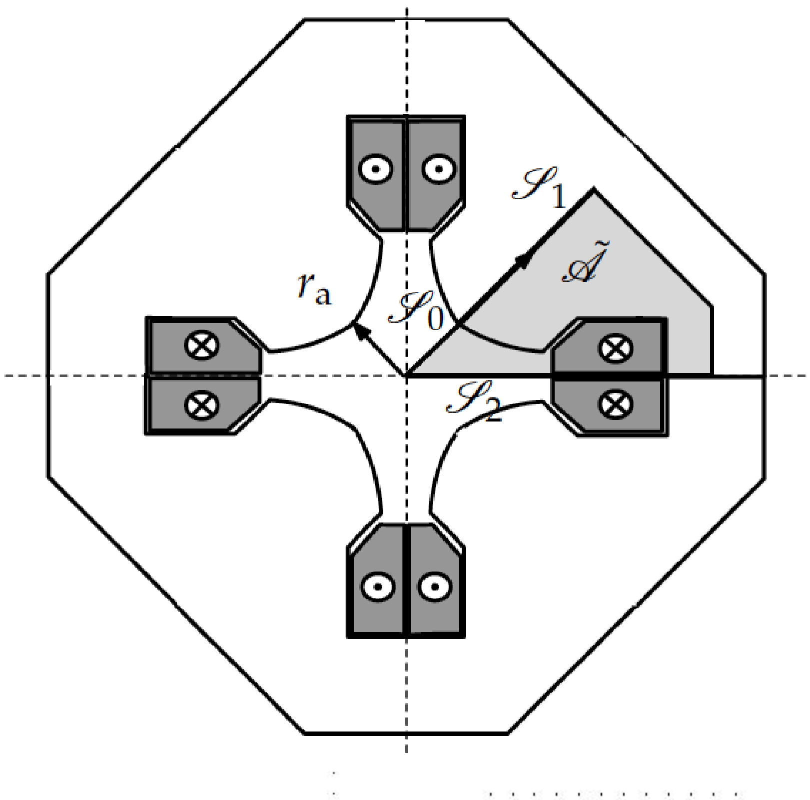
$$\frac{1}{\mu_0\mu_r} B_{\text{iron}}\, s_{\text{iron}} + \frac{1}{\mu_0} B_0\, s_0 = N\,I\,,$$

$$B_0 = \frac{\mu_0 N\, I}{s_0}\,.$$

Interrupt: Ampere-turns

Gradient in Normal Conducting Quadrupole

$$\int_{\partial \tilde{\mathscr{A}}} \mathbf{H} \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{S}_0} \mathbf{H}_0 \cdot \mathrm{d}\mathbf{r} + \int_{\mathscr{S}_1} \mathbf{H}_1 \cdot \mathrm{d}\mathbf{r} + \int_{\mathscr{S}_2} \mathbf{H}_2 \cdot \mathrm{d}\mathbf{r} = N\,I\,.$$



$$B_x = gy, \qquad B_y = gx;$$

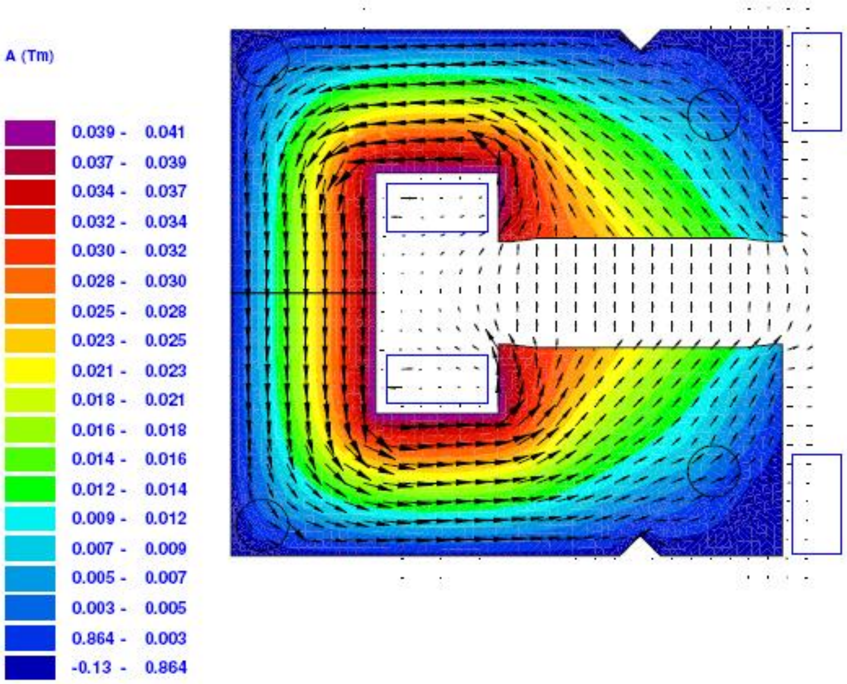
$$H = \frac{g}{\mu_0}\sqrt{x^2 + y^2} = \frac{g}{\mu_0} r.$$

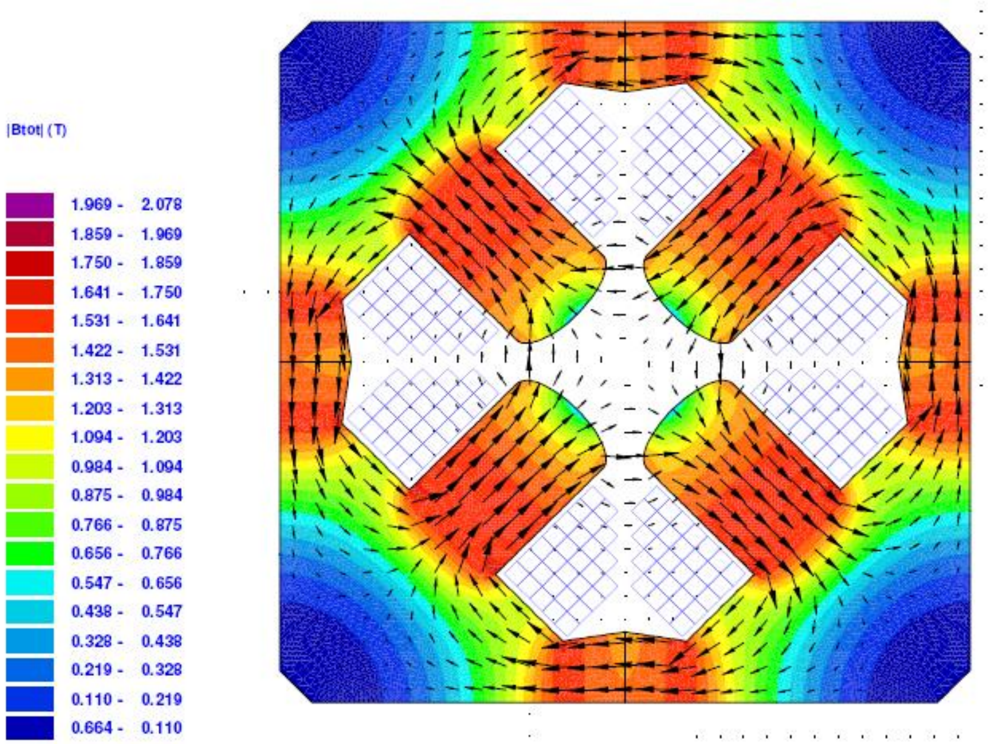
$$\int_0^{r_a} H \mathrm{d}r = \frac{g}{\mu_0} \int_0^{r_a} r \mathrm{d}r = \frac{g}{\mu_0} \frac{r_a^2}{2} = N\,I,$$

or

$$g = \frac{2\mu_0 N I}{r_a^2}.$$

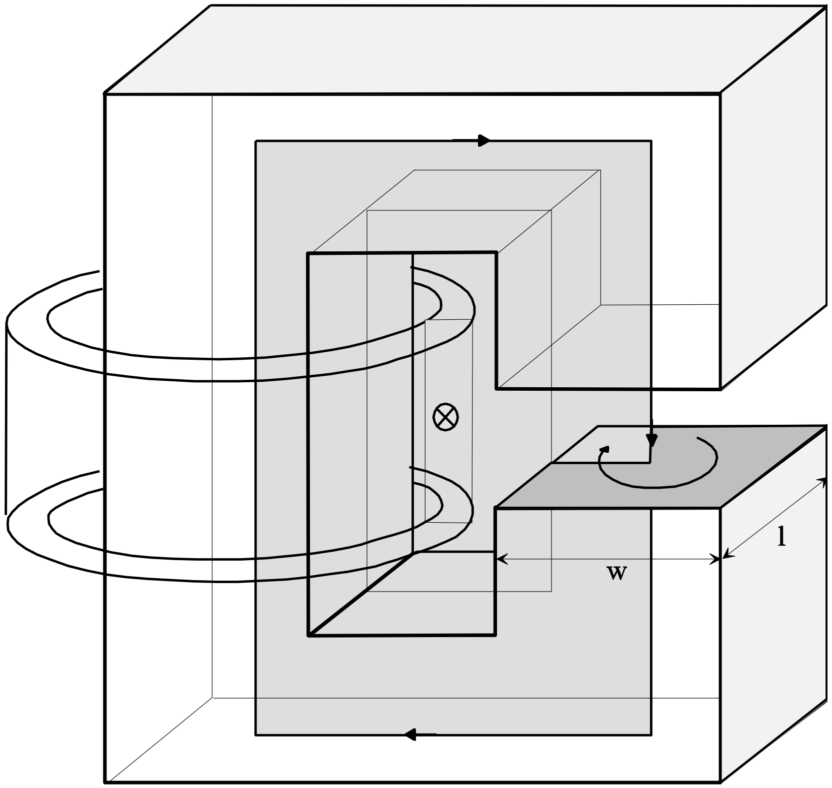
LEP Dipole and Quadrupole





Interrupt: Exercise c-core dipole; comparison to ROXIE simulations

Dipole with Varying Cut-Section



$$\sum_{i=0}^{n} H_i s_i = N\,I$$

$$H_i = \frac{B_i}{\mu_i} = \frac{\Phi}{a_i\,\mu_i}$$

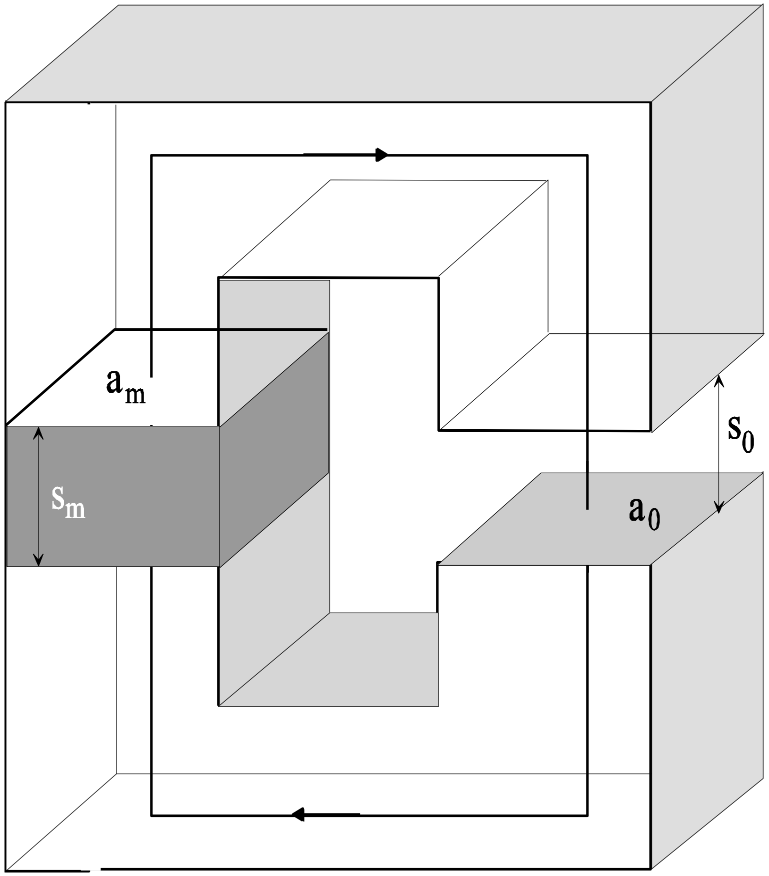
$$\Phi \sum_{i=0}^{n} \frac{s_i}{a_i\,\mu_i} = N\,I = V_{\mathrm{m}}$$

Ohm's law: $I \sum_{i=0}^{n} \frac{s_i}{a_i\,\kappa_i} = U$

$$N\,I = \Phi \sum_{i=0}^{n} \frac{s_i}{a_i\,\mu_i} = \Phi \left(\frac{s_0}{a_0\,\mu_0} + \sum_{i=1}^{n} \frac{s_i}{a_i\,\mu_i} \right)$$

Conclusion: Magnet with large air-gap is stabilized against variations in permeability

Permanent Magnet Excitation



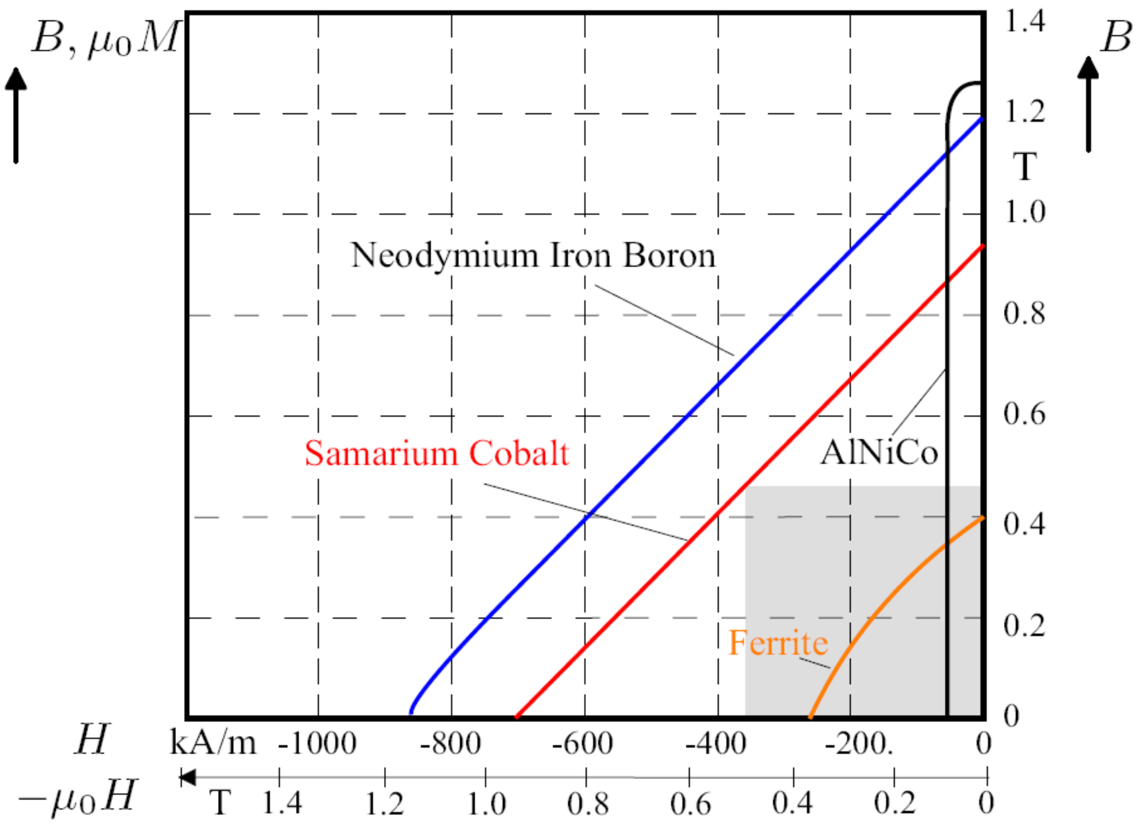
$$H_0\, s_0 + H_m\, s_m = 0$$

$$B_m a_m = B_0 a_0 = \mu_0 H_0 a_0$$

$$\begin{aligned} H_0 s_0 &= -H_m s_m, \\ \frac{1}{\mu_0} B_m \frac{a_m}{a_0} s_0 &= -H_m s_m, \\ B_m &= -\mu_0 \frac{s_m}{s_0} \frac{a_0}{a_m} H_m, \\ \frac{B_m}{\mu_0 H_m} &= -\frac{s_m}{s_0} \frac{a_0}{a_m} = P \end{aligned}$$

$$s = \frac{B_m}{H_m} = \mu_0 P = -\mu_0 \frac{s_m}{s_0} \frac{a_0}{a_m} = \mu_0 \frac{M\,(1-N)}{H_m - N\,M}$$

BH Maximum



$$B_m a_m = B_0 a_0 = \mu_0 H_0 a_0$$

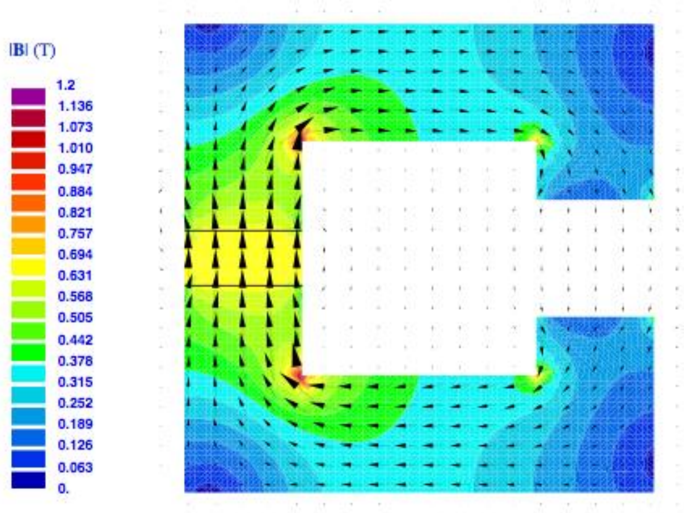
$$H_0 s_0 = -H_m s_m$$

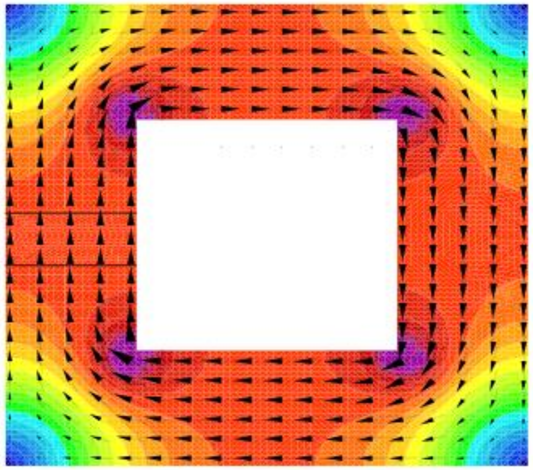
$$B_m a_m s_m = \mu_0 H_0 a_0 \frac{-H_0 s_0}{H_m}$$

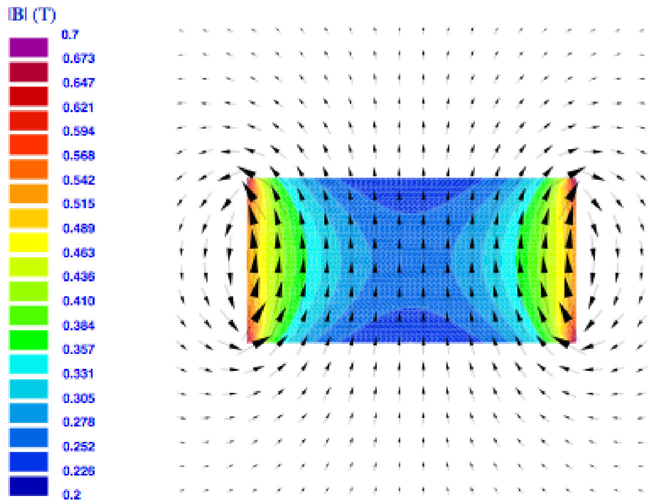
$$(BH)^{\text{id}}_{\text{max}} := \frac{B_r^2}{4\mu_0},$$

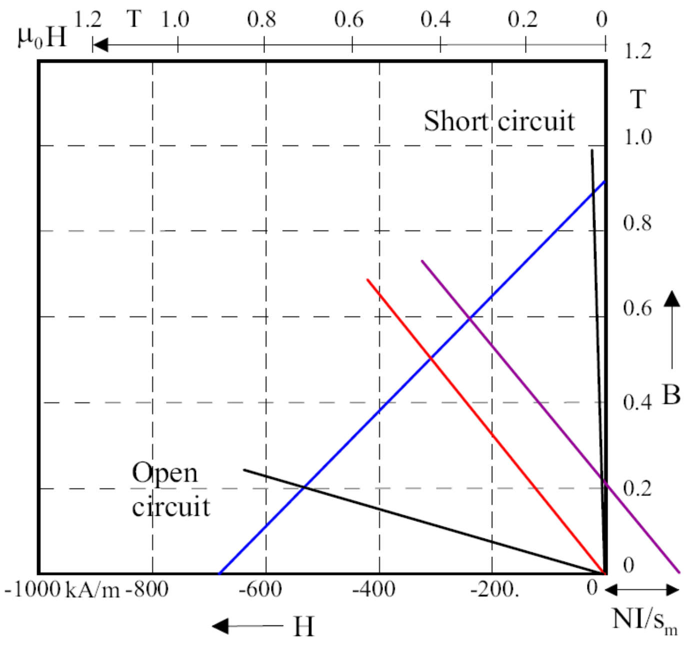
$$H_0 = \sqrt{\frac{(a_m s_m)(-B_m H_m)}{\mu_0 (a_0 s_0)}} = \sqrt{\frac{V_m(-B_m H_m)}{\mu_0 V_0}}$$

Permanent Magnet Circuits





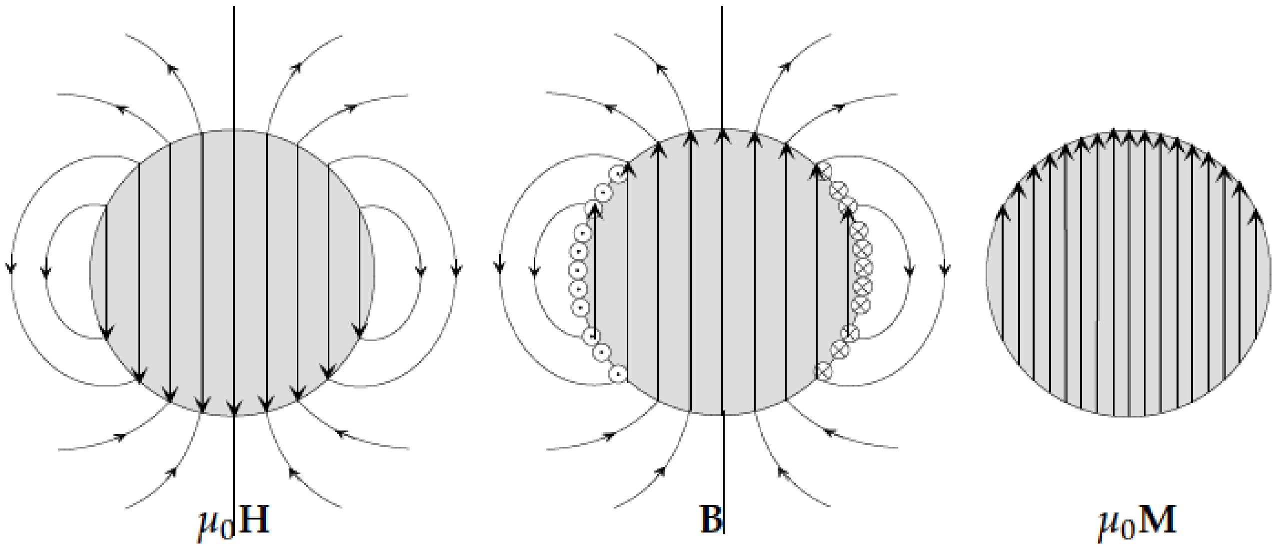


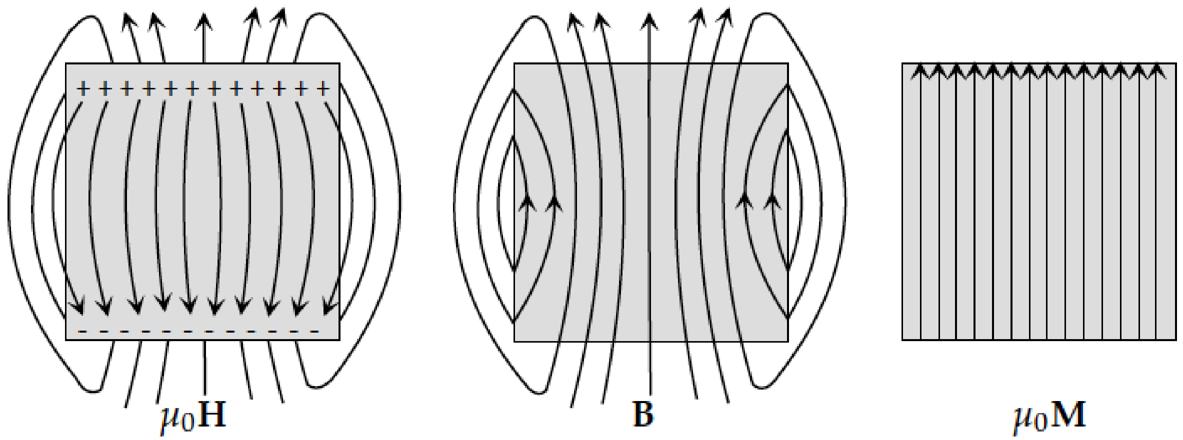


$$s = \frac{B_m}{H_m} = \mu_0 P = -\mu_0 \frac{s_m}{s_0} \frac{a_0}{a_m}$$

Surface charges and Surface Currents

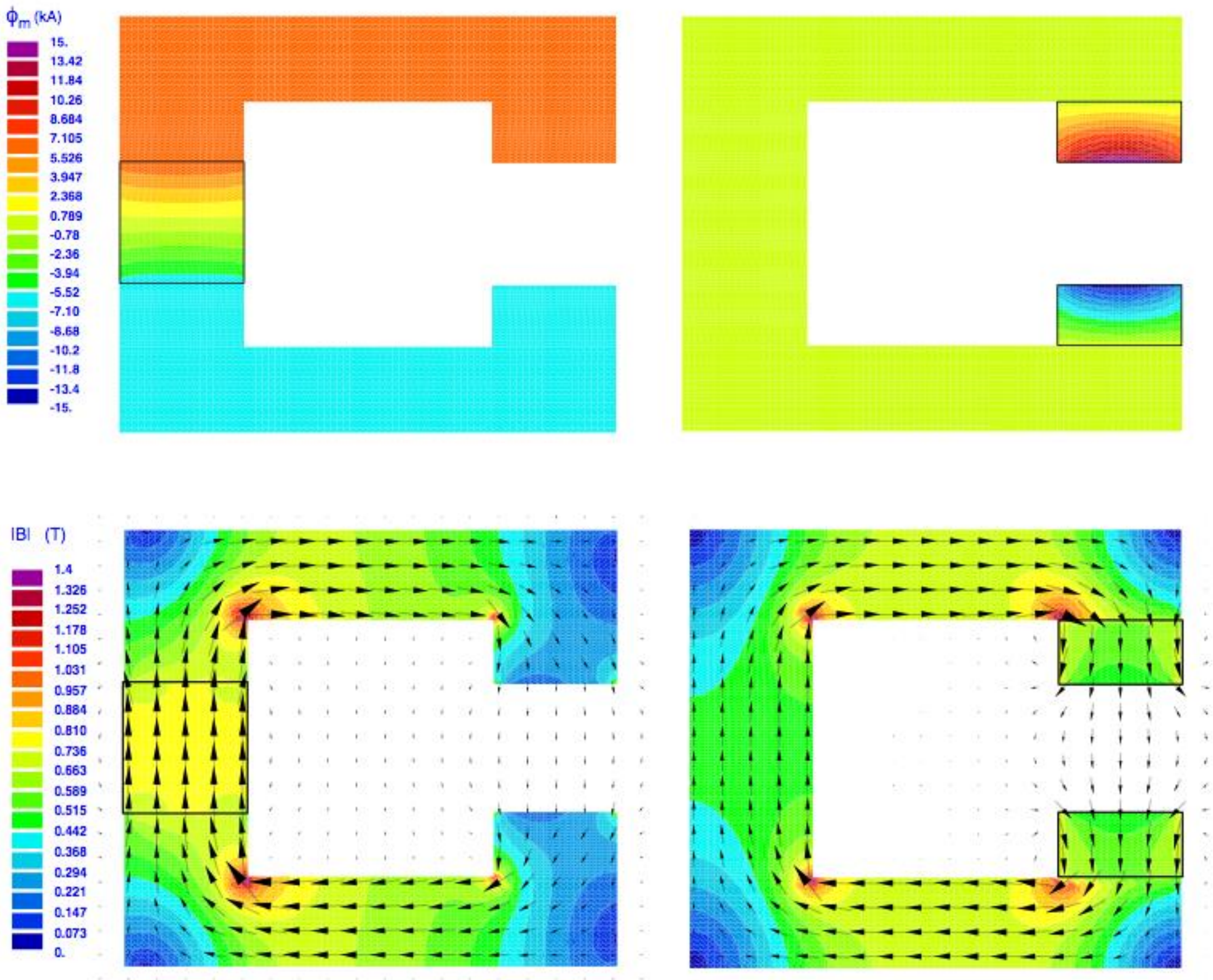
$$\boldsymbol{\alpha} = -\mathbf{n} \times [\![\mathbf{M}]\!]_{12}$$



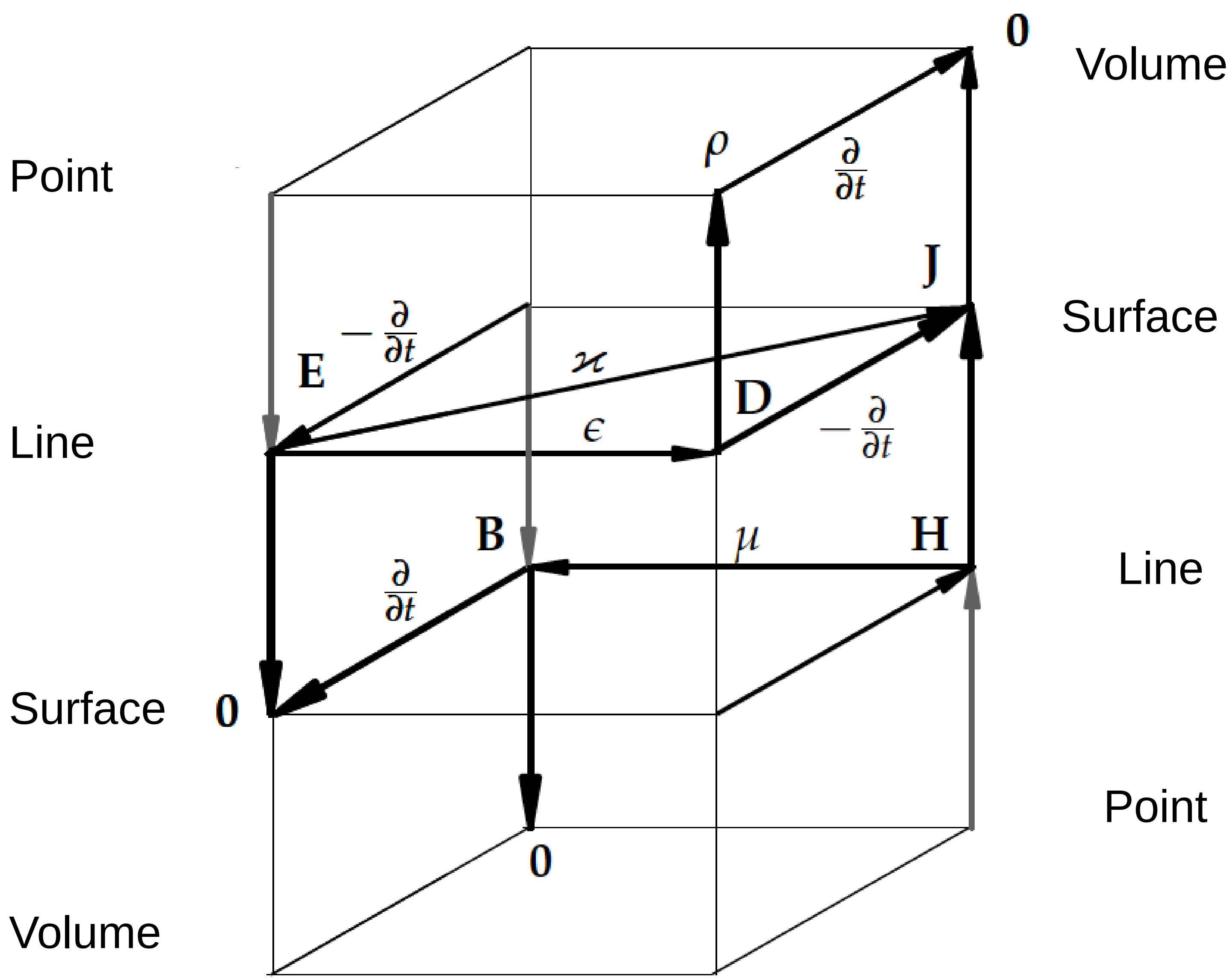


$$\sigma_{\mathrm{m}} = -\mu_0 \mathbf{n} \cdot [\![\mathbf{M}]\!]_{12}$$

Optimal Position of Permanent Magnets



Maxwell's House Again



Coordinate Free Definition of Grad, Curl, and Div

$$\int_{\mathscr{P}_1}^{\mathscr{P}_2} \mathbf{a} \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{P}_1}^{\mathscr{P}_2} \operatorname{grad} \phi \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{P}_1}^{\mathscr{P}_2} \mathrm{d}\phi = \phi(\mathscr{P}_2) - \phi(\mathscr{P}_1),$$

$$\mathbf{n} \cdot \operatorname{curl} \mathbf{g} = \lim_{a \to 0} \frac{\int_{\partial \mathscr{A}} \mathbf{g} \cdot \mathrm{d}\mathbf{r}}{a},$$

$$\operatorname{div} \mathbf{g} = \lim_{V \to 0} \frac{\int_{\partial \mathscr{V}} \mathbf{g} \cdot \mathrm{d}\mathbf{a}}{V},$$

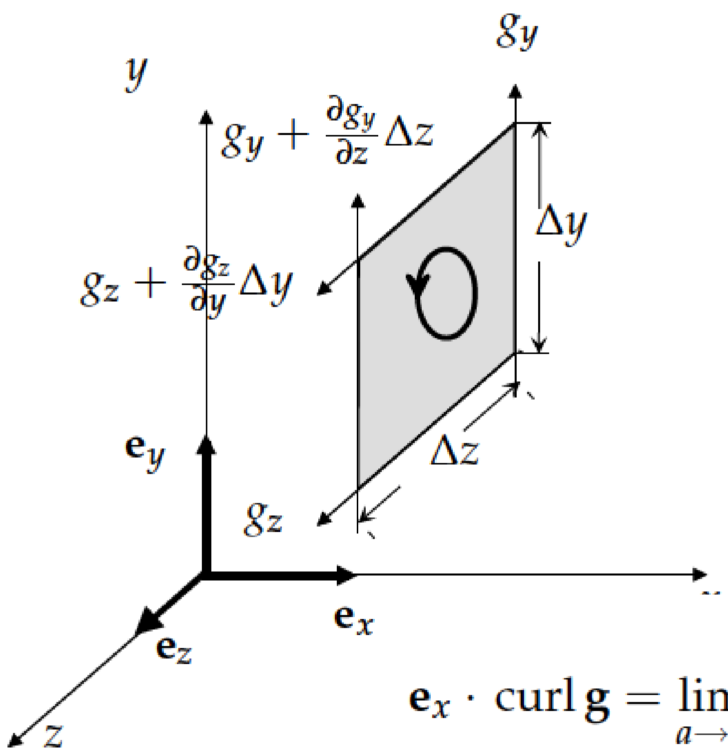
Grad, Curl and Div in Cartesian Coordinates

$$\operatorname{grad}\phi := \frac{\partial\phi}{\partial x}\mathbf{e}_x + \frac{\partial\phi}{\partial y}\mathbf{e}_y + \frac{\partial\phi}{\partial z}\mathbf{e}_z$$

$$\operatorname{curl}\mathbf{g} = \left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y}\right)\mathbf{e}_z .$$

$$\operatorname{div}\mathbf{g} = \frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} .$$

Curl in Cartesian Coordinates



$$\mathbf{n} \cdot \operatorname{curl} \mathbf{g} = \lim_{a \to 0} \frac{\int_{\partial \mathscr{A}} \mathbf{g} \cdot \mathrm{d}\mathbf{r}}{a},$$

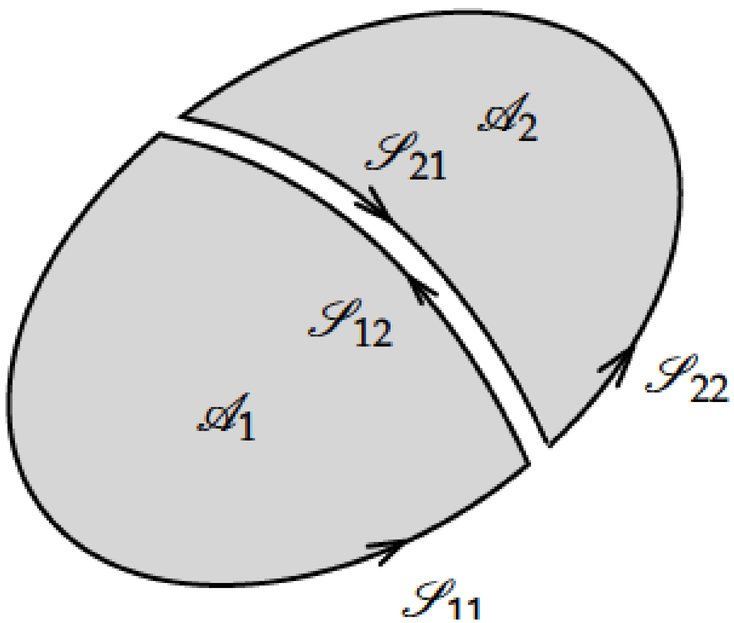
$$\begin{aligned}
\mathbf{e}_x \cdot \operatorname{curl} \mathbf{g} &= \lim_{a \to 0} \frac{\int_{\partial \mathscr{A}} \mathbf{g} \cdot \mathrm{d}\mathbf{r}}{a} \\
&= \lim_{\Delta y, \Delta z \to 0} \frac{g_y \Delta y + \left(g_z + \frac{\partial g_z}{\partial y} \Delta y \right) \Delta z - \left(g_y + \frac{\partial g_y}{\partial z} \Delta z \right) \Delta y - g_z \Delta z}{\Delta y \, \Delta z} \\
&= \frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z}.
\end{aligned}$$

$$\operatorname{curl} \mathbf{g} = \left(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z} \right) \mathbf{e}_x + \left(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x} \right) \mathbf{e}_y + \left(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y} \right) \mathbf{e}_z.$$

Differential Operators in Coordinates

	Cartesian	Cylindrical
u^1, u^2, u^3	x, y, z	r, φ, z
$\mathrm{d}s^2$	$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$	$\mathrm{d}r^2 + r^2 \mathrm{d}\varphi^2 + \mathrm{d}z^2$
$\mathrm{d}V$	$\mathrm{d}x\mathrm{d}y\mathrm{d}z$	$r\mathrm{d}r\mathrm{d}\varphi\mathrm{d}z$
$\mathrm{grad}\,\phi$	$\frac{\partial \phi}{\partial x}\mathbf{e}_x + \frac{\partial \phi}{\partial y}\mathbf{e}_y + \frac{\partial \phi}{\partial z}\mathbf{e}_z$	$\frac{\partial \phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \varphi}\mathbf{e}_\varphi + \frac{\partial \phi}{\partial z}\mathbf{e}_z$
$\mathrm{div}\,\mathbf{g}$	$\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z}$	$\frac{1}{r}\frac{\partial}{\partial r}(r g_r) + \frac{1}{r}\frac{\partial g_\varphi}{\partial \varphi} + \frac{\partial g_z}{\partial z}$
$\mathrm{curl}\,\mathbf{g}$	$(\frac{\partial g_z}{\partial y} - \frac{\partial g_y}{\partial z})\mathbf{e}_x$ $+(\frac{\partial g_x}{\partial z} - \frac{\partial g_z}{\partial x})\mathbf{e}_y$ $+(\frac{\partial g_y}{\partial x} - \frac{\partial g_x}{\partial y})\mathbf{e}_z$	$(\frac{1}{r}\frac{\partial g_z}{\partial \varphi} - \frac{\partial g_\varphi}{\partial z})\mathbf{e}_r$ $+(\frac{\partial g_r}{\partial z} - \frac{\partial g_z}{\partial r})\mathbf{e}_\varphi$ $+(\frac{1}{r}\frac{\partial}{\partial r}(r g_\varphi) - \frac{1}{r}\frac{\partial g_r}{\partial \varphi})\mathbf{e}_z$
$\nabla^2 \phi$	$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$	$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2}$

Kelvin-Stokes Theorem



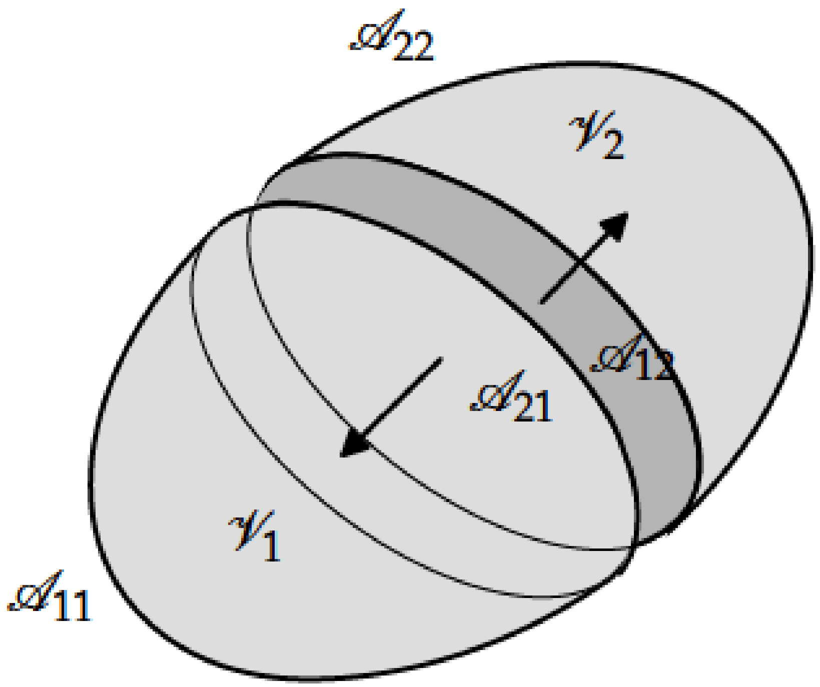
Smooth vector fields, smooth surfaces with simply connected, closed, piecewise-smooth and consistently oriented boundaries, and volumes with piecewise-smooth, closed and consistently oriented surfaces.

$$\int_{\partial\mathscr{A}} \mathbf{g}\cdot d\mathbf{r} = \int_{\mathscr{S}_1} \mathbf{g}\cdot d\mathbf{r} + \int_{\mathscr{S}_2} \mathbf{g}\cdot d\mathbf{r} = \int_{\mathscr{S}_{11}} \mathbf{g}\cdot d\mathbf{r} + \int_{\mathscr{S}_{22}} \mathbf{g}\cdot d\mathbf{r},$$

$$\int_{\partial\mathscr{A}} \mathbf{g}\cdot d\mathbf{r} = \lim_{I\to\infty}\sum_{i=1}^{I}\int_{\partial\mathscr{A}_i} \mathbf{g}\cdot d\mathbf{r} = \lim_{I\to\infty}\sum_{i=1}^{I}\Delta a_i \frac{1}{\Delta a_i}\int_{\partial\mathscr{A}_i} \mathbf{g}\cdot d\mathbf{r}$$

$$= \lim_{I\to\infty}\sum_{i=1}^{I}(\operatorname{curl}\mathbf{g})_i\cdot\mathbf{n}\,\Delta a_i = \int_{\mathscr{A}} \operatorname{curl}\mathbf{g}\cdot d\mathbf{a}\,.$$

Gauss' Theorem



Smooth vector fields, smooth surfaces with simply connected, closed, piecewise-smooth and consistently oriented boundaries, and volumes with piecewise-smooth, closed and consistently oriented surfaces.

$$\int_{\partial\mathscr{V}} \mathbf{g}\cdot\mathrm{d}\mathbf{a} = \int_{\mathscr{A}_1} \mathbf{g}\cdot\mathrm{d}\mathbf{a} + \int_{\mathscr{A}_2} \mathbf{g}\cdot\mathrm{d}\mathbf{a} = \int_{\mathscr{A}_{11}} \mathbf{g}\cdot\mathrm{d}\mathbf{a} + \int_{\mathscr{A}_{22}} \mathbf{g}\cdot\mathrm{d}\mathbf{a}$$

$$\int_{\partial\mathscr{V}} \mathbf{g}\cdot\mathrm{d}\mathbf{a} = \lim_{I\to\infty}\sum_{i=1}^{I}\int_{\partial\mathscr{V}_i} \mathbf{g}\cdot\mathrm{d}\mathbf{a} = \lim_{I\to\infty}\sum_{i=1}^{I}\Delta V_i\frac{1}{\Delta V_i}\int_{\partial\mathscr{V}_i} \mathbf{g}\cdot\mathrm{d}\mathbf{a}$$

$$= \lim_{I\to\infty}\sum_{i=1}^{I}(\operatorname{div}\mathbf{g})_i\,\Delta V_i = \int_{\mathscr{V}} \operatorname{div}\mathbf{g}\,\mathrm{d}V\,.$$

Maxwell's Equations in Differential Form

$$\int_{\mathscr{A}} \operatorname{curl} \mathbf{g} \cdot \mathrm{d}\mathbf{a} = \int_{\partial\mathscr{A}} \mathbf{g} \cdot \mathrm{d}\mathbf{r},$$

$$\int_{\mathscr{V}} \operatorname{div} \mathbf{g} \,\mathrm{d}V = \int_{\partial\mathscr{V}} \mathbf{g} \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{A}} \mathbf{H} \cdot \mathrm{d}\mathbf{r} = \int_{\mathscr{A}} \mathbf{J} \cdot \mathrm{d}\mathbf{a} + \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{A}} \mathbf{D} \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{A}} \mathbf{E} \cdot \mathrm{d}\mathbf{r} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{A}} \mathbf{B} \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial\mathscr{V}} \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0,$$

$$\int_{\partial\mathscr{V}} \mathbf{D} \cdot \mathrm{d}\mathbf{a} = \int_{\mathscr{V}} \rho \,\mathrm{d}V.$$

$$\int_{\mathscr{A}} \operatorname{curl} \mathbf{H} \cdot \mathrm{d}\mathbf{a} = \int_{\mathscr{A}} \left(\mathbf{J} + \frac{\partial}{\partial t} \mathbf{D} \right) \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\mathscr{A}} \operatorname{curl} \mathbf{E} \cdot \mathrm{d}\mathbf{a} = -\int_{\mathscr{A}} \frac{\partial}{\partial t} \mathbf{B} \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\mathscr{V}} \operatorname{div} \mathbf{B} \,\mathrm{d}V = 0,$$

$$\int_{\mathscr{V}} \operatorname{div} \mathbf{D} \,\mathrm{d}V = \int_{\mathscr{V}} \rho \,\mathrm{d}V.$$

$$\operatorname{curl} \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t} \mathbf{D},$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B},$$

$$\operatorname{div} \mathbf{B} = 0,$$

$$\operatorname{div} \mathbf{D} = \rho.$$

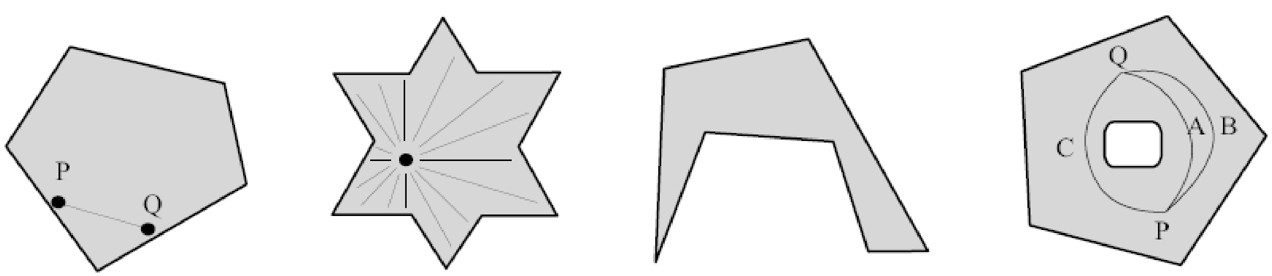
Lemmata of Poincare

$$\operatorname{div}\,\operatorname{curl}\mathbf{g} = 0. \qquad \operatorname{curl}\,\operatorname{grad}\phi = 0,$$

$$\partial(\partial\mathscr{V}) = \emptyset, \qquad \partial(\partial\mathscr{A}) = \emptyset,$$

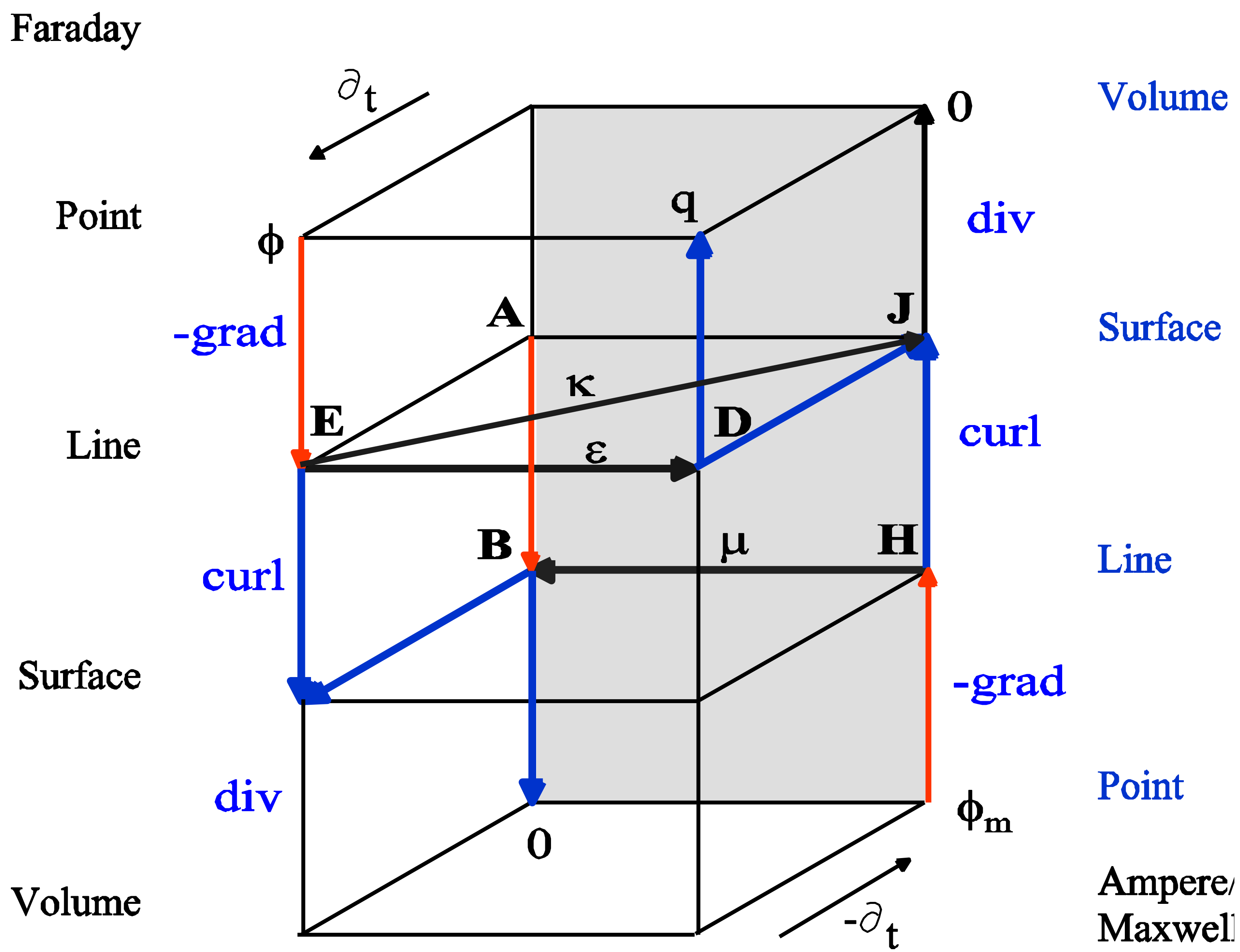
$$\int_{\mathscr{A}} \operatorname{curl}\,\operatorname{grad}\phi\cdot d\mathbf{a} = \int_{\partial\mathscr{A}} \operatorname{grad}\phi\cdot d\mathbf{r} = \phi|_{\partial(\partial\mathscr{A})} = 0,$$

$$\int_{\mathscr{V}} \operatorname{div}\,\operatorname{curl}\mathbf{g}\, dV = \int_{\partial\mathscr{V}} \operatorname{curl}\mathbf{g}\cdot d\mathbf{a} = \int_{\partial(\partial\mathscr{V})} \mathbf{g}\cdot d\mathbf{r} = 0,$$

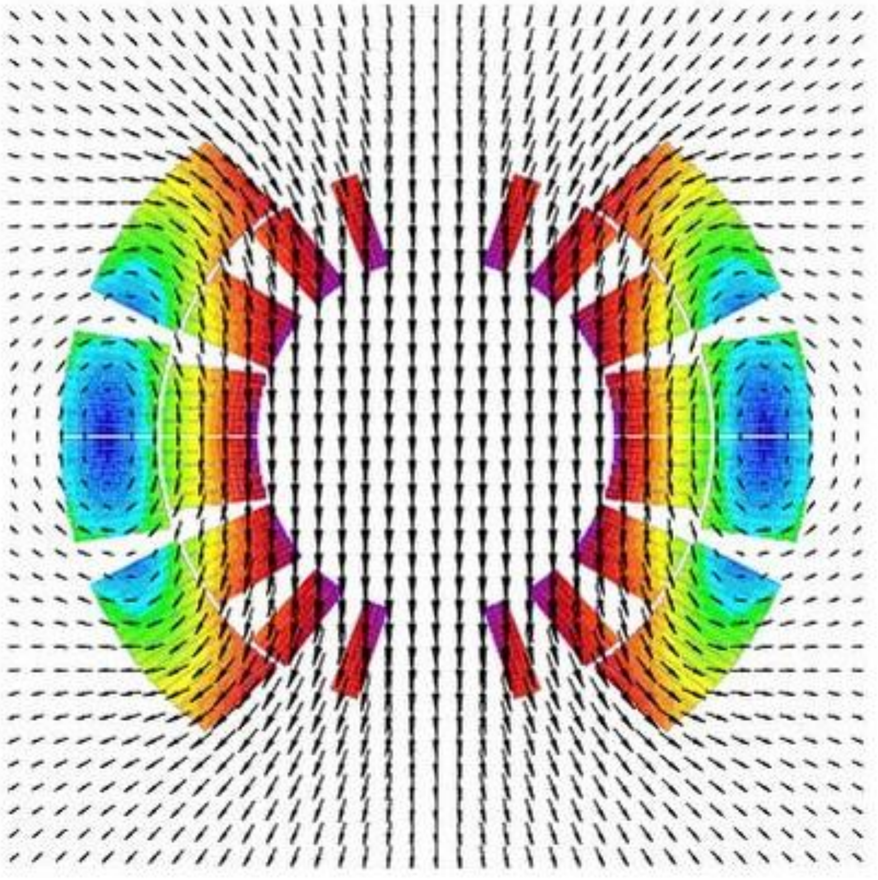


In 3D: also connected boundaries

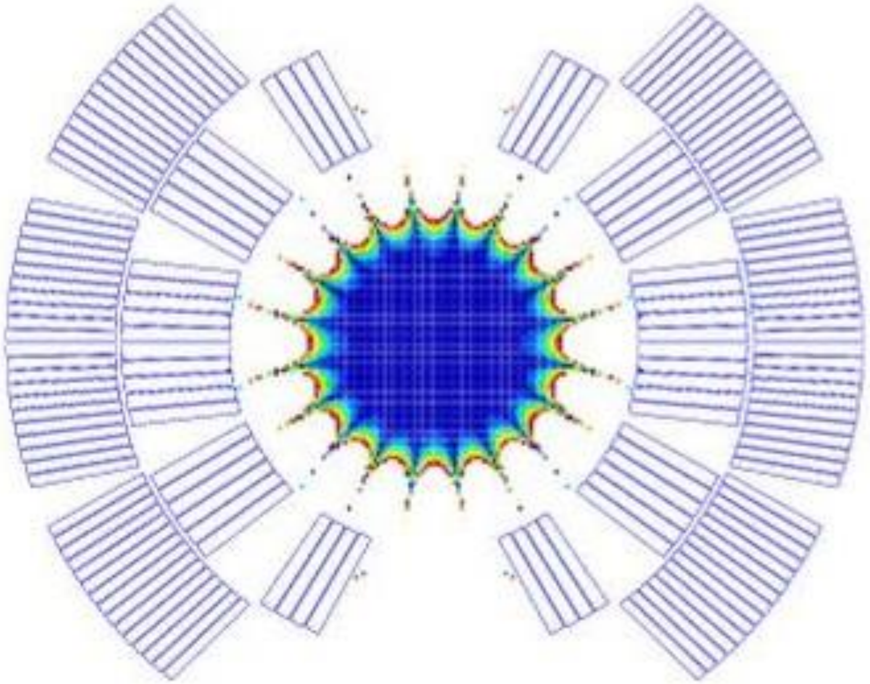
Maxwell’s House



Field Quality



Field map



Good field region

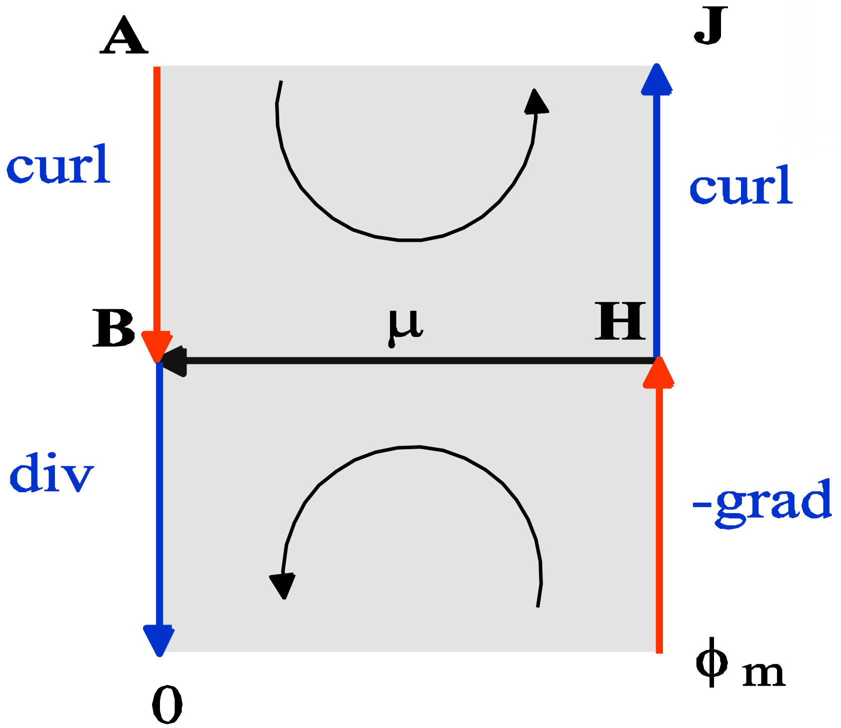
Maxwell's Facade

$$\mathrm{curl}\,\frac{1}{\mu}\,\mathrm{curl}\,\mathbf{A} = \mathbf{J}$$

$$\frac{1}{\mu_0}\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{A} = \mathbf{J}$$

$$\nabla^2\mathbf{A} - \mathrm{grad}\,\mathrm{div}\,\mathbf{A} = 0$$

$$\nabla^2 A_z = 0$$



$$\mathrm{div}\,\mu\,\mathrm{grad}\,\phi_{\mathrm{m}} = 0$$

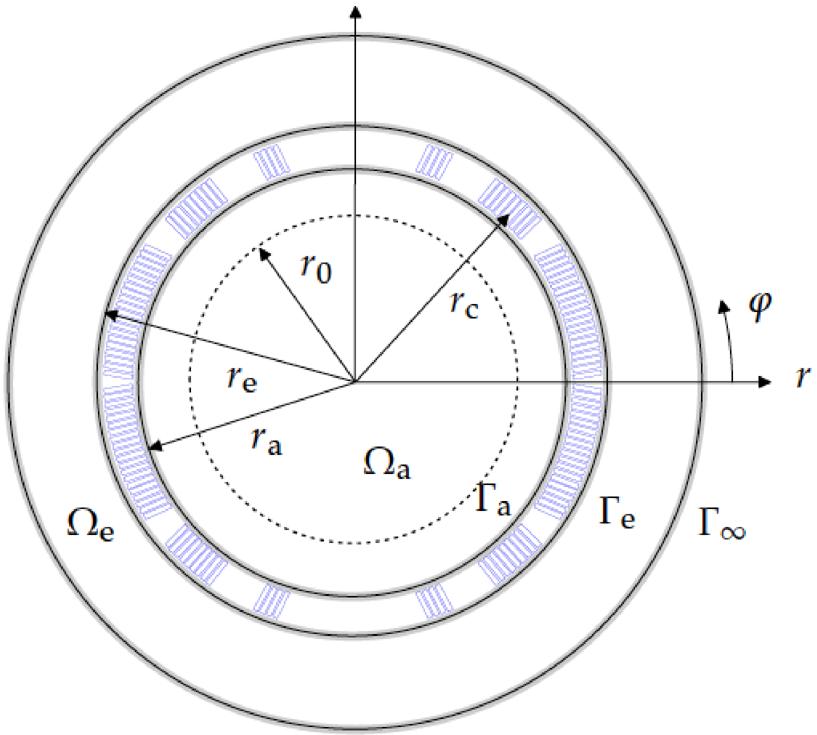
$$\mu_0\,\mathrm{div}\,\mathrm{grad}\,\phi_{\mathrm{m}} = 0$$

$$\nabla^2\phi_{\mathrm{m}} = 0$$

1. Governing equation in the air domain

$$\nabla^2 A_z = 0,$$

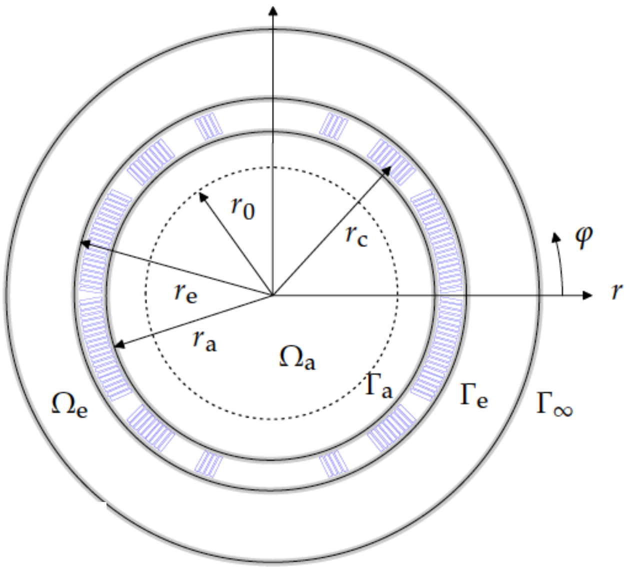
2. Chose a suitable coordinate system



$$r^2 \frac{\partial^2 A_z}{\partial r^2} + r \frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial \varphi^2} = 0,$$

3. Make a guess, look it up in a book, use the method of separation:
That is: find eigenfunctions. Coefficients are not know yet

$$A_z(r,\varphi) = \sum_{n=1}^{\infty} (\mathcal{E}_n r^n + \mathcal{F}_n r^{-n})(\mathcal{G}_n \sin n\varphi + \mathcal{H}_n \cos n\varphi).$$



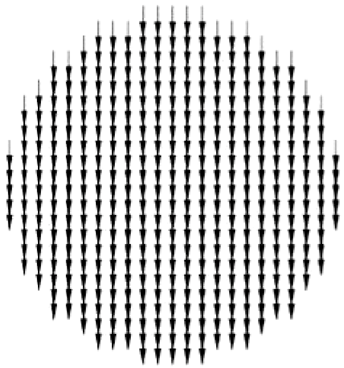
4. Incorporate a bit of knowledge and rename

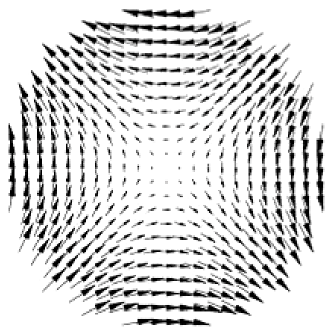
$$A_z(r,\varphi) = \sum_{n=1}^{\infty} r^n (\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi).$$

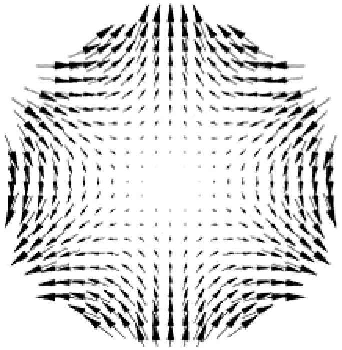
2. Calculate a field component

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

$$B_\varphi(r,\varphi) = -\frac{\partial A_z}{\partial r} = -\sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi),$$





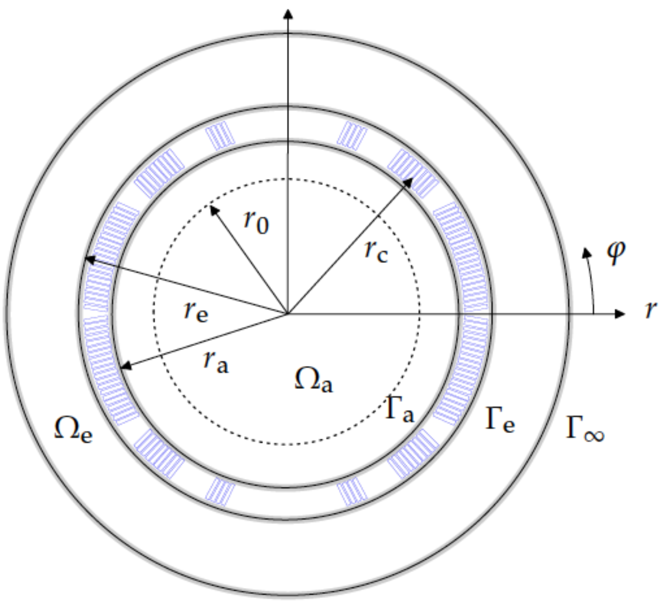


Solving of Boundary Value Problems

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

3. Measure or calculate the field on a reference radius and perform Fourier analysis (develop into the eigenfunctions). Coefficients known here.

$$B_r(r_0,\varphi) = \sum_{n=1}^{\infty} (B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi),$$



Solving the Boundary Value Problem

4: Compare the known and unknown coefficients

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi),$$

$$B_r(r_0,\varphi) = \sum_{n=1}^{\infty} (B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi),$$

$$\mathcal{A}_n = \frac{1}{n\, r_0^{n-1}} A_n(r_0)\,, \qquad \mathcal{B}_n = \frac{-1}{n\, r_0^{n-1}} B_n(r_0)\,.$$

5. Put this into the original solution for the entire air domain

$$A_z(r,\varphi) = -\sum_{n=1}^{\infty} \frac{r_0}{n} \left(\frac{r}{r_0}\right)^n (B_n(r_0) \cos n\varphi - A_n(r_0) \sin n\varphi).$$

6: Calculate fields and potential in the entire air domain

$$A_z(r,\varphi) = -\sum_{n=1}^{\infty} \frac{r_0}{n}\left(\frac{r}{r_0}\right)^n (B_n(r_0)\cos n\varphi - A_n(r_0)\sin n\varphi).$$

$$B_r(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi)$$

$$B_\varphi(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0)\cos n\varphi - A_n(r_0)\sin n\varphi)$$

$$B_x(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0)\sin(n-1)\varphi + A_n(r_0)\cos(n-1)\varphi)$$

$$B_y(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0)\cos(n-1)\varphi - A_n(r_0)\sin(n-1)\varphi)$$

Fourier Series (an Infinite Dimensional Vector Space)

$$B_r(r_0,\varphi) = \sum_{n=1}^{\infty} (B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi),$$

$$A_n(r_0) = \frac{1}{\pi}\int_0^{2\pi} B_r(r_0,\varphi)\cos n\varphi \,\mathrm{d}\varphi, \qquad n = 1,2,3,\dots,$$

$$B_n(r_0) = \frac{1}{\pi}\int_0^{2\pi} B_r(r_0,\varphi)\sin n\varphi \,\mathrm{d}\varphi, \qquad n = 1,2,3,\dots.$$

And on the computer: Discrete setting (don't bother with the FFT)

$$\varphi_k = \frac{2\pi k}{N}, \qquad k = 0,1,2,\dots,N-1.$$

$$A_n(r_0) \approx \frac{2}{N}\sum_{k=0}^{N-1} B_r(r_0,\varphi_k)\cos n\varphi_k,$$

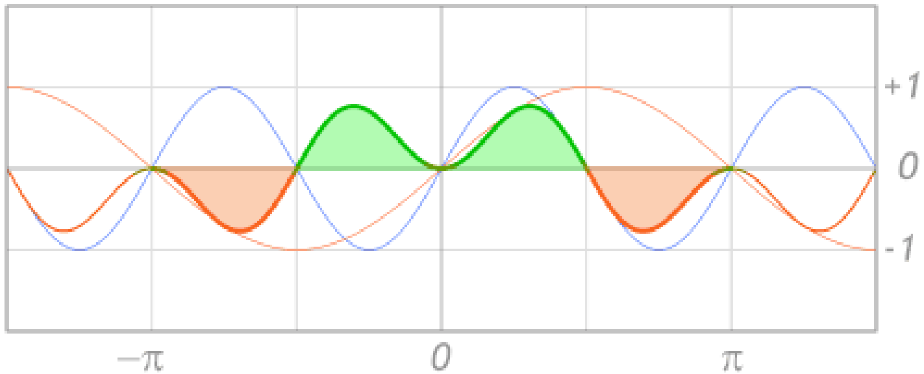
$$B_n(r_0) \approx \frac{2}{N}\sum_{k=0}^{N-1} B_r(r_0,\varphi_k)\sin n\varphi_k.$$

Interrupt: Expansions; orthogonality, completeness, convergence

The Road Map to Convergence of Fourier Series

- ➔ The trigonometric functions are orthogonal
- ➔ The Fourier polynomial Pn of grade n is the best approximation of f in V_n
- ➔ The projections onto the trigonometric functions (scalar product) induces a norm (the RMS error)
- ➔ Riemann Lebesque Lemma: Within this norm, the coefficients converge to zero.
- ➔ 3 Convergence theorems
  - For a C^1 function Pn converges uniformly to f(x)
  - For "clean jumps" Pn converges pointwise to 0.5 $(f_+(x) + f_-(x))$
  - The Pn converges for every square integrable function in the RMS sense

Trigonometric Functions as Orthogonal Function Set (from Wikipedia)



$$\int_{-\pi}^{+\pi} \sin(2x)\sin(1x)\; dx = 0$$

$$\int_{-\pi}^{\pi} \cos(mx)\, \cos(nx)\, dx = \pi\delta_{mn},$$

$$\int_{-\pi}^{\pi} \sin(mx)\, \sin(nx)\, dx = \pi\delta_{mn},$$

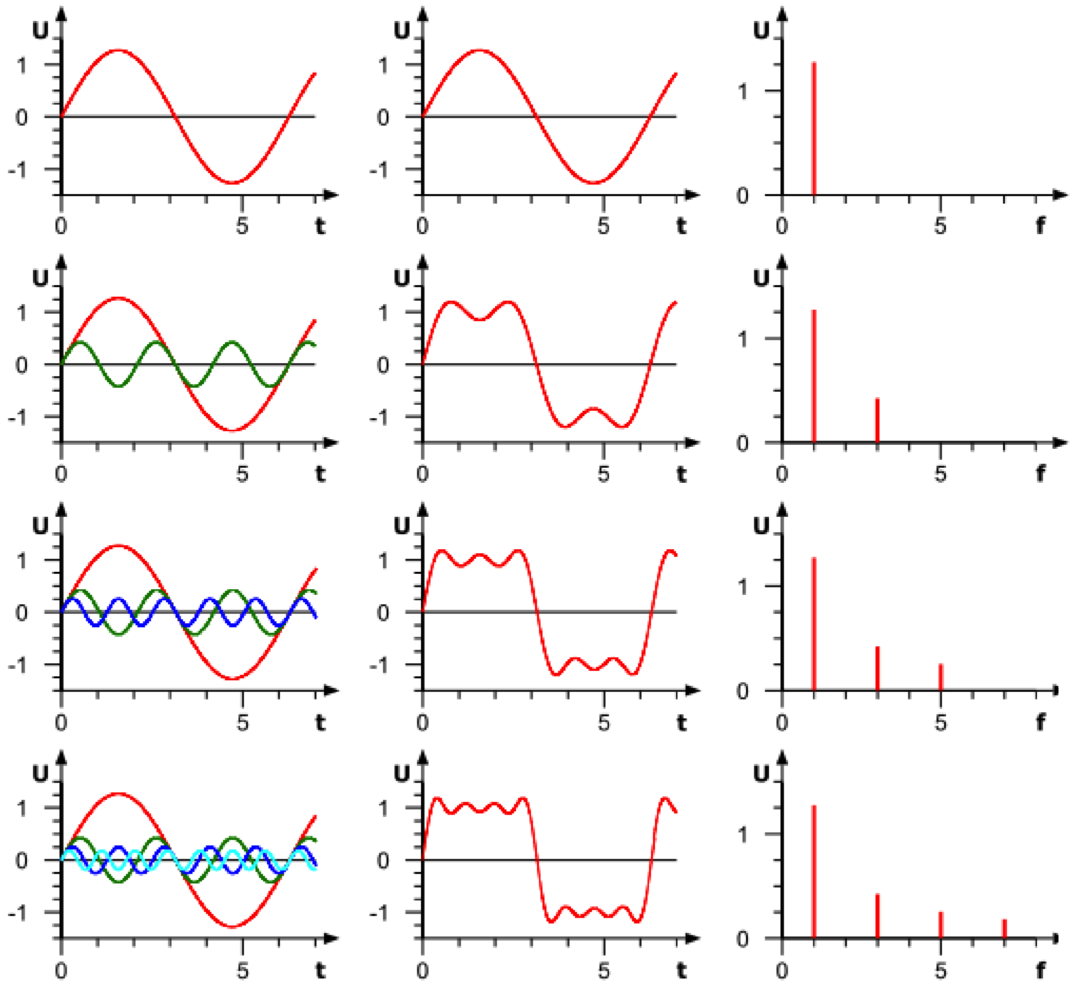
(where δ_{mn} is the Kronecker delta), and

$$\int_{-\pi}^{\pi} \cos(mx)\, \sin(nx)\, dx = 0\,;$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left(\sin(\alpha+\beta) - \sin(\alpha-\beta)\right)$$

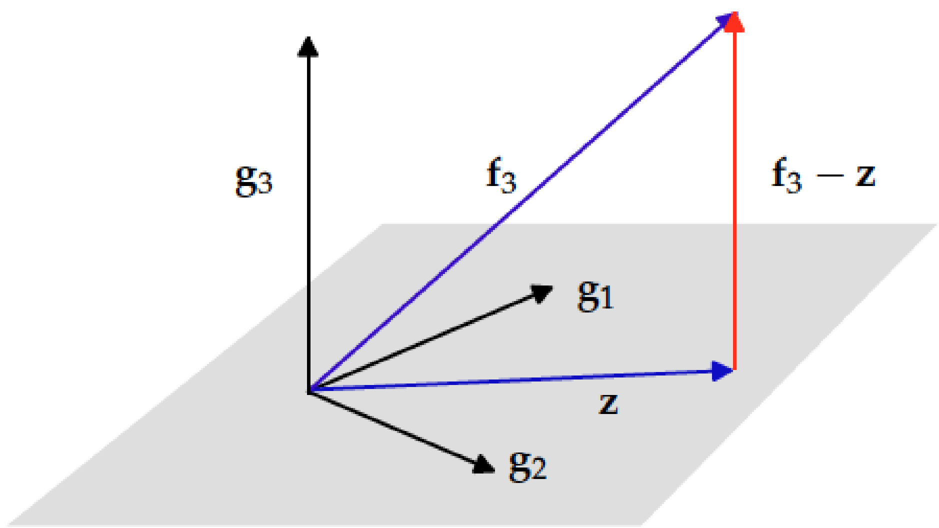
$$\begin{aligned}\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx &= \frac{1}{2}\int_{-L}^{L}\sin\frac{(m+n)\pi x}{L}dx - \frac{1}{2}\int_{-L}^{L}\sin\frac{(m-n)\pi x}{L}dx \\ &= \frac{1}{2}\left(\frac{-\cos\frac{(m+n)\pi x}{L}}{\frac{(m+n)\pi}{L}}\right)\Bigg|_{-L}^{L} - \frac{1}{2}\left(\frac{-\cos\frac{(m-n)\pi x}{L}}{\frac{(m-n)\pi}{L}}\right)\Bigg|_{-L}^{L} = 0\end{aligned}$$

The Fourier Polynomial Pn is the best Approximation in Vn

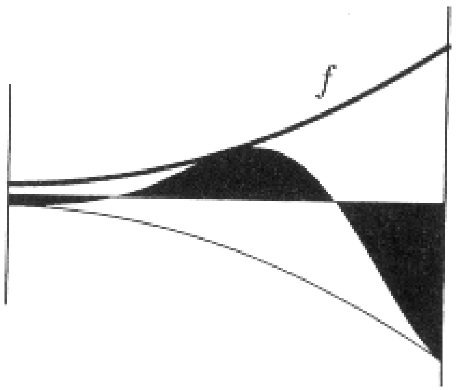


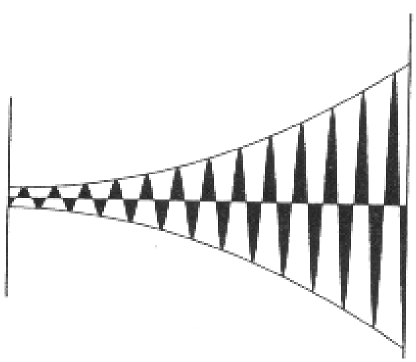
Projection of the square wave onto the “shape” of the trigonometric functions

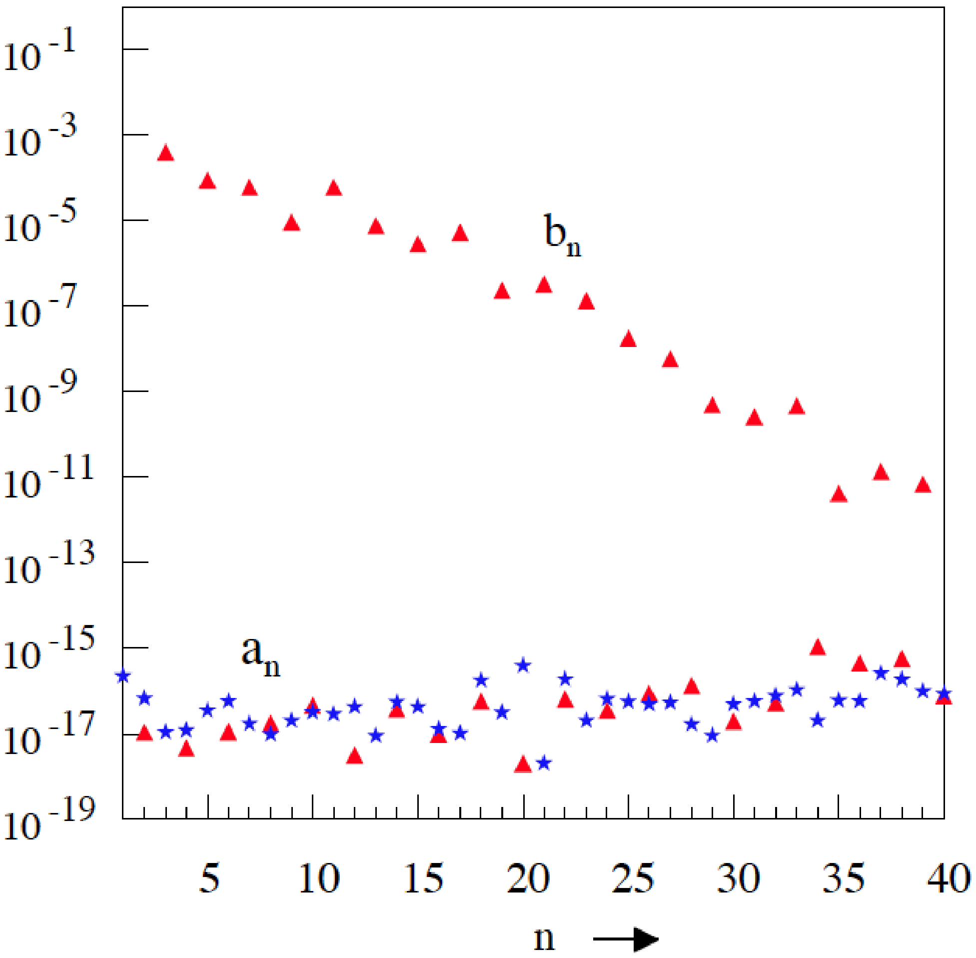
f_3-z is the shortest distance to the projective plane



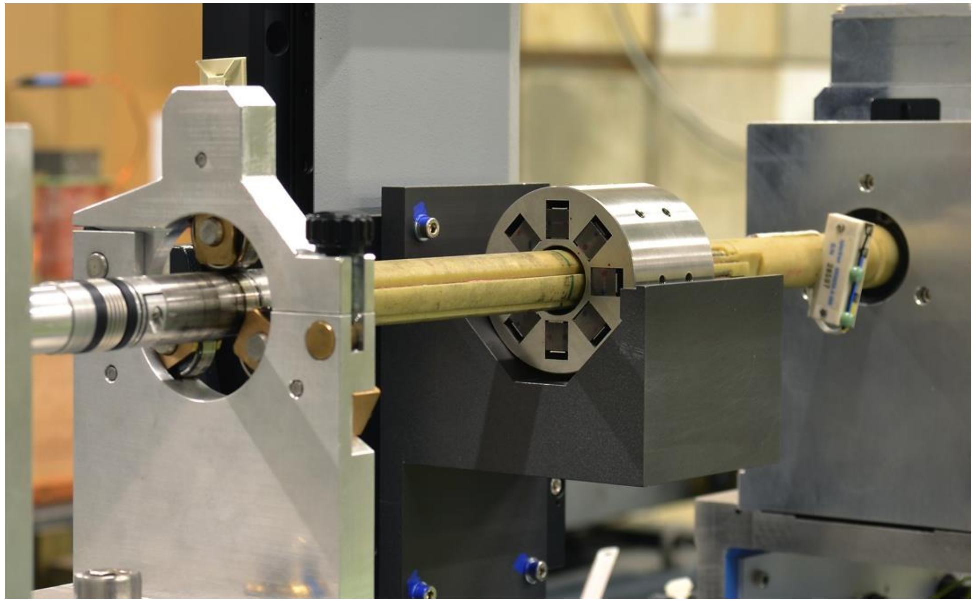
The Fourier coefficients tend to zero as n goes to infinity

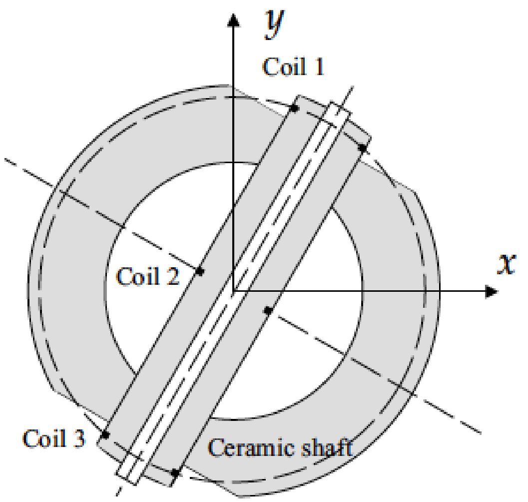


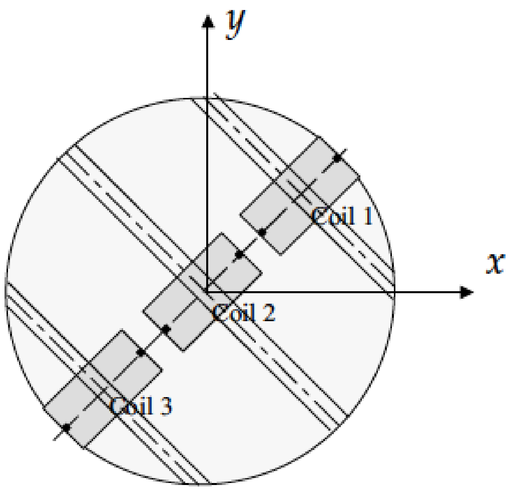


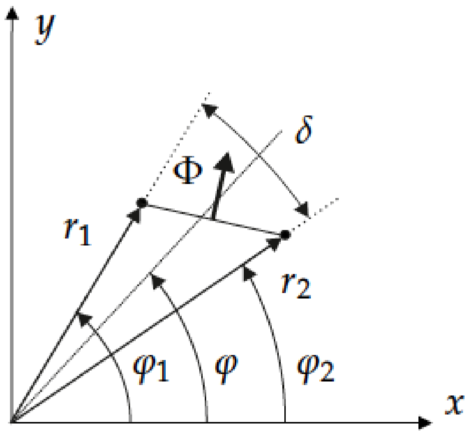


Series Measurements of the LHC Magnets









Tangential coil
Radial flux

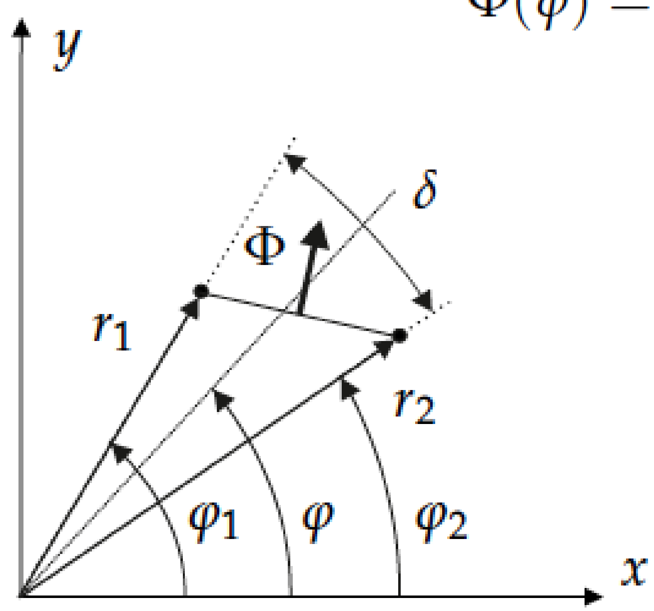
Radial coil
Tangential flux

Rotating Coil Measurements

$$\varphi = \omega t + \Theta,$$

$$\Phi(\varphi) = N \int_{\mathscr{A}} \mathbf{B} \cdot \mathrm{d}\mathbf{a} = N \int_{\mathscr{A}} \operatorname{curl} \mathbf{A} \cdot \mathrm{d}\mathbf{a} = N \int_{\partial\mathscr{A}} \mathbf{A} \cdot \mathrm{d}\mathbf{r} = N\ell \left[A_z(\mathscr{P}_1) - A_z(\mathscr{P}_2) \right],$$

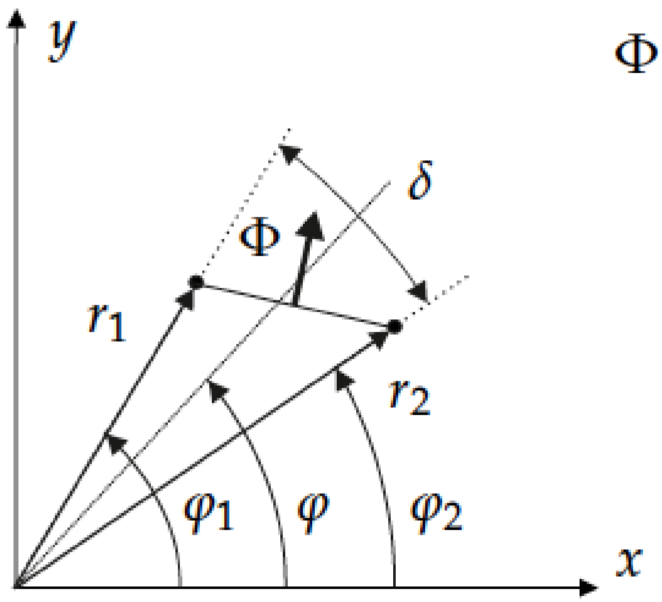
2 Dim version of Stoke's Theorem



$$\Phi(\varphi) = N\ell \left[\sum_{n=1}^{\infty} \frac{r_0}{n} \left(\frac{r_2}{r_0} \right)^n \left(B_n(r_0) \cos n\varphi_2 - A_n(r_0) \sin n\varphi_2 \right) - \sum_{n=1}^{\infty} \frac{r_0}{n} \left(\frac{r_1}{r_0} \right)^n \left(B_n(r_0) \cos n\varphi_1 - A_n(r_0) \sin n\varphi_1 \right) \right],$$

Rotating Coil Measurements

$$\Phi(\varphi) = N\ell \left[\sum_{n=1}^{\infty} \frac{r_0}{n} \left(\frac{r_2}{r_0}\right)^n (B_n(r_0)\cos n\varphi_2 - A_n(r_0)\sin n\varphi_2) - \sum_{n=1}^{\infty} \frac{r_0}{n} \left(\frac{r_1}{r_0}\right)^n (B_n(r_0)\cos n\varphi_1 - A_n(r_0)\sin n\varphi_1) \right],$$

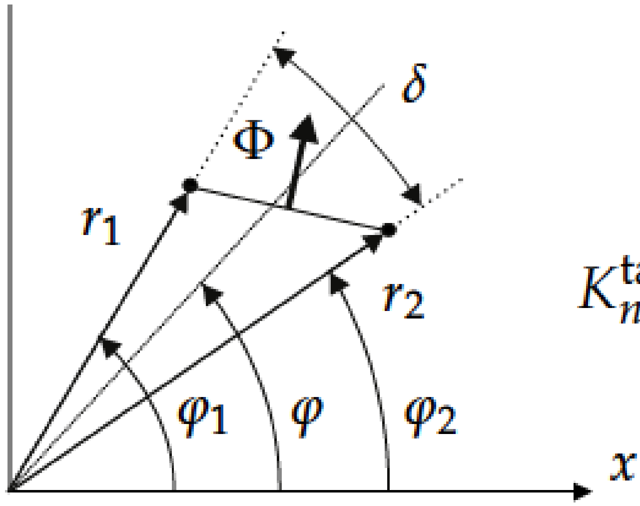


$$\Phi(\varphi) = \sum_{n=1}^{\infty} \frac{\ell}{r_0^{n-1}} \left[K_n^{\text{rad}} \left(B_n(r_0)\cos n\varphi - A_n(r_0)\sin n\varphi\right) + K_n^{\text{tan}} \left(B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi\right) \right],$$

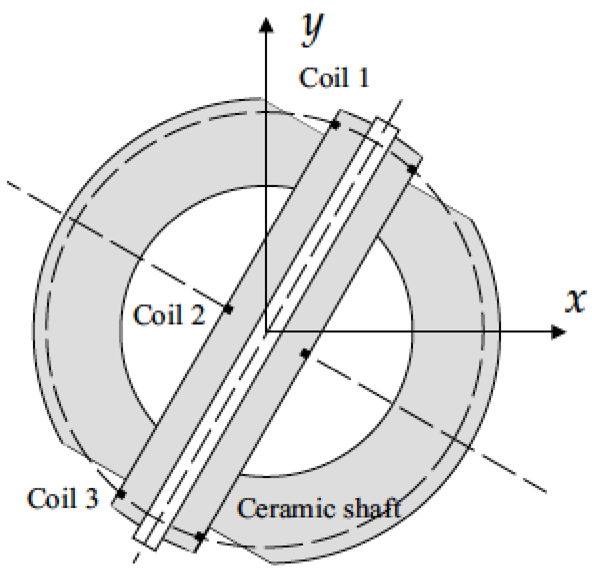
$$K_n^{\text{rad}} = \frac{N}{n}\left[r_2^n \cos n(\varphi_2 - \varphi) - r_1^n \cos n(\varphi_1 - \varphi) \right],$$

$$K_n^{\text{tan}} = -\frac{N}{n}\left[r_2^n \sin n(\varphi_2 - \varphi) - r_1^n \sin n(\varphi_1 - \varphi) \right],$$

Rotating Coil Measurements



$$K_n^{\tan} = \frac{2N}{n} r_c^n \sin\left(\frac{n\delta}{2}\right),$$



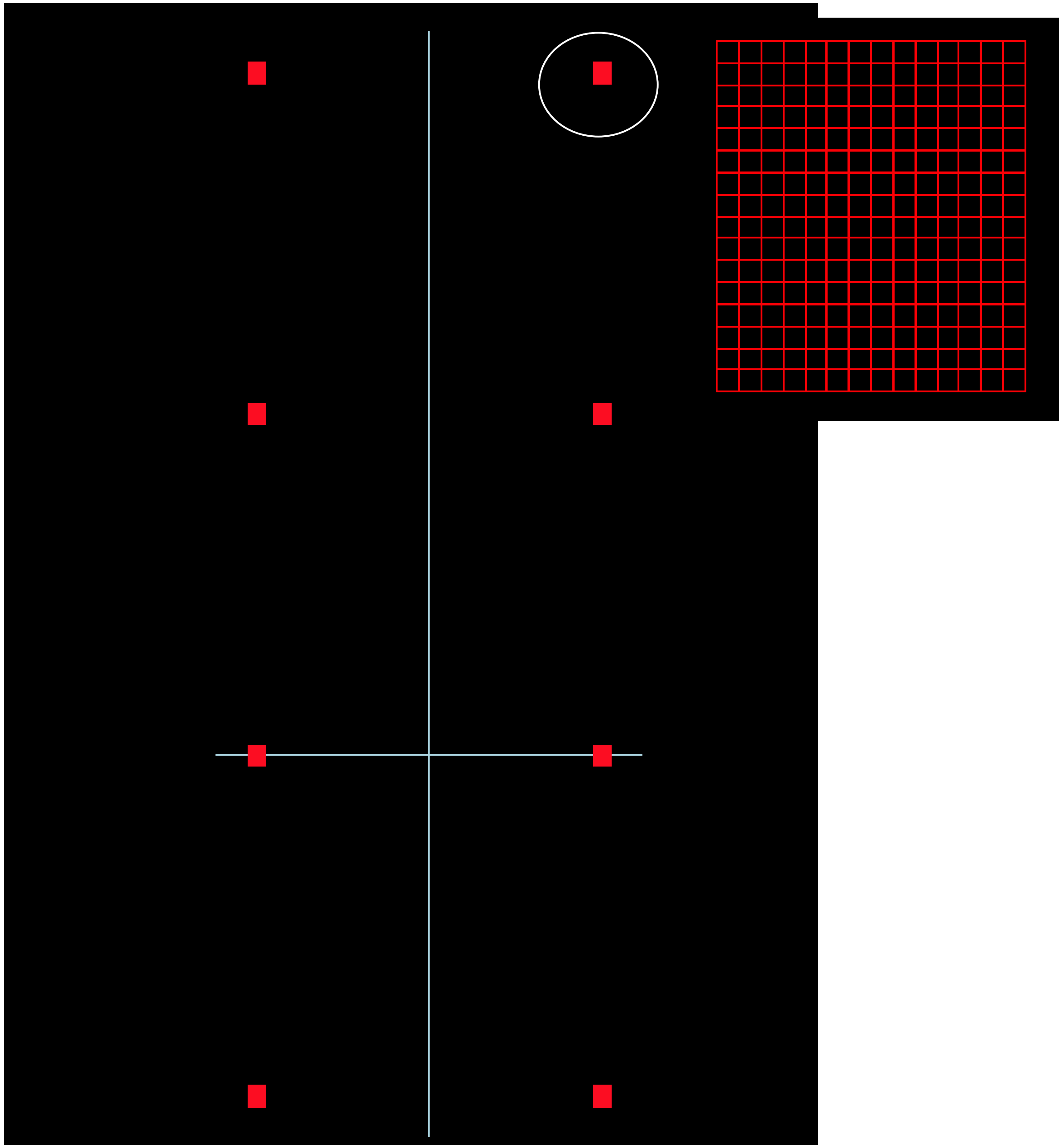
$$\Phi = N\ell \int_{\varphi-\delta/2}^{\varphi+\delta/2} B_r(r_c, \varphi)\, r_c \,\mathrm{d}\varphi$$

$$= \sum_{n=1}^{\infty} K_n^{\tan} \frac{\ell}{r_0^{n-1}} \Big[B_n(r_0)\sin(n\omega t + n\Theta) + A_n(r_0)\cos(n\omega t + n\Theta)\Big],$$

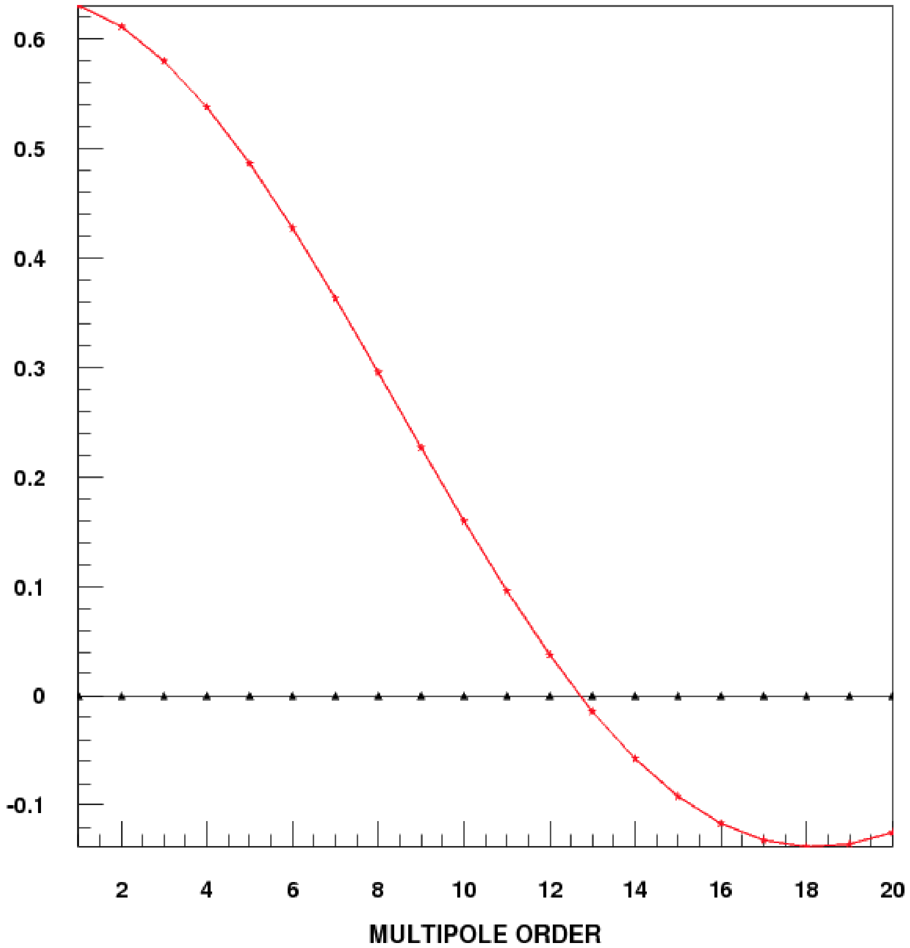
$$U = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

$$= \sum_{n=1}^{\infty} K_n^{\tan} \frac{n\omega\ell}{r_0^{n-1}} \Big[-B_n(r_0)\cos(n\omega t + n\Theta) + A_n(r_0)\sin(n\omega t + n\Theta)\Big].$$

Rotating Coil Measurements



$$S_n := \frac{\ell}{r_0^{n-1}} K_n \, .$$



Multipoles and Scaling Laws

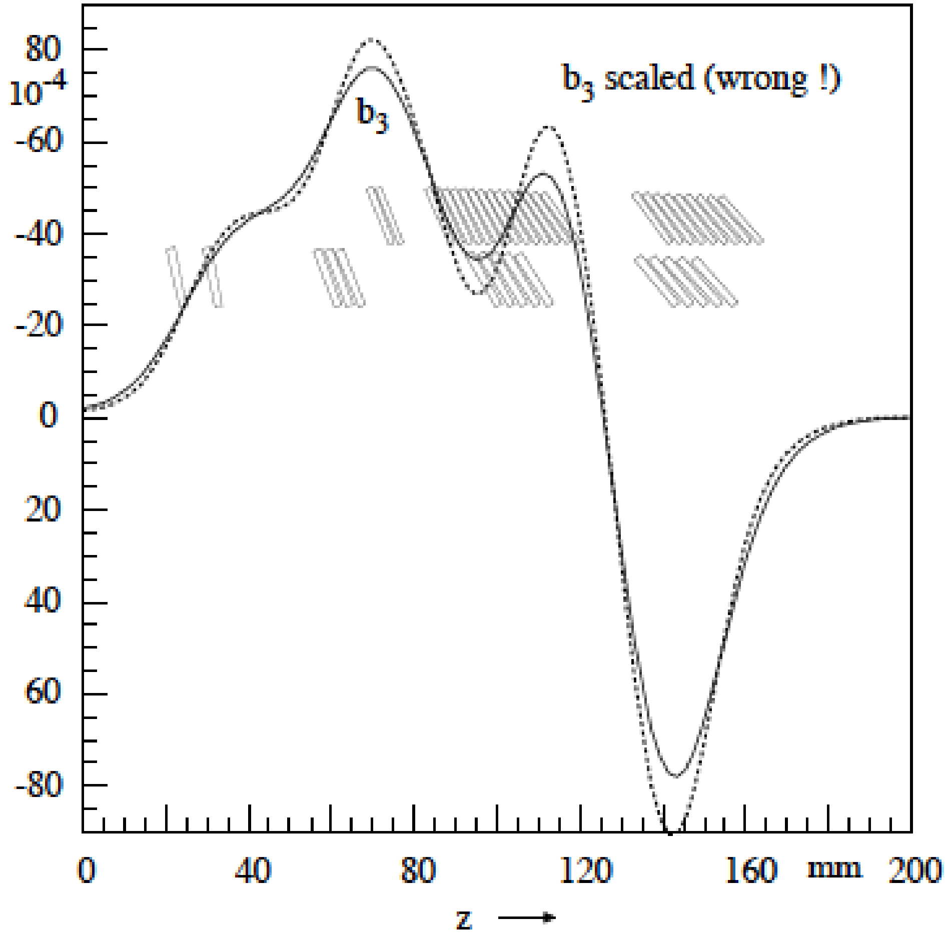
$$B_r(r,\varphi) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} (B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi)$$

$$B_r(r,\varphi) = B_N \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-N} (b_n(r_0)\sin n\varphi + a_n(r_0)\cos n\varphi)\,.$$

$$A_n(r_1) = \left(\frac{r_1}{r_0}\right)^{n-1} A_n(r_0), \qquad B_n(r_1) = \left(\frac{r_1}{r_0}\right)^{n-1} B_n(r_0),$$

$$b_n(r_1) = \frac{B_n(r_1)}{B_N(r_1)} = \frac{\left(\frac{r_1}{r_0}\right)^{n-1} B_n(r_0)}{\left(\frac{r_1}{r_0}\right)^{N-1} B_N(r_0)} = \left(\frac{r_1}{r_0}\right)^{n-N} b_n(r_0),$$

Integrated Harmonics



Local transverse harmonics calculated at different reference radii and scaled with the 2D laws

$$b_n(r_1) = \left(\frac{r_1}{r_0}\right)^{n-N} b_n(r_0),$$

wrong

Integrated Harmonics

$$\nabla^2 \phi_{\mathrm{m}}(x,y,z) = \frac{\partial^2 \phi_{\mathrm{m}}(x,y,z)}{\partial x^2} + \frac{\partial^2 \phi_{\mathrm{m}}(x,y,z)}{\partial y^2} + \frac{\partial^2 \phi_{\mathrm{m}}(x,y,z)}{\partial z^2} = 0\,.$$

$$\overline{\phi}_{\mathrm{m}}(x,y) := \int_{-z_0}^{z_0} \phi_{\mathrm{m}}(x,y,z)\mathrm{d}z.$$

$$\begin{aligned}
\frac{\partial^2 \overline{\phi}_{\mathrm{m}}(x,y)}{\partial x^2} + \frac{\partial^2 \overline{\phi}_{\mathrm{m}}(x,y)}{\partial y^2} &= \int_{-z_0}^{z_0} \left(\frac{\partial^2 \phi_{\mathrm{m}}}{\partial x^2} + \frac{\partial^2 \phi_{\mathrm{m}}}{\partial y^2} \right) \mathrm{d}z \\
&= \int_{-z_0}^{z_0} \left(-\frac{\partial^2 \phi_{\mathrm{m}}}{\partial z^2} \right) \mathrm{d}z = - \left. \frac{\partial \phi_{\mathrm{m}}}{\partial z} \right|_{-z_0}^{z_0} \\
&= H_z(-z_0) - H_z(z_0) \stackrel{!}{=} 0\,.
\end{aligned}$$

The 2D scaling laws hold for the integrated harmonics

Separation (in Cylindrical Coordinates)

$$\nabla^2 A_z = 0,$$

$$A_z = \rho(r)\phi(\varphi)$$

$$\frac{\partial A_z}{\partial r} = \frac{\mathrm{d}\rho(r)}{\mathrm{d}r}\phi(\varphi),$$
$$\frac{\partial^2 A_z}{\partial r^2} = \frac{\mathrm{d}^2\rho(r)}{\mathrm{d}r^2}\phi(\varphi),$$
$$\frac{\partial^2 A_z}{\partial \varphi^2} = \frac{\mathrm{d}^2\phi(\varphi)}{\mathrm{d}\varphi^2}\rho(r).$$

$$r^2\frac{\partial^2 A_z}{\partial r^2} + r\frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial \varphi^2} = 0,$$

$$\underbrace{\frac{1}{\rho(r)}\left(r^2\frac{\mathrm{d}^2\rho(r)}{\mathrm{d}r^2} + r\frac{\mathrm{d}\rho(r)}{\mathrm{d}r}\right)}_{n^2} = \underbrace{-\frac{1}{\phi(\varphi)}\frac{\mathrm{d}^2\phi(\varphi)}{\mathrm{d}\varphi^2}}_{n^2}.$$

$$r^2\frac{\mathrm{d}^2\rho(r)}{\mathrm{d}r^2} + r\frac{\mathrm{d}\rho(r)}{\mathrm{d}r} - n^2\rho(r) = 0,$$
$$\frac{\mathrm{d}^2\phi(\varphi)}{\mathrm{d}\varphi^2} + n^2\phi(\varphi) = 0,$$

$$\rho_n(r) = \mathcal{A}_n\, r^n + \mathcal{B}_n\, r^{-n},$$
$$\phi_n(\varphi) = \mathcal{C}_n \sin n\varphi + \mathcal{D}_n \cos n\varphi.$$

Cartesian Components

$$\nabla^2 \phi_{\mathrm{m}} = 0.$$

$$\frac{\partial^2 \phi_{\mathrm{m}}}{\mathrm{d}x^2} + \frac{\partial^2 \phi_{\mathrm{m}}}{\mathrm{d}y^2} + \frac{\partial^2 \phi_{\mathrm{m}}}{\mathrm{d}z^2} = 0$$

$$\phi_{\mathrm{m}} = X(x)Y(y)\,.$$

$$\underbrace{\frac{1}{X(x)}\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2}}_{p^2} + \underbrace{\frac{1}{Y(y)}\frac{\mathrm{d}^2 Y(y)}{\mathrm{d}y^2}}_{-p^2} = 0\,.$$

$$X_p(x) = \mathcal{C}_p \cos px + \mathcal{D}_p \sin px\,,$$

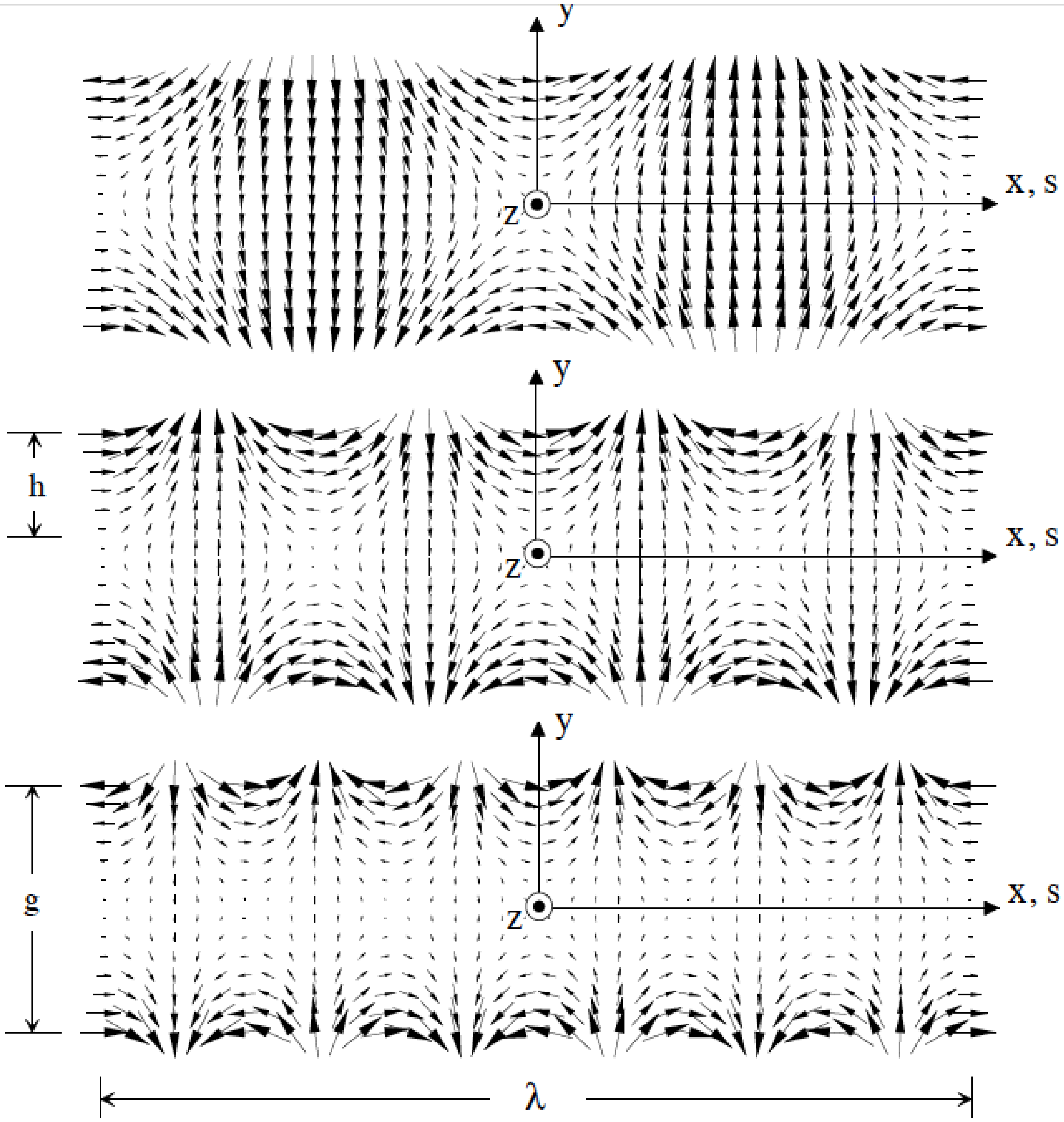
$$Y_p(y) = \mathcal{E}_p \cosh py + \mathcal{F}_p \sinh py\,,$$

$$p = n\,\frac{2\pi}{\lambda} =: nk_0\,,$$

$$B_x(x,y) = \mu_0 \sum_{n=1}^{\infty} \left(-\mathcal{A}_n \sin(nk_0x) + \mathcal{B}_n \cos(nk_0x)\right) \sinh(nk_0y)\,,$$

$$B_y(x,y) = \mu_0 \sum_{n=1}^{\infty} \left(\mathcal{A}_n \cos(nk_0x) + \mathcal{B}_n \sin(nk_0x)\right) \cosh(nk_0y)\,.$$

Cartesian Components (Eigenfunctions of the Wiggler Magnet)



Ideal Pole Shape of Conventional Magnets

Cauchy Schwarz inequality

$$| \langle \mathbf{a}, \mathbf{b} \rangle | \leq \| \mathbf{a} \| \| \mathbf{b} \|,$$

Thus for the directional derivative

$$|\partial_{\mathbf{v}} \phi| \leq |\operatorname{grad} \phi| \, |\mathbf{v}|$$

This implies that the directional derivative is maximal when v points in the direction of the gradient. But the gradient points in the direction of the steepest ascent of *Φ* and is thus normal to the surface of equipotential.

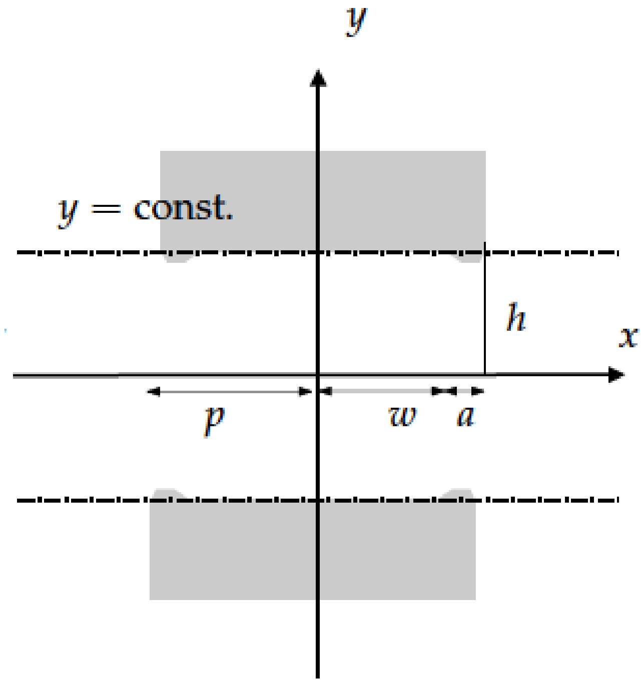
Remember also, that **B** exits a highly permeable surface in normal direction. Therefore the pole shape can be seen as an equipotential of the magnetic scalar potential.

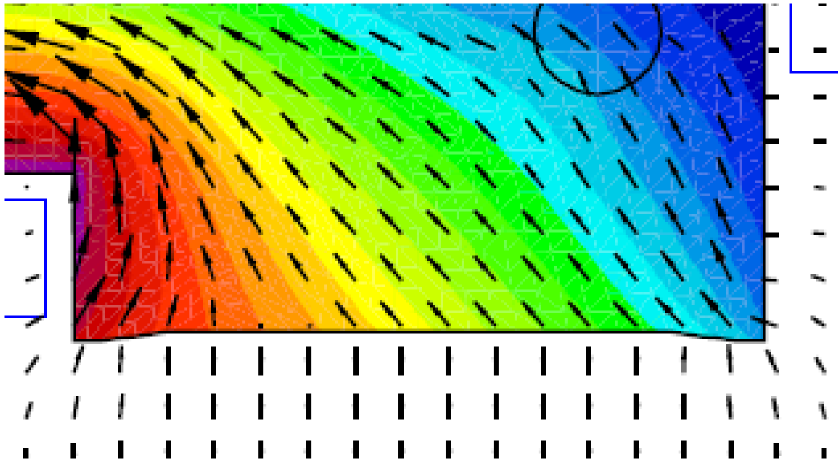
Ideal Pole Shape of Conventional Magnets

$$\phi_{\mathrm{m}}(r,\varphi) = -\sum_{n=1}^{\infty} \frac{r^n}{\mu_0}(\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi).$$

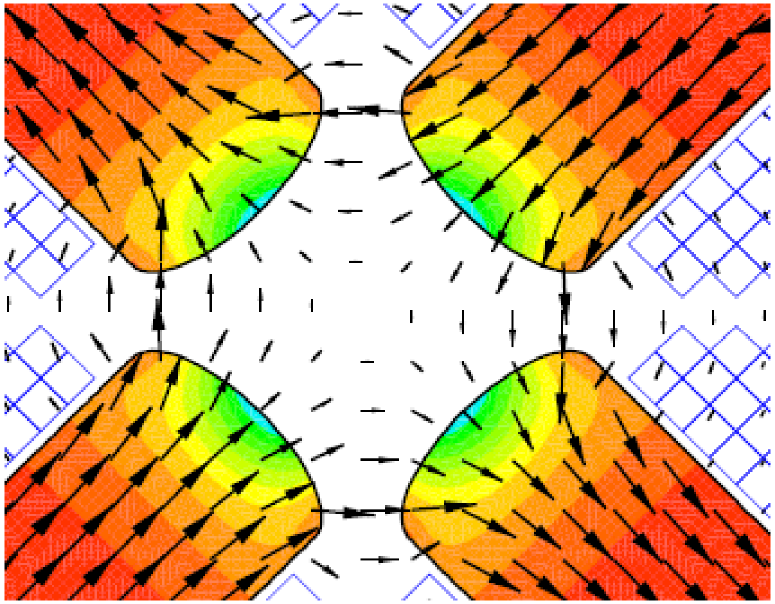
$$\phi_{\mathrm{m}}(r,\varphi) = -\sum_{n=1}^{\infty} \frac{r_0}{n\mu_0}\left(\frac{r}{r_0}\right)^n (A_n(r_0)\cos n\varphi + B_n(r_0)\sin n\varphi).$$

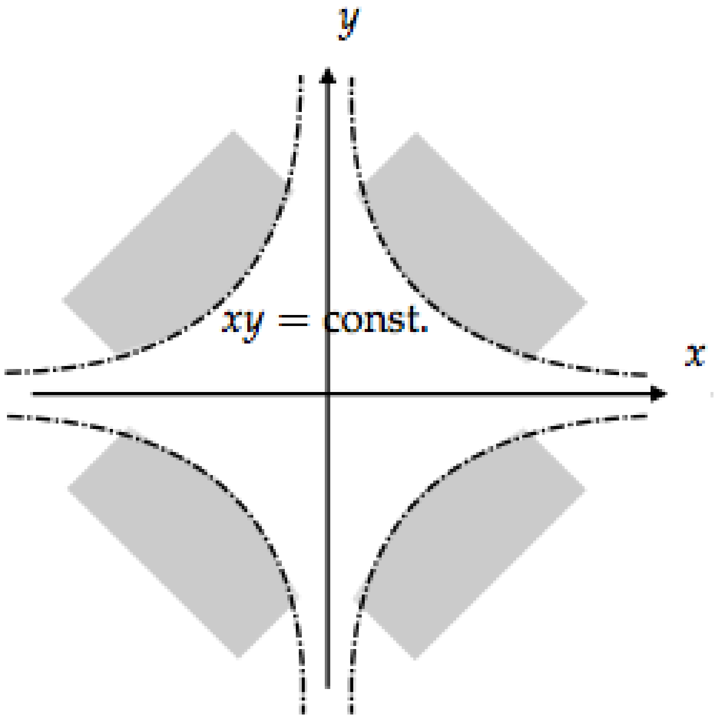
$$\phi_{\mathrm{m}}(x,y) = -\frac{1}{\mu_0}(B_1 y + A_1 x).$$





Ideal Pole Shape of Conventional Magnets





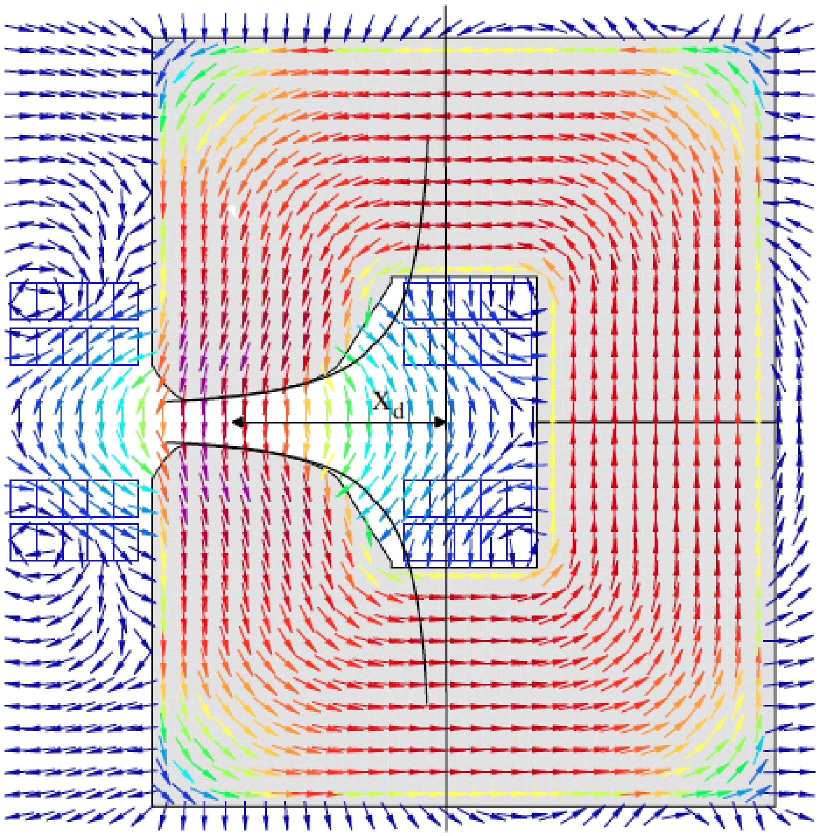
$$\phi_\mathrm{m}(r,\varphi) = \frac{1}{2\mu_0 r_0}\left(B_2(r_0)\,2xy + A_2(r_0)\,(x^2 - y^2)\right).$$

Combined Function Magnet (Feed Down)

The magnet has a field gradient of 5 T m^{-1} and a dipole field of 1.5 T at nominal excitation of 6000 A.

$x' = x - x_{\mathrm{d}}$:

$$B_y(x') = \frac{1}{r_0}\left(B_2(r_0)x' + B_2(r_0)x_{\mathrm{d}}\right).$$



Theorem 9.2 *Real and imaginary parts of a holomorphic function are harmonic functions.*

Proof. If $f(z) = f(x,y) = u(x,y) + iv(x,y)$ is holomorphic, the Cauchy–Riemann equations yield

$$\nabla^2 u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = 0, \qquad (9.55)$$

$$\nabla^2 v = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 0.$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \qquad f^{(1)}(z_0) = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}\frac{1}{i}$$

Complex Potentials

$$\mathbf{H} = -\operatorname{grad}\phi = -\frac{\partial \phi}{\partial x}\mathbf{e}_x - \frac{\partial \phi}{\partial y}\mathbf{e}_y,$$

$$\mathbf{B} = \operatorname{curl}(\mathbf{e}_z A_z) = \frac{\partial A_z}{\partial y}\mathbf{e}_x - \frac{\partial A_z}{\partial x}\mathbf{e}_y \,.$$

This implies

$$\frac{\partial A_z}{\partial y} = -\mu_0 \frac{\partial \phi}{\partial x} \quad \text{and} \quad \frac{\partial A_z}{\partial x} = \mu_0 \frac{\partial \phi}{\partial y},$$

Wich are the Cauchy Riemann equations of

$$w(z) := u(x,y) + iv(x,y) = A_z(x,y) + i\mu_0\phi(x,y) \,.$$

$$-\frac{\mathrm{d}w}{\mathrm{d}z} = -\frac{\partial A_z}{\partial x} - i\mu_0\frac{\partial \phi}{\partial x} = i\frac{\partial A_z}{\partial y} - \mu_0\frac{\partial \phi}{\partial y} = B_y(x,y) + iB_x(x,y) =: B(z).$$

Complex Representation of the Field in Accelerator Magnets

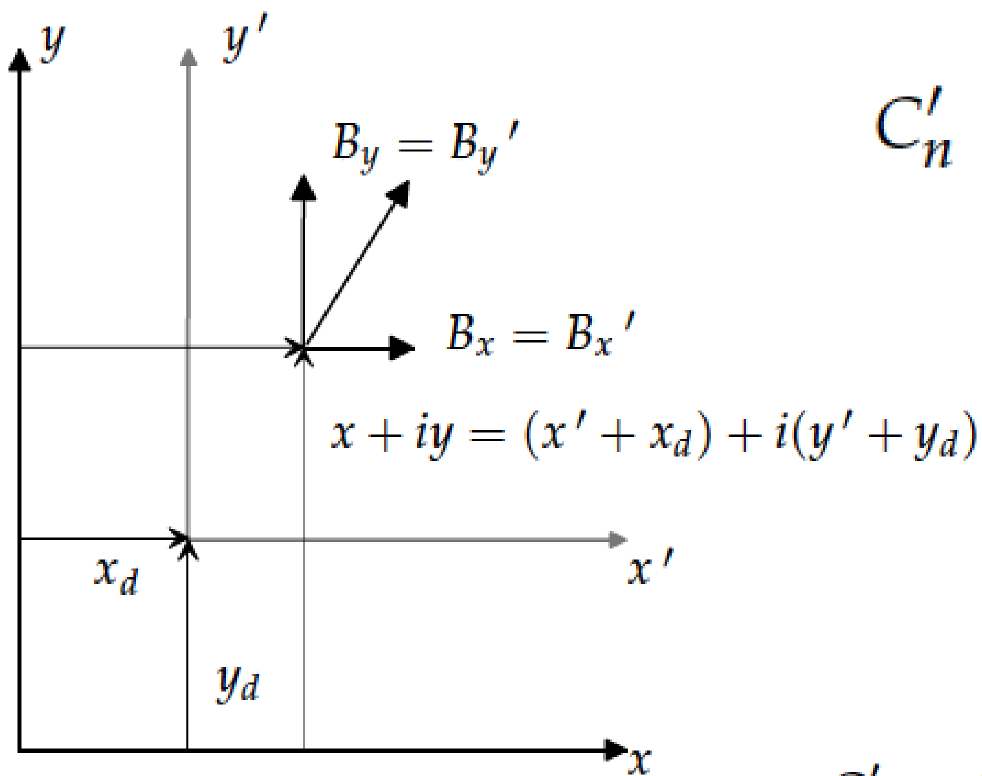
$$B_x = B_r \cos\varphi - B_\varphi \sin\varphi, \qquad B_y = B_r \sin\varphi + B_\varphi \cos\varphi,$$

$$B_y + iB_x = (B_\varphi + iB_r)e^{-i\varphi}.$$

$$\begin{aligned} B_y + iB_x &= \sum_{n=1}^{\infty} (B_n(r_0) + i\,A_n(r_0)) \left(\frac{r}{r_0}\right)^{n-1} e^{i(n-1)\varphi} \\ &= \sum_{n=1}^{\infty} (B_n(r_0) + i\,A_n(r_0)) \left(\frac{z}{r_0}\right)^{n-1} \\ &= B_N \sum_{n=1}^{\infty} (b_n(r_0) + i\,a_n(r_0)) \left(\frac{z}{r_0}\right)^{n-1}, \end{aligned}$$

Feed-down (Holomorphic Continuation)

$$\sum_{n=1}^{\infty} C_n \left(\frac{z}{r_0}\right)^{n-1} \stackrel{!}{=} \sum_{n=1}^{\infty} C'_n \left(\frac{z'}{r_0}\right)^{n-1},$$



$$C'_n = \sum_{k=n}^{\infty} C_k \binom{k-1}{n-1} \left(\frac{z_\mathrm{d}}{r_0}\right)^{k-n}.$$

$$\binom{n}{p} = \frac{n!}{p!(n-p)!} \text{ for } 0 \le p \le n$$

$$C'_2 = C_2 + 2\,C_3 \left(\frac{z_\mathrm{d}}{r_0}\right) + 3\,C_4 \left(\frac{z_\mathrm{d}}{r_0}\right)^2 + \cdots,$$

Feed-down: Proof

$$\sum_{n=1}^{\infty} C_n \left(\frac{z}{r_0}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{C_n}{r_0{}^{n-1}} \left(z' + z_{\mathrm{d}}\right)^{n-1}$$

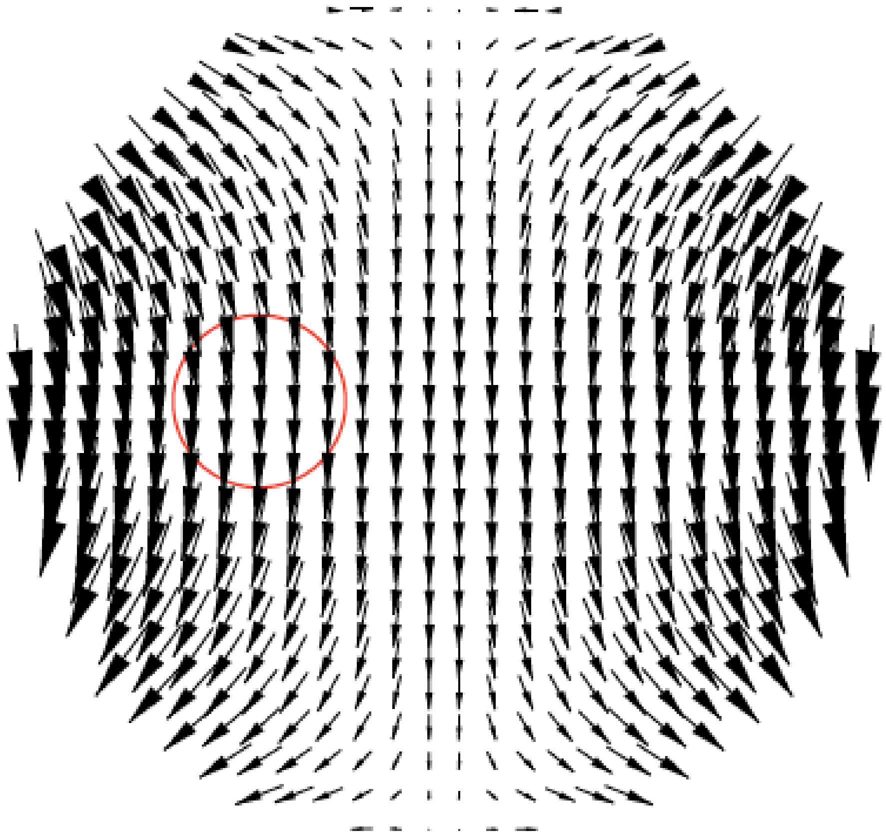
$$= \sum_{n=1}^{\infty} \frac{C_n}{r_0{}^{n-1}} \sum_{k=1}^{n} \binom{n-1}{k-1} (z')^{k-1} z_{\mathrm{d}}{}^{n-k}$$

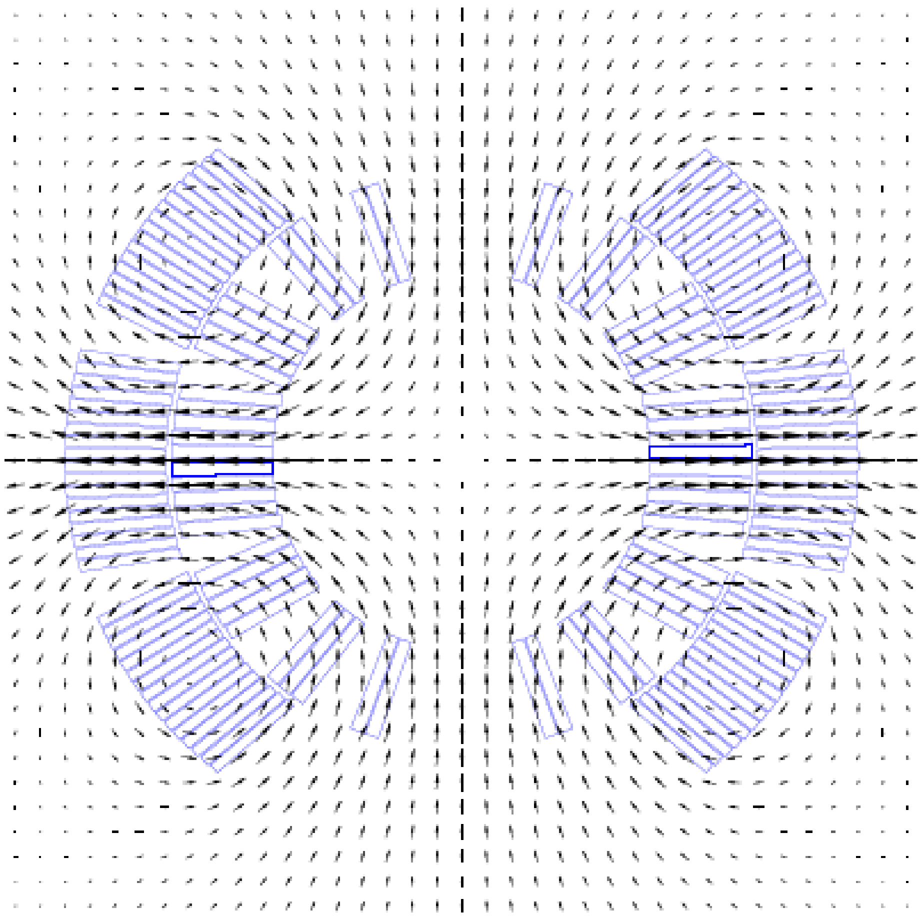
$$= \sum_{k=1}^{\infty} \left[\sum_{n=1}^{n} \frac{C_n}{r_0{}^{n-1}} \binom{n-1}{k-1} z_{\mathrm{d}}{}^{n-k} \right] (z')^{k-1}$$

$$= \sum_{k=1}^{\infty} \left[\sum_{n=k}^{\infty} C_n \binom{n-1}{k-1} \left(\frac{z_{\mathrm{d}}}{r_0}\right)^{n-k} \right] \left(\frac{z'}{r_0}\right)^{k-1}$$

$$= \sum_{n=1}^{\infty} \left[\sum_{k=n}^{\infty} C_k \binom{k-1}{n-1} \left(\frac{z_{\mathrm{d}}}{r_0}\right)^{k-n} \right] \left(\frac{z'}{r_0}\right)^{n-1} .$$

Feed-down: Enemy and Friend

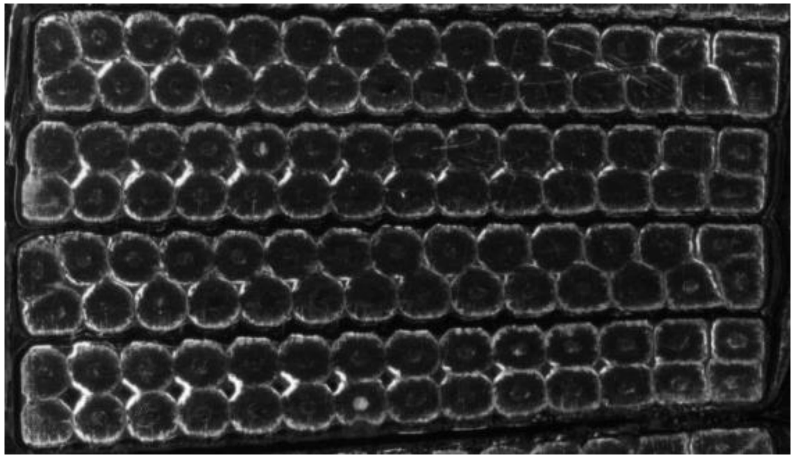


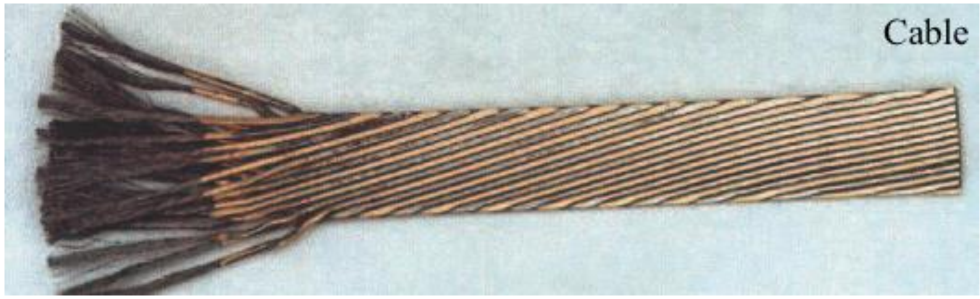


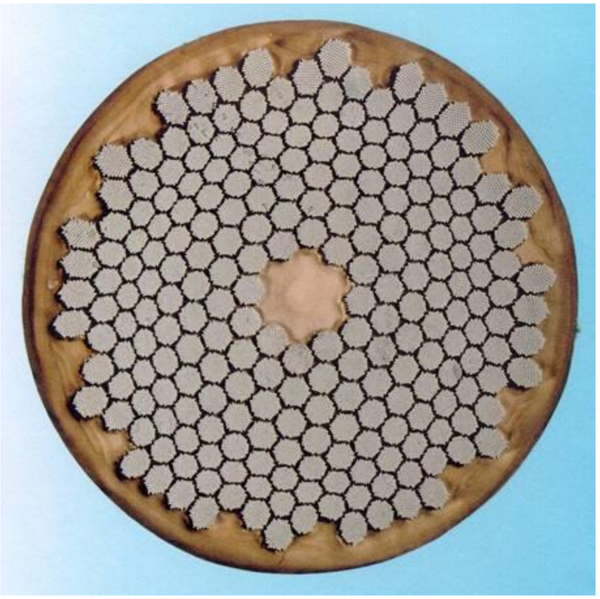
Where we are

- ➔ We have studied the mathematical foundations of magnetic fields
  - Vectorfields
  - One dimensional techniques for normal conducting magnets
  - The Laplace equation
  - Field harmonics in accelerator magnets
- ➔ So far we assumed that the fields on the domain boundary where known (from measurements or calculations)
- ➔ Now we need to address how to calculate these fields
  - The field of line currents and the Biot Savart law
  - Development of these field solutions into the Eigenfunctions

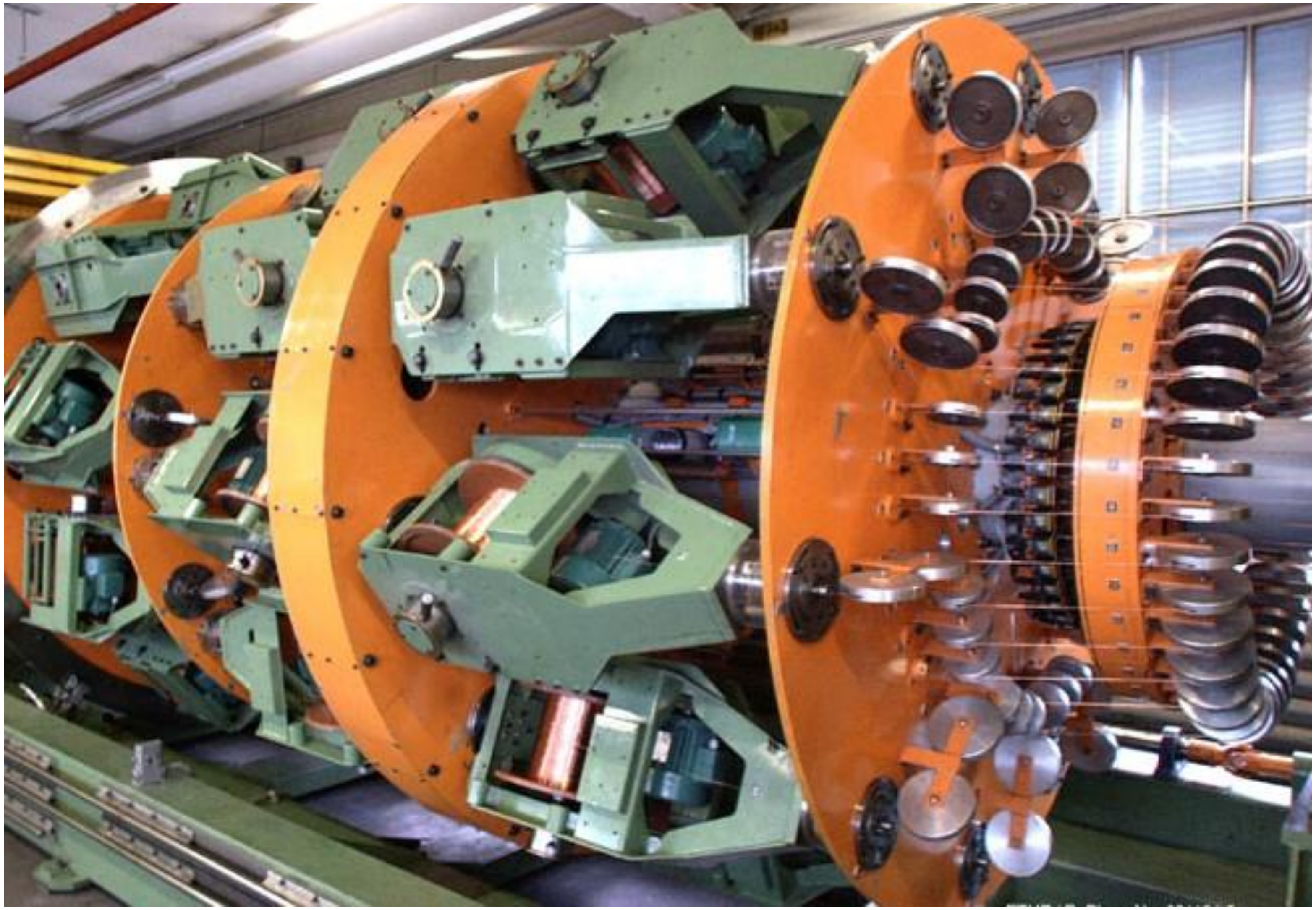
Rutherford (Roebel) Kabel, Strand, Nb-Ti Filament



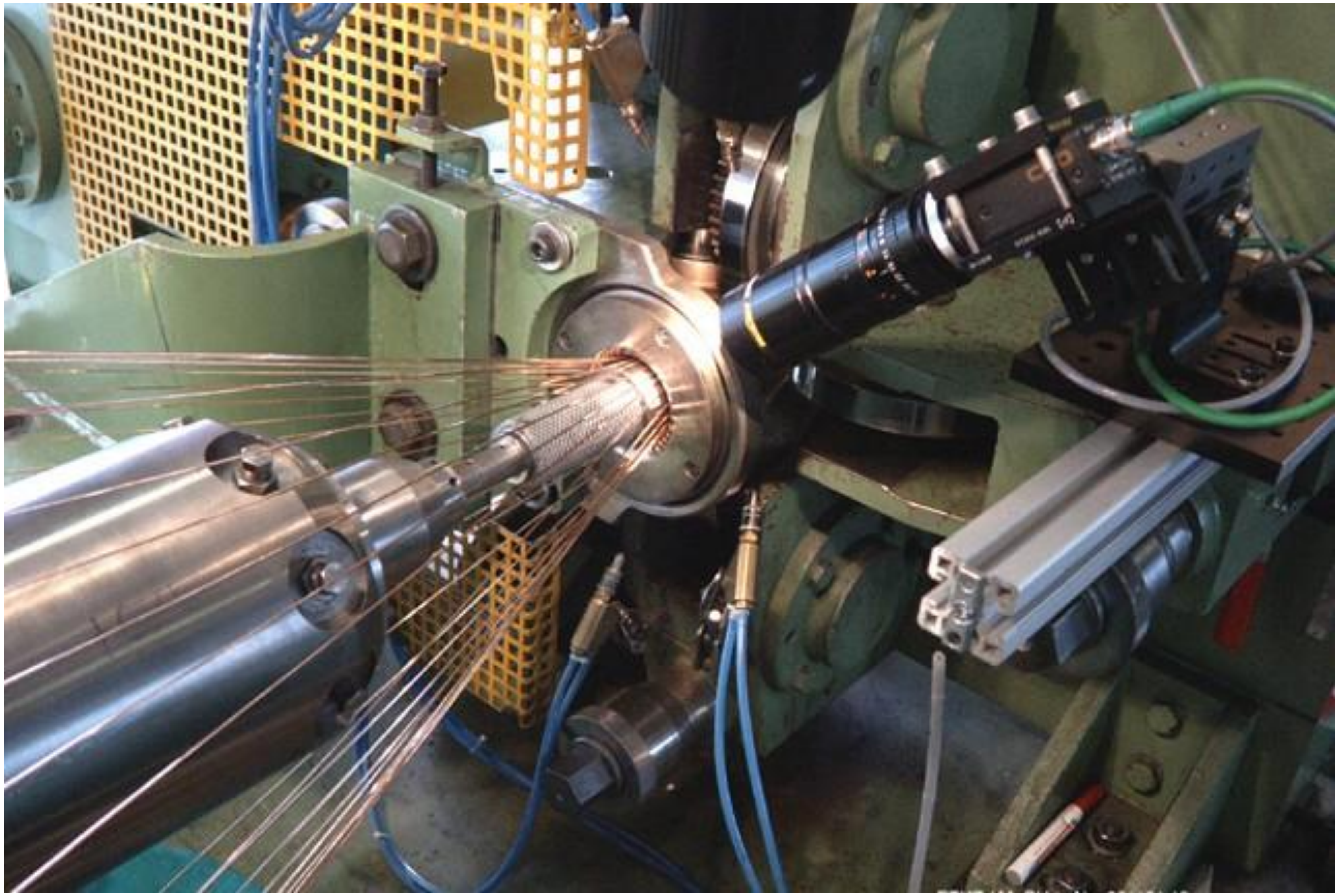




Cabling Machine for Rutherford Cables



Turk's Head

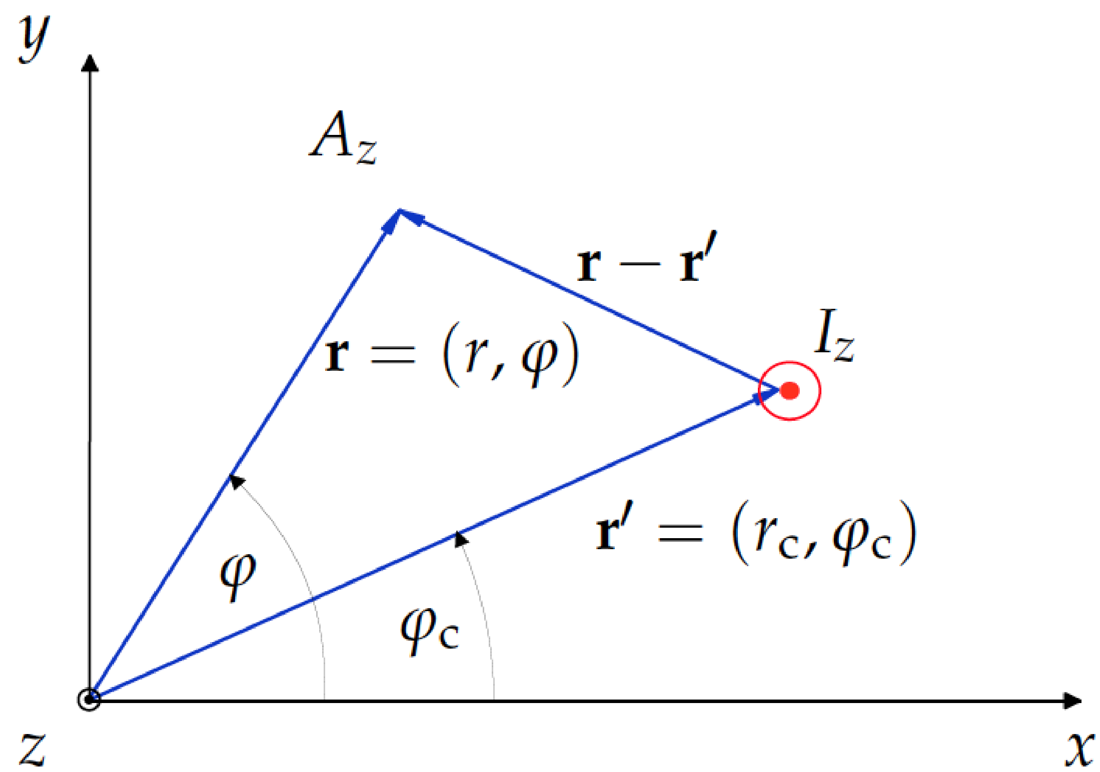


The Field of Line Currents

$$\mathbf{r} \mapsto \phi(|\mathbf{r}-\mathbf{r}'|)$$
$$\mathbf{r}' \mapsto \phi(|\mathbf{r}-\mathbf{r}'|)$$

$$\operatorname{grad} \phi(|\mathbf{r}-\mathbf{r}'|) = -\operatorname{grad}_{\mathbf{r}'} \phi(|\mathbf{r}-\mathbf{r}'|),$$
$$\operatorname{div} \mathbf{a}(|\mathbf{r}-\mathbf{r}'|) = -\operatorname{div}_{\mathbf{r}'} \mathbf{a}(|\mathbf{r}-\mathbf{r}'|),$$
$$\operatorname{curl} \mathbf{a}(|\mathbf{r}-\mathbf{r}'|) = -\operatorname{curl}_{\mathbf{r}'} \mathbf{a}(|\mathbf{r}-\mathbf{r}'|),$$
$$\nabla^2 \phi(|\mathbf{r}-\mathbf{r}'|) = \nabla^2_{\mathbf{r}'} \phi(|\mathbf{r}-\mathbf{r}'|).$$

Why bother?
Reciprocity; except for sign it does not matter if we exchange the source and field points



Interrupt: Diracs δ function

Greens Functions of Free Space

$$\mathcal{L}_{\mathbf{r}'}\phi(\mathbf{r}') = -f(\mathbf{r}')$$

$$\mathcal{L}_{\mathbf{r}'}G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'),$$

$$\int_{\mathcal{V}} \mathcal{L}_{\mathbf{r}'}G(\mathbf{r},\mathbf{r}')\,f(\mathbf{r})\mathrm{d}V = -\int_{\mathcal{V}} \delta(\mathbf{r}-\mathbf{r}')f(\mathbf{r})\mathrm{d}V = -f(\mathbf{r}').$$

$$\mathcal{L}_{\mathbf{r}'}\phi(\mathbf{r}') = \int_{\mathcal{V}} \mathcal{L}_{\mathbf{r}'}G(\mathbf{r},\mathbf{r}')f(\mathbf{r})\mathrm{d}V = \mathcal{L}_{\mathbf{r}'}\int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}')f(\mathbf{r})\mathrm{d}V,$$

$$\phi(\mathbf{r}') = \int_{\mathcal{V}} G(\mathbf{r},\mathbf{r}')f(\mathbf{r})\mathrm{d}V.$$

$$G_2(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi}\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{r_{\mathrm{ref}}}\right), \qquad G_3(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Green's Functions of Free Space

$$\phi(\mathbf{r}') = \int_{\mathcal{V}} G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}) \mathrm{d}V \, .$$

$$\boxed{\phi(\mathbf{r}) = \int_{\mathcal{V}} G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}') \mathrm{d}V' \, .}$$

But what if boundaries are present?

$$\phi(\mathbf{r}) = \int_{\mathcal{V}} G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}') \mathrm{d}V' + \int_{\partial\mathcal{V}} \Big(-\phi(\mathbf{r}') \, \partial_{\mathbf{n}'} G(\mathbf{r}, \mathbf{r}') + G(\mathbf{r}, \mathbf{r}') \, \partial_{\mathbf{n}'} \phi(\mathbf{r}') \Big) \mathrm{d}a' \, .$$

Biot-Savart's Law

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}\,, \qquad G_3(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V'\,,$$

$$\mathbf{A}(\mathbf{r}) = A_x\mathbf{e_x} + A_y\mathbf{e_y} + A_z\mathbf{e_z} = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V'.$$

This works only in Cartesian Coordinates

$$\mathbf{B}(\mathbf{r}) = \operatorname{curl}\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \operatorname{curl}\left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V'$$

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{1}{|\mathbf{r}-\mathbf{r}'|}\sum_{k=1}^{3} J_k(\mathbf{r}')(\mathbf{e}_i(\mathbf{r})\cdot\mathbf{e}_k(\mathbf{r}'))\,\mathrm{d}V'. \quad \mathrm{d}V'$$

$$= \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\mathrm{d}V'.$$

Biot Savart's Law

But wait a minute: Are we finished? Are we sure that the divergence of the vector potential is zero as it was required for the Laplace equation?

Interrupt: Why all this fuzz about gauging?

$$\operatorname{div} \mathbf{A}(\mathbf{r})$$

$$= -\frac{\mu_0}{4\pi} \int_{\partial \mathscr{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \cdot \mathrm{d}\mathbf{a}' .$$

Current loops must always be closed and must not leave the problem domain

Biot-Savart's Law for Line Currents

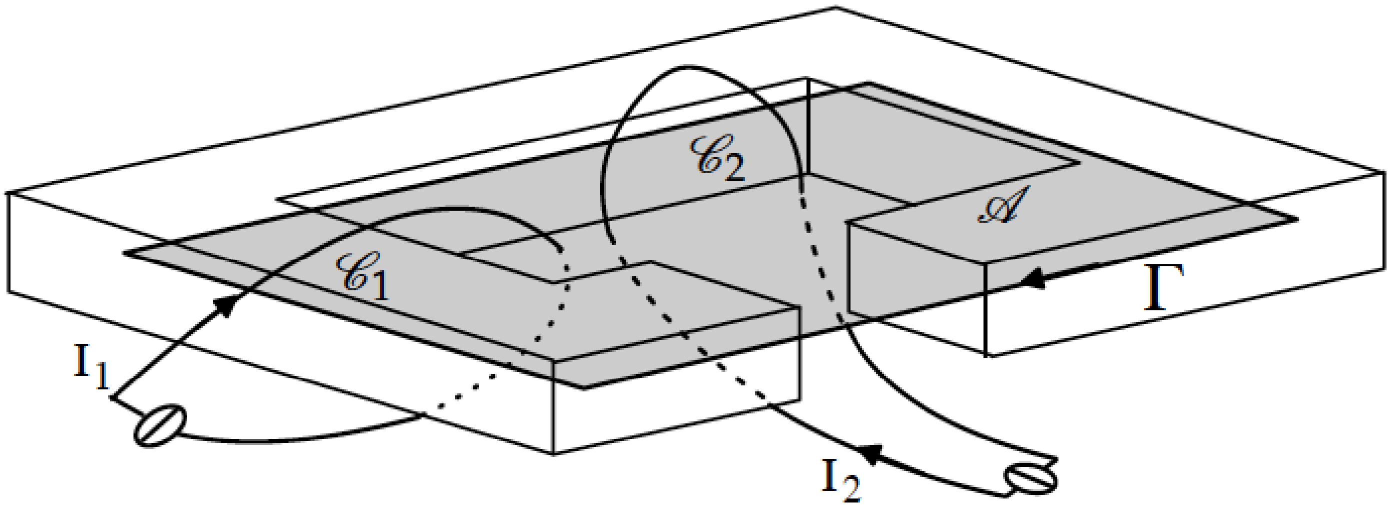
$$\mathbf{A}(\mathbf{r}) = A_x \mathbf{e_x} + A_y \mathbf{e_y} + A_z \mathbf{e_z} = \frac{\mu_0}{4\pi} \int_{\mathscr{V}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V'.$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_{\mathscr{C}} \frac{\mathrm{d}\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_{\mathscr{C}} \frac{\mathrm{d}\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3},$$

Biot-Savart's Law and the Linking Number

$$N\,I = I\sum_{k=1}^{K}\operatorname{link}(\partial\mathscr{A},\mathscr{C}_k) = I\sum_{k=1}^{K}\operatorname{int}(\mathscr{A},\mathscr{C}_k),$$



$$\operatorname{int}(\mathscr{A},\mathscr{C}_k) = \sum_{\mathscr{A}\cap\mathscr{C}_k} \pm 1$$

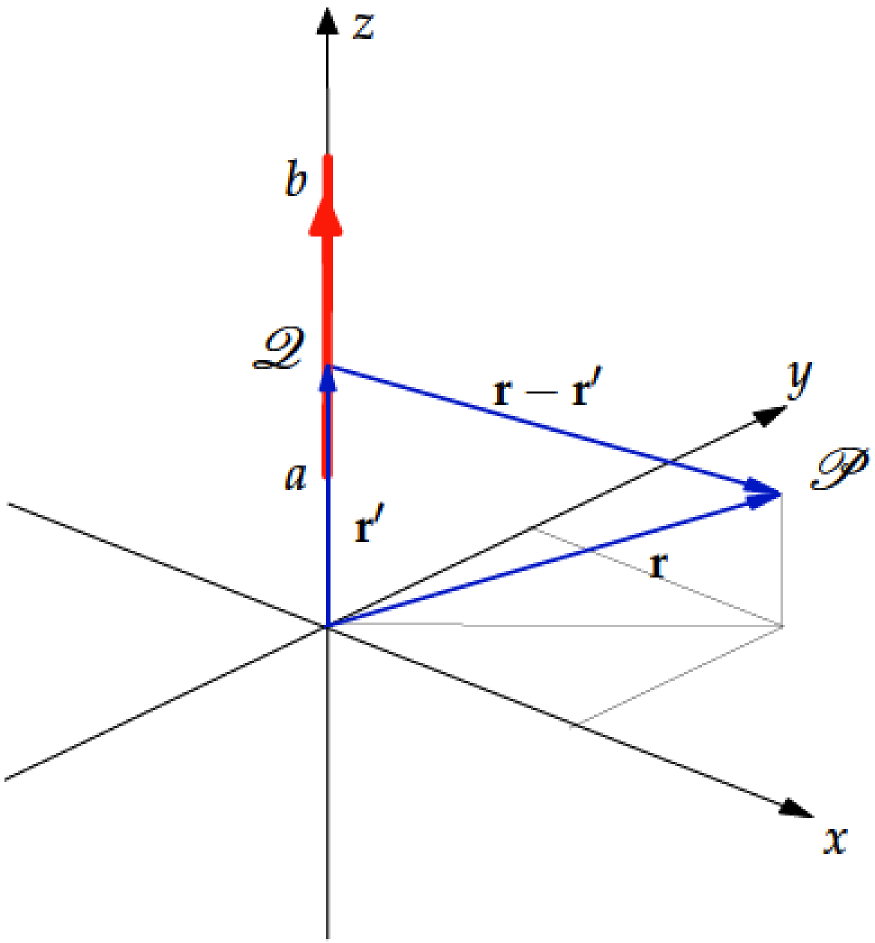
$$\operatorname{link}(\partial\mathscr{A},\mathscr{C}_k) = \frac{1}{4\pi}\int_{\partial\mathscr{A}}\int_{\mathscr{C}_k}\frac{\mathrm{d}\mathbf{r}'\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\cdot\mathrm{d}\mathbf{r},$$

Vector Potential of a Line Current

$$A_z(x,y,z) = \frac{\mu_0 I}{4\pi}\int_a^b \frac{\mathrm{d}z_c}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0 I}{4\pi}\int_a^b \frac{\mathrm{d}z_c}{\sqrt{x^2+y^2+(z-z_c)^2}}$$

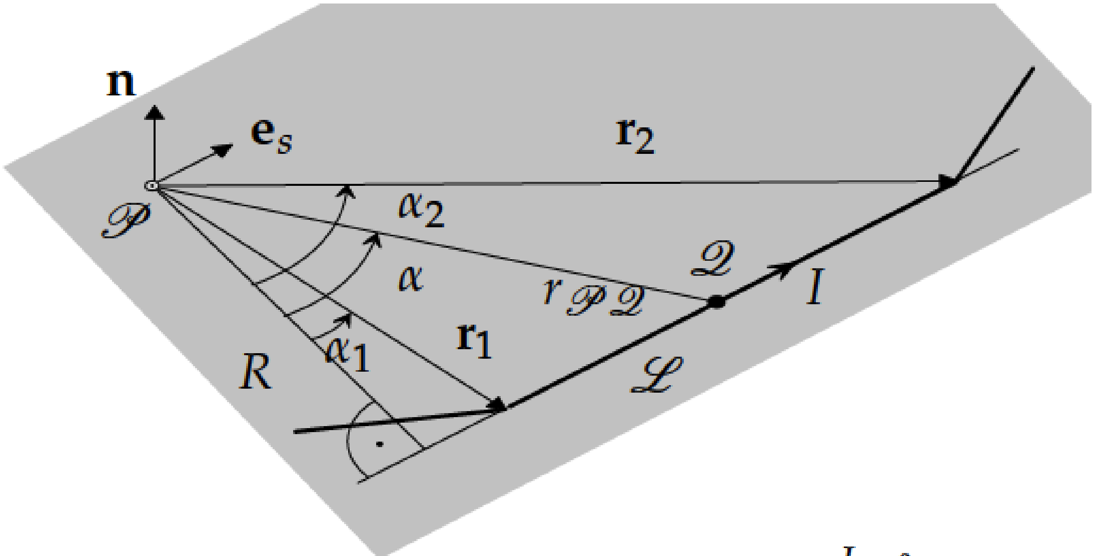
$$\frac{-\mu_0 I}{4\pi}\ln\left((z-z_c)+\sqrt{x^2+y^2+(z-z_c)^2}\right)\Big|_a^b$$

$$\frac{\mu_0 I}{4\pi}\ln\frac{z-a+\sqrt{x^2+y^2+(z-a)^2}}{z-b+\sqrt{x^2+y^2+(z-b)^2}}\,.$$



Caution: Infinitely long line currents have infinite energy

Field of a Line Current Segment



$$\mathbf{B}(\mathscr{P}) = \frac{\mu_0 I}{4\pi}\int_{\mathscr{L}} \frac{\cos\alpha}{r^2_{\mathscr{P}\mathscr{Q}}}\mathrm{d}r' = \frac{\mu_0 I}{4\pi R}\mathbf{n}\int_{\alpha_1}^{\alpha_2}\cos\alpha\,\mathrm{d}\alpha = \frac{\mu_0 I}{4\pi R}(\sin\alpha_2 - \sin\alpha_1)\,\mathbf{n}$$

$$= \frac{\mu_0 I}{4\pi}\frac{\cos\alpha_2+\cos\alpha_1}{R}\frac{\sin\alpha_2-\sin\alpha_1}{\cos\alpha_2+\cos\alpha_1}\mathbf{n}$$

$$= \frac{\mu_0 I}{4\pi}\left(\frac{1}{|\mathbf{r}_1|}+\frac{1}{|\mathbf{r}_2|}\right)\frac{\sin(\alpha_2-\alpha_1)}{1+\cos(\alpha_2-\alpha_1)}\mathbf{n}$$

$$= \frac{\mu_0 I}{4\pi}\left(\frac{1}{|\mathbf{r}_1|}+\frac{1}{|\mathbf{r}_2|}\right)\frac{\sin(\alpha_2-\alpha_1)}{1+\frac{\mathbf{r}_1\cdot\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|}}\frac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|\sin(\alpha_2-\alpha_1)}$$

$$= \frac{\mu_0 I}{4\pi}\frac{|\mathbf{r}_1|+|\mathbf{r}_2|}{|\mathbf{r}_1|\,|\mathbf{r}_2|+\mathbf{r}_1\cdot\mathbf{r}_2}\frac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1|\,|\mathbf{r}_2|},$$

Field of a Line Current

$$\lim_{a,b\to\pm\infty} \ln \frac{z-a+\sqrt{x^2+y^2+(z-a)^2}}{z-b+\sqrt{x^2+y^2+(z-b)^2}} = \lim_{a,b\to\pm\infty} \ln \frac{-a+|a|\sqrt{1+\frac{x^2+y^2}{a^2}}}{-b+|b|\sqrt{1+\frac{x^2+y^2}{b^2}}}$$

$$= \lim_{a,b\to\pm\infty} \ln \frac{-a-a(1+\frac{x^2+y^2}{2a^2}+\cdots)}{-b+b(1+\frac{x^2+y^2}{2b^2}+\cdots)} = \lim_{a,b\to\pm\infty} \ln \frac{-2a}{-b+b+\frac{x^2+y^2}{2b}}$$

$$= \lim_{a,b\to\pm\infty} \ln \frac{-4ab}{x^2+y^2}\,.$$

$$A_z(x,y) = \lim_{a,b\to\pm\infty} \frac{\mu_0 I}{4\pi} \ln\left(\frac{-4ab}{x_0^2+y_0^2}\right) - \frac{\mu_0 I}{4\pi} \ln\left(\frac{x^2+y^2}{x_0^2+y_0^2}\right)\,.$$

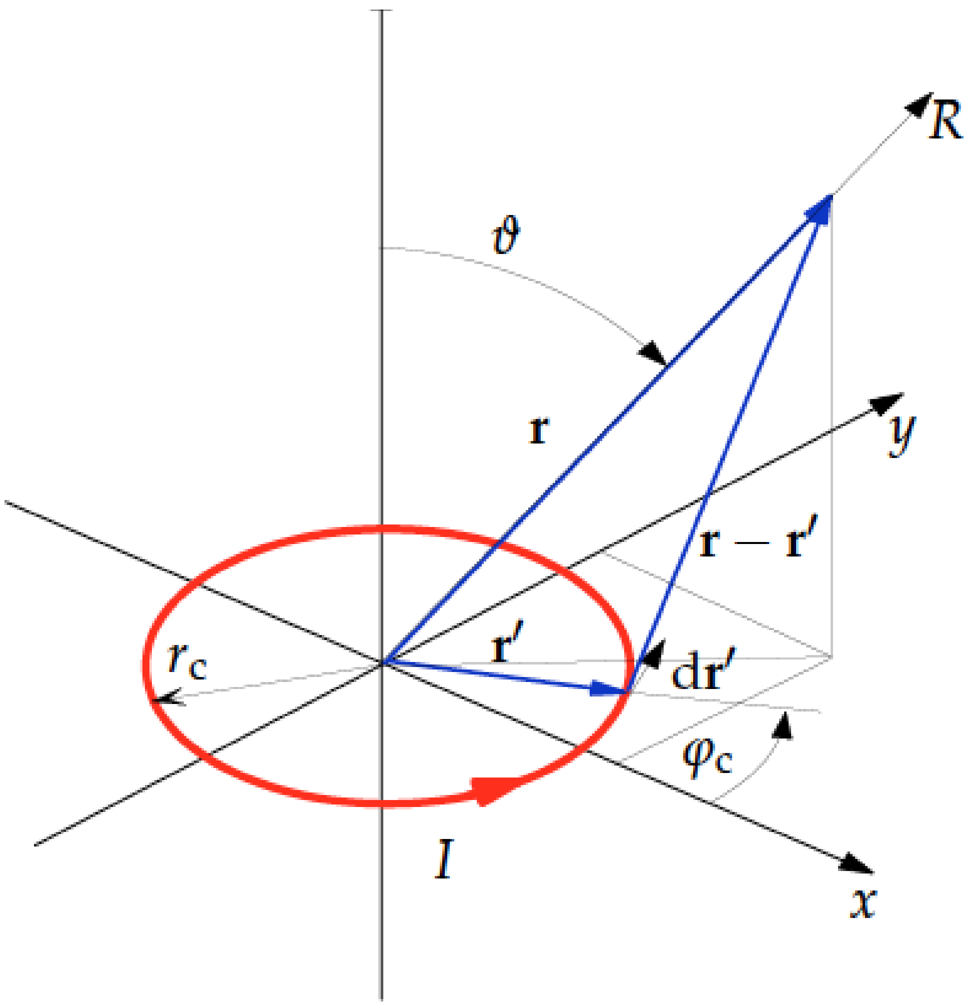
$$\mathbf{A}(x,y) = -\frac{\mu_0 I}{4\pi} \ln\left(\frac{x^2+y^2}{x_0^2+y_0^2}\right) \mathbf{e}_z = -\frac{\mu_0 I}{2\pi} \ln\left(\frac{r}{r_{\text{ref}}}\right) \mathbf{e}_z\,,$$

Problem solved, but reference radius has physical significance:
Return path for sum-currents

Field of a Line Current

$$A_\varphi(r,z) = \frac{\mu_0 I r_c}{\pi\sqrt{(r+r_c)^2+z^2}} \int_0^{\pi/2} \frac{2\sin^2\psi - 1}{\sqrt{1-k^2\sin^2\psi}}\,d\psi\,.$$

Appearance of elliptic integrals:
To be solved numerically.



On axis:

$$A_\varphi(r,z) = \frac{\mu_0 I r_c^2}{4}\frac{r}{(r_c^2+z^2)^{\frac{3}{2}}}\,,$$

$$B_z(z) = \frac{\mu_0 I}{2}\frac{r_c^2}{(r_c^2+z^2)^{\frac{3}{2}}}\,.$$

In the center:

$$B_z(z=0) = \frac{\mu_0 I}{2 r_c}\,.$$

Magnetic Dipole Moment

Far field approximation

$$A_\varphi(R,\vartheta) \approx \frac{\mu_0 I r_c^2 \pi}{4\pi}\frac{\sin\vartheta}{R^2} = \frac{\mu_0 m}{4\pi}\frac{\sin\vartheta}{R^2},$$

$$R = \sqrt{r^2+z^2} \text{ and } \sin\vartheta = r/R,$$

$$[m] = 1\,\mathrm{A\,m^2}.$$

Definition

$$m := I r_c^2 \pi$$

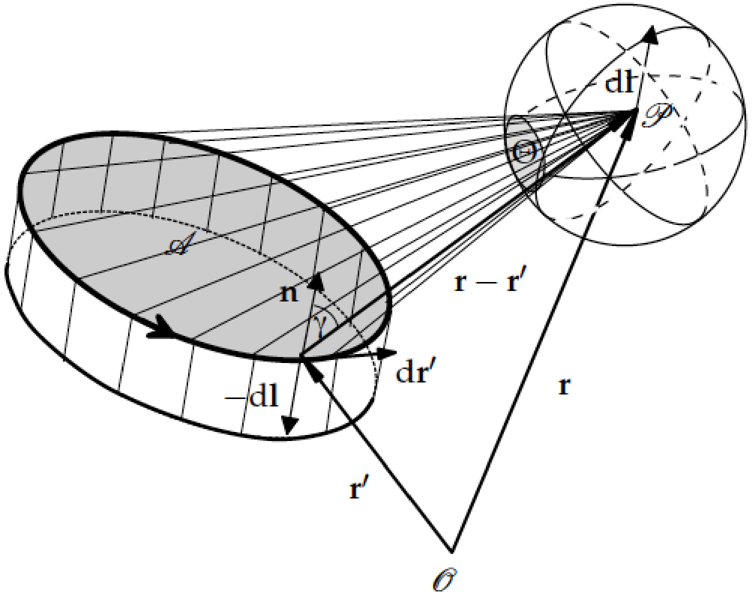
$$\mathbf{m} = I\mathbf{a},$$

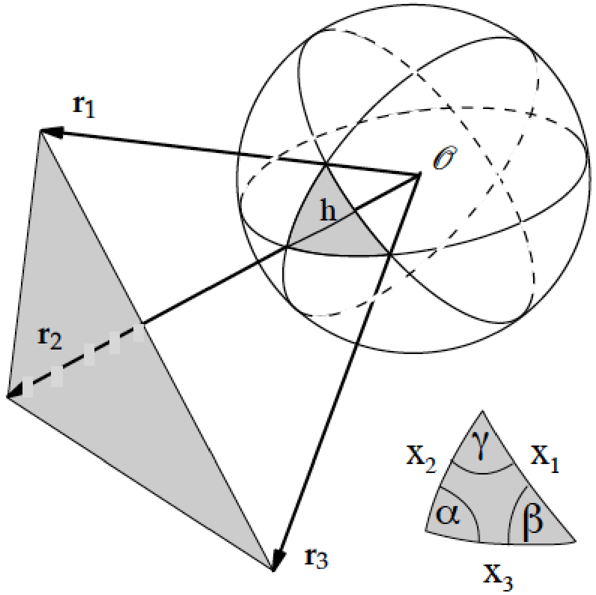
$$\mathbf{m} = \frac{I}{2}\int_{\mathscr{C}} \mathbf{r}\times \mathrm{d}\mathbf{r},$$

$$\mathbf{M}(\mathbf{r}) := \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}V} = \frac{1}{2}\mathbf{r}\times\mathbf{J}(\mathbf{r}),$$

Solid Angle and Magnetic Scalar Potential

Solid angle (easy to compute) gives the magnetic scalar potential of a current loop

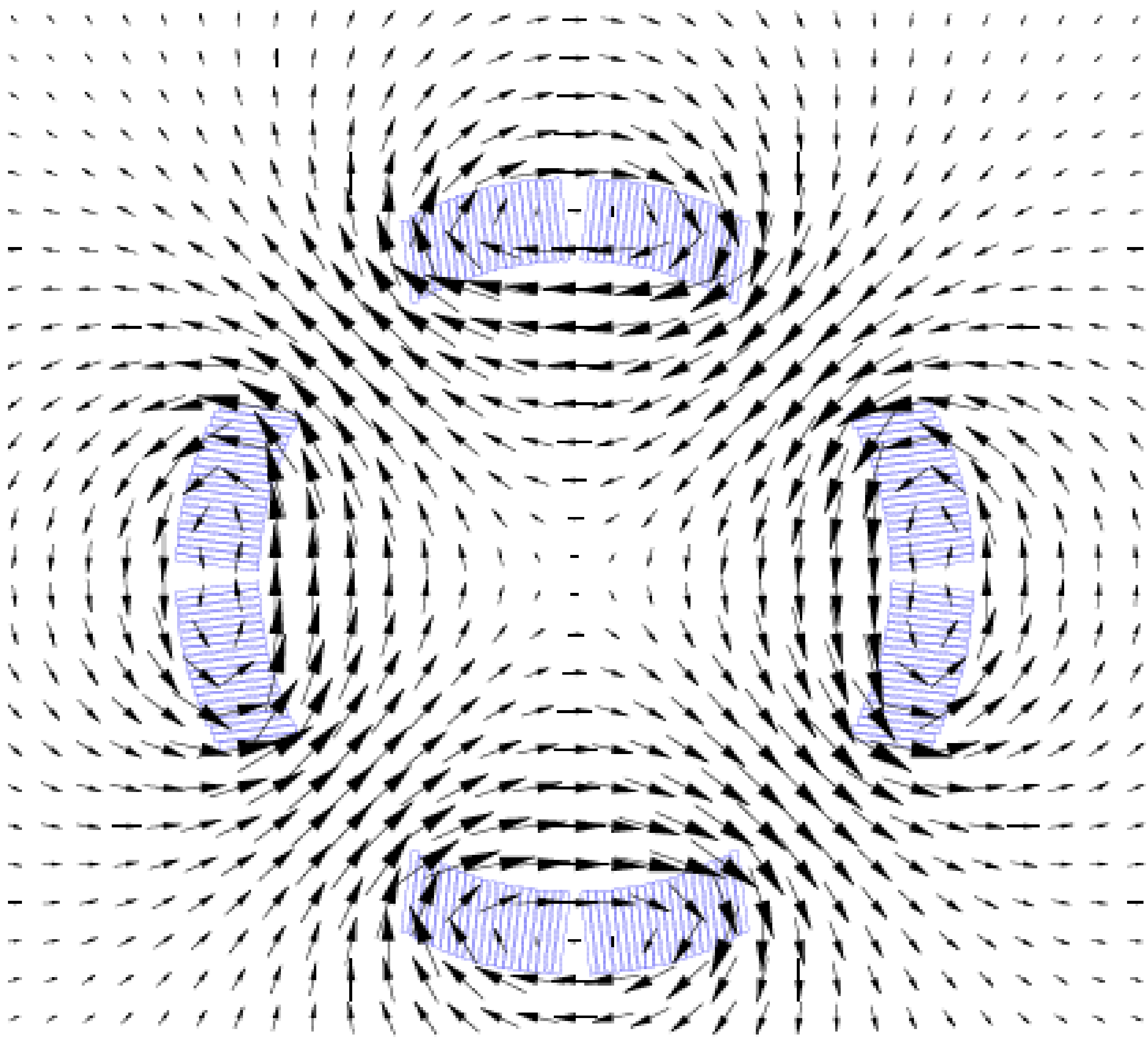




$$\phi_{\mathrm{m}}(\mathbf{r}) = \frac{I}{4\pi}\Theta .$$

$$\tan\left(\frac{\Theta}{2}\right) = \frac{\mathbf{r}_1 \cdot (\mathbf{r}_2 \times \mathbf{r}_3)}{r_1 r_2 r_3 + (\mathbf{r}_1 \cdot \mathbf{r}_2) r_3 + (\mathbf{r}_1 \cdot \mathbf{r}_3) r_2 + (\mathbf{r}_2 \cdot \mathbf{r}_3) r_1} .$$

The Imaging Current Method

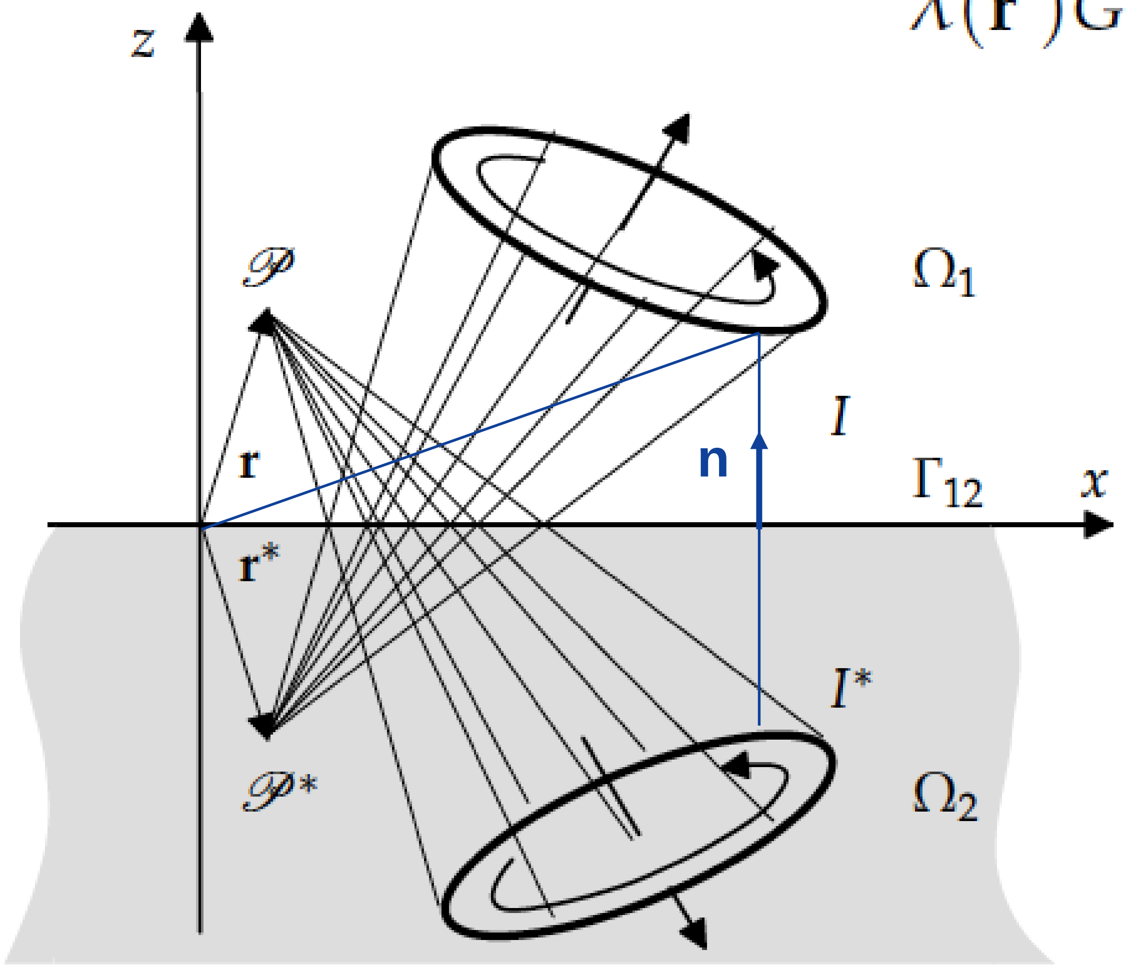


Notes on the Imaging Method

- Domain 1: Domain with current sources
- Domain 2: Highly permeable material
  - All imaging currents must be in domain 2
  - The sources and the images must create a field that satisfies the continuity conditions at the interface between domains 1 and 2
  - The image of the image must be the original source
  - The field generated in domain 1 is identical to the source field plus the field from the (iron) magnetization.
  - The field generated in domain 2 has no physical significance

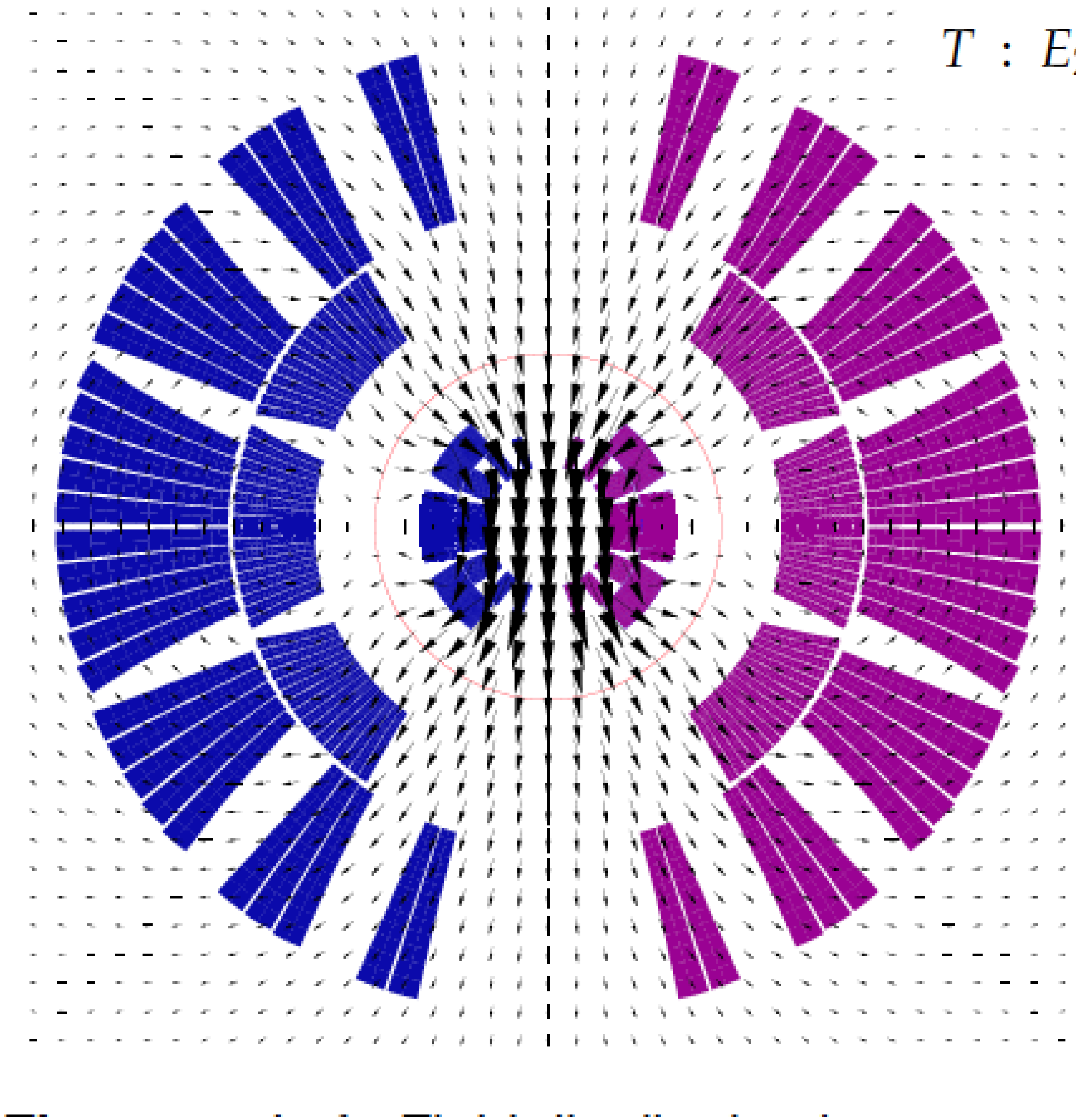
The Imaging Current Method

$$\lambda(\mathbf{r}')G(\mathbf{r},T\mathbf{r}') = \lambda(\mathbf{r})G(T\mathbf{r},\mathbf{r}')$$



$$T : E_3 \to E_3 : \mathbf{r}' \mapsto T\mathbf{r}' = \mathbf{r}' - 2\mathbf{n}\,(\mathbf{n}\cdot\mathbf{r}')\,,$$

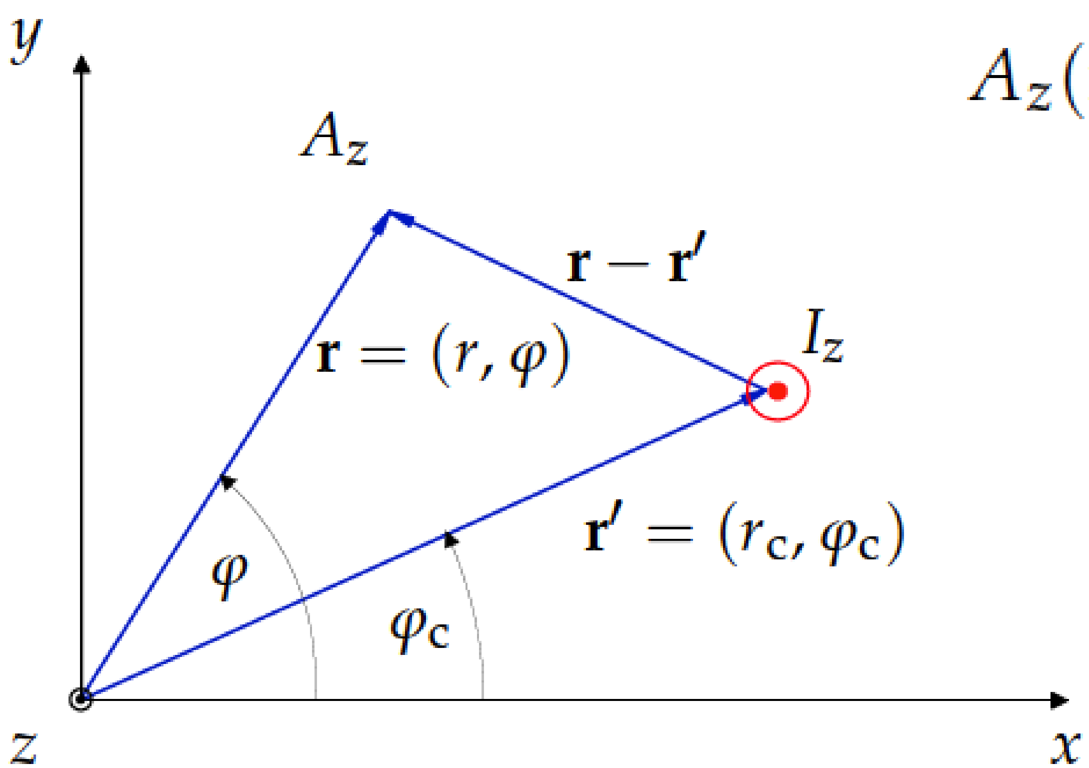
The Imaging Current Method



$$T : E_2 \to E_2 : \mathbf{r}' \mapsto T\mathbf{r}' = \frac{r_y^2}{|\mathbf{r}'|^2}\mathbf{r}' ,$$

$$I^* = \lambda_\mu I := \frac{\mu_r - 1}{\mu_r + 1} I .$$

The Field of a Line Current (2D)



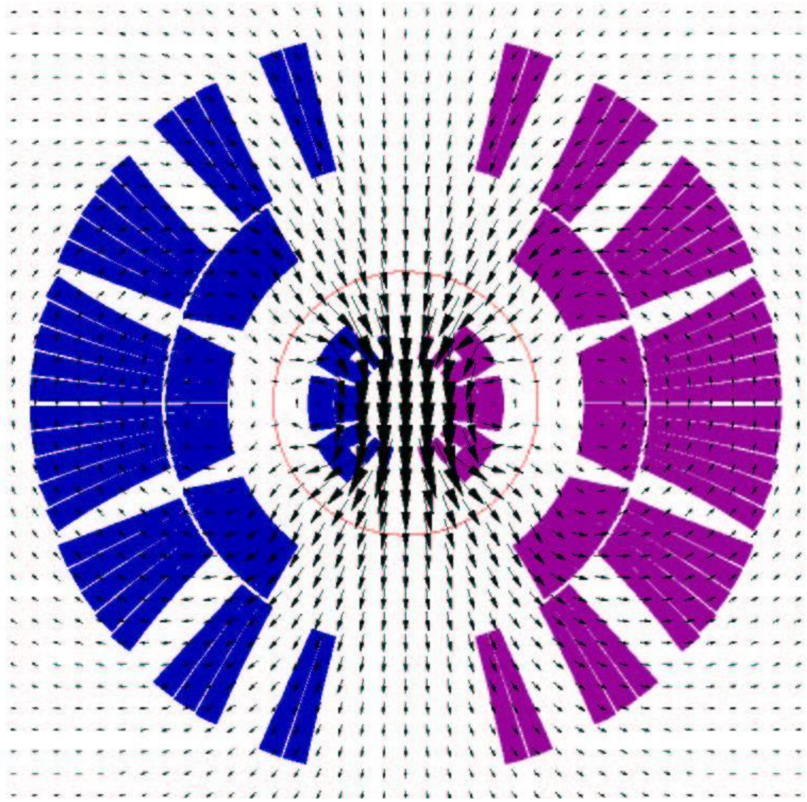
$$A_z(\mathbf{r}) = -\frac{\mu_0 I}{2\pi} \ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{r_{\text{ref}}}\right)$$

$$A_z(r,\varphi) = -\frac{\mu_0 I}{2\pi} \ln\left(\frac{r_c}{r_{\text{ref}}}\right) + \frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{r_c}\right)^n \cos n(\varphi - \varphi_c)$$

$$B_n(r_0) = -\frac{\mu_0 I}{2\pi r_c}\left(\frac{r_0}{r_c}\right)^{n-1} \cos n\varphi_c, \qquad A_n(r_0) = \frac{\mu_0 I}{2\pi r_c}\left(\frac{r_0}{r_c}\right)^{n-1} \sin n\varphi_c.$$

Imaging Method

$$B_n(r_0) = -\sum_{k=1}^{K} \frac{\mu_0 I_k}{2\pi} \frac{r_0^{n-1}}{r_{c,k}^n} \left(1 + \lambda_\mu \left(\frac{r_{c,k}}{r_y}\right)^{2n}\right) \cos n\varphi_{c,k}\,,$$



$$\lambda_\mu I := \frac{\mu_r - 1}{\mu_r + 1} I\,.$$

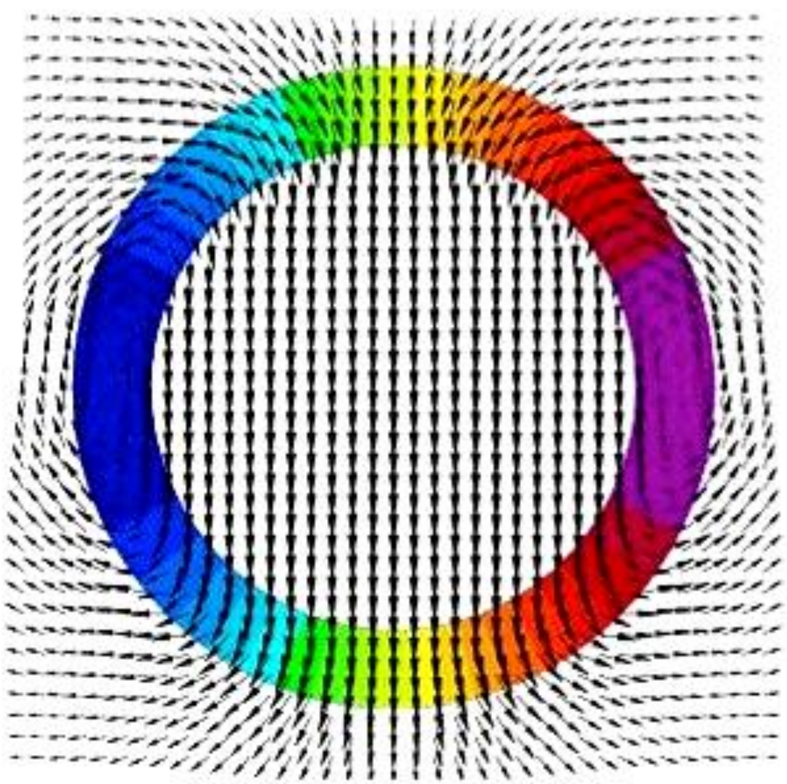
$$\frac{B_N^{\text{imag}}}{B_N + B_N^{\text{imag}}} \approx \left(1 + \left(\frac{r_y}{r}\right)^{2N}\right)^{-1}.$$

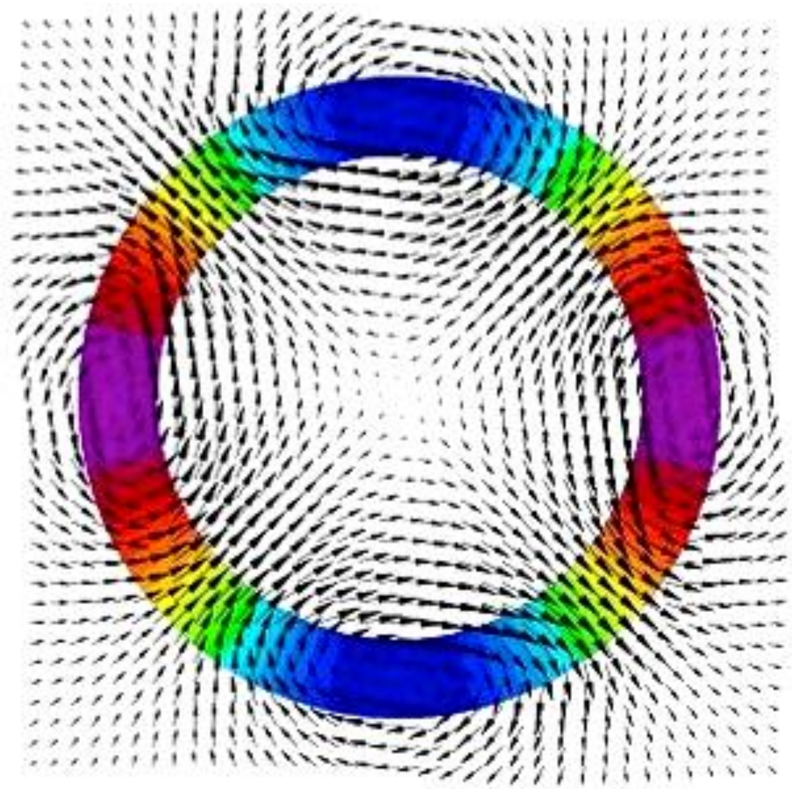
Interrupt: Numerical example

Ideal Current Distributions

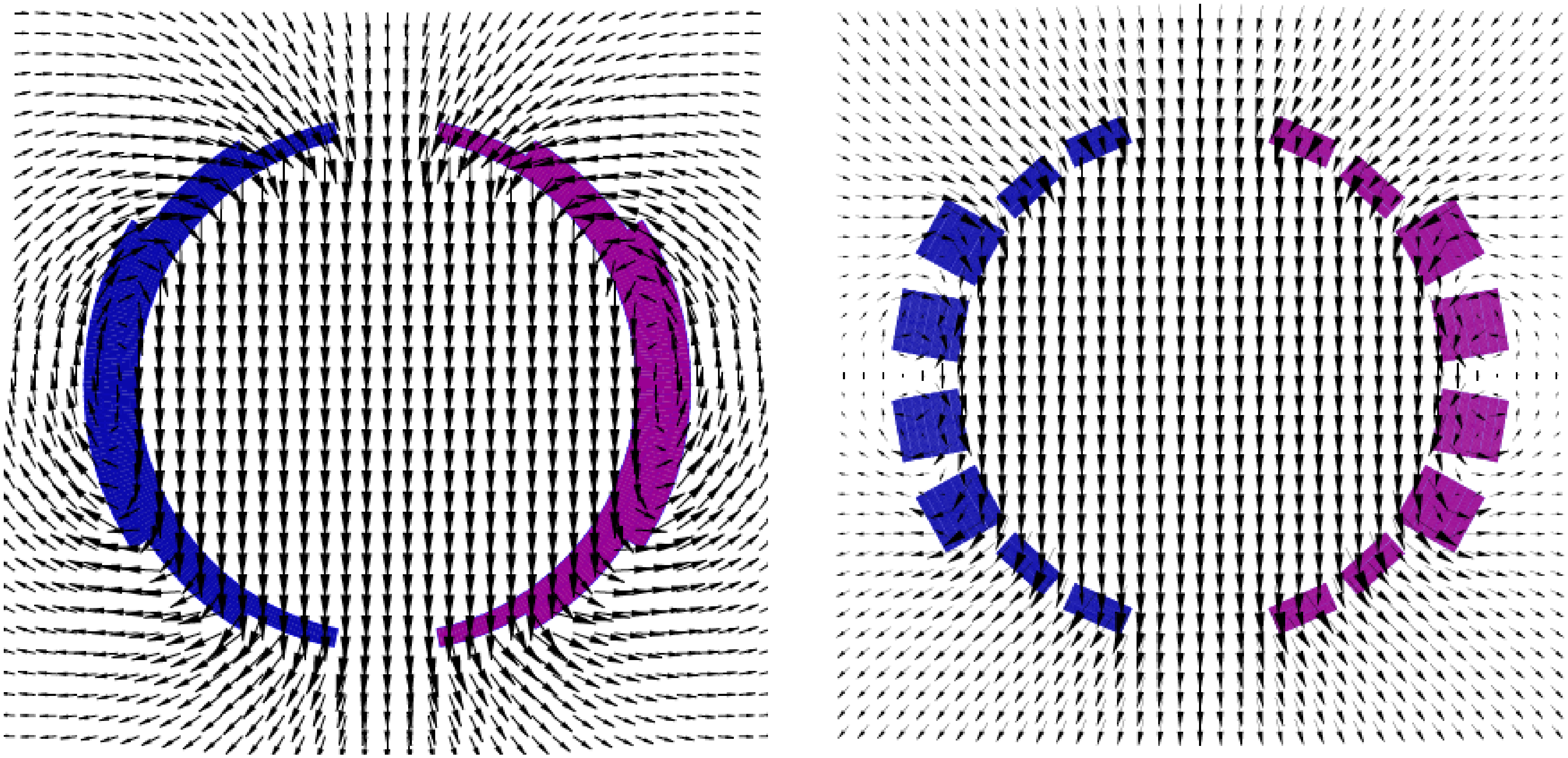
$$B_n(r_0) = \int_{r_a}^{r_e} \int_0^{2\pi} -\frac{\mu_0 J_E}{2\pi} \quad J_c(B) = d\,(B_{c2} - B) \qquad \cos n\varphi_c \, r_c \mathrm{d}\varphi_c \, \mathrm{d}r_c$$

$$B = \frac{\mu_0}{2} \lambda_{tot} J_c \,(r_e - r_a) = \frac{\mu_0}{2} \lambda_{tot} \, d\,(B_{c2} - B)\,(r_e - r_a)\,,$$

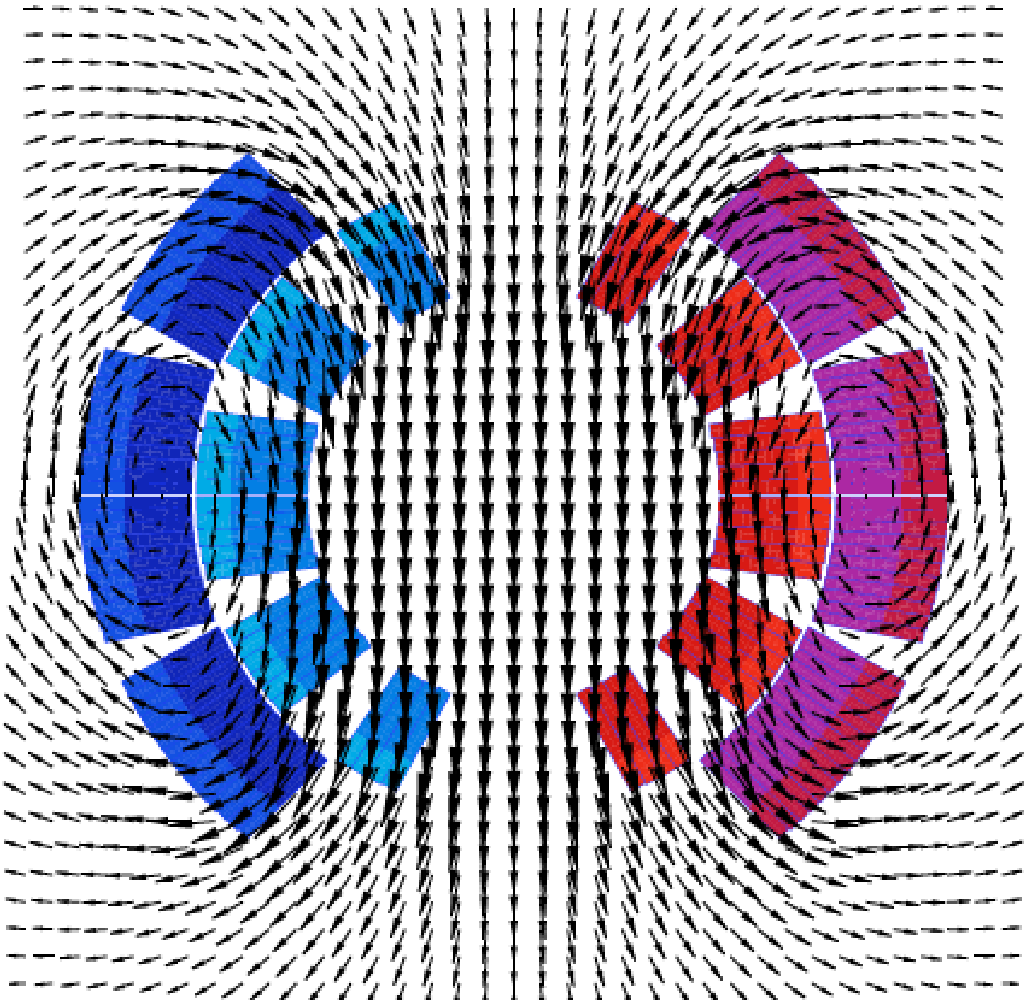


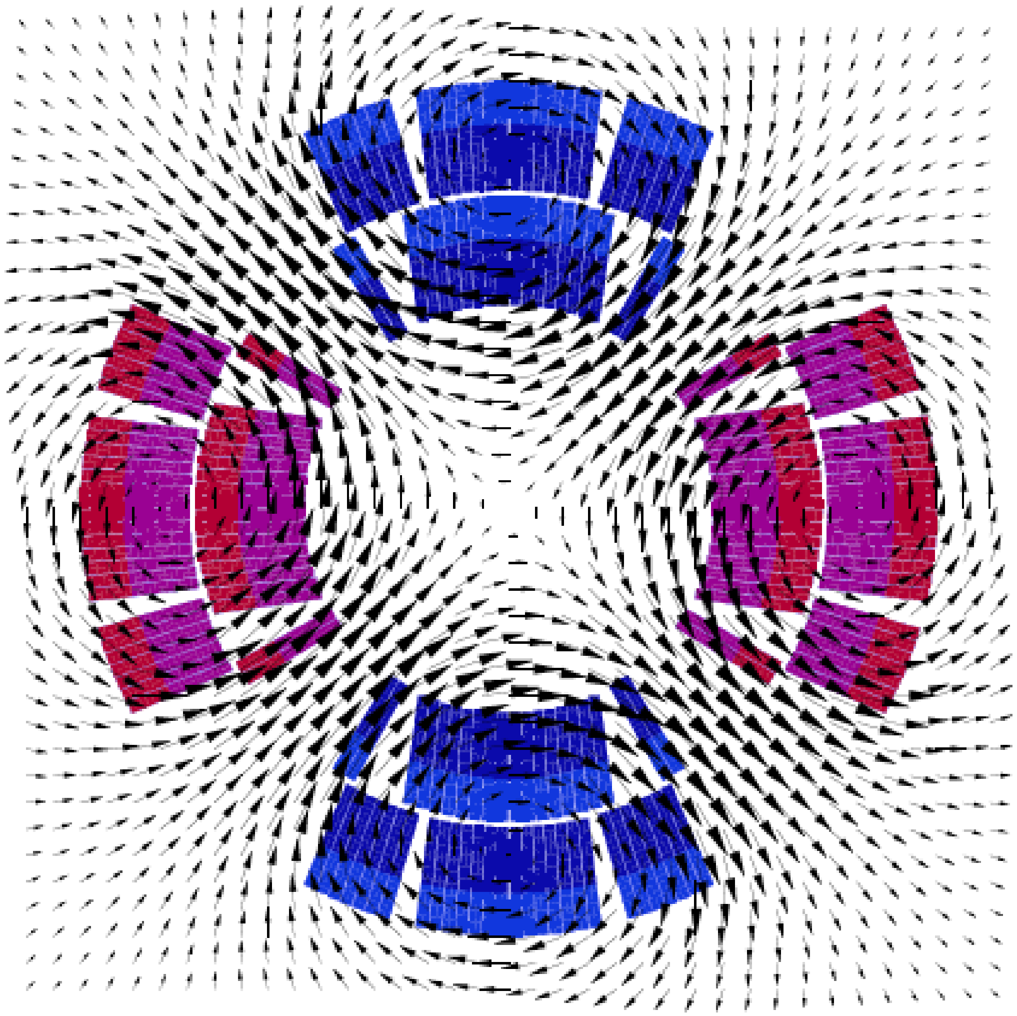


Coil-Block Approximations



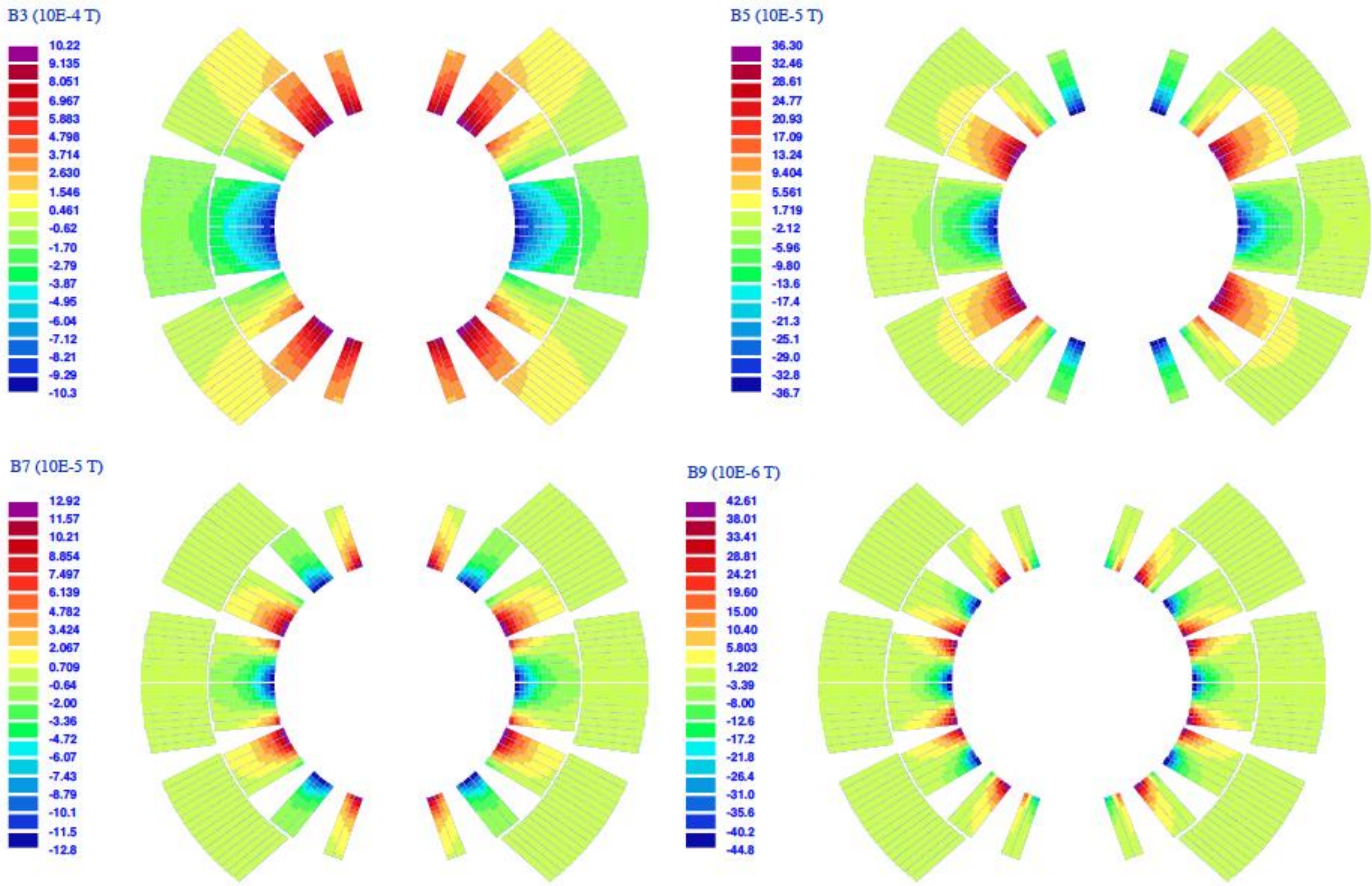
Coil-Block Approximations





Generation of Multipole Field Errors

$$B_n(r_0) = -\sum_{k=1}^{K} \frac{\mu_0 I_k}{2\pi} \frac{r_0^{n-1}}{r_{c,k}^n} \left(1 + \lambda_\mu \left(\frac{r_{c,k}}{r_y}\right)^{2n}\right) \cos n\varphi_{c,k} \,,$$



Sensitivity to Manufacturing Errors

$$B_n(r_0) = -\sum_{k=1}^{K} \frac{\mu_0 I_k}{2\pi} \frac{r_0^{n-1}}{r_{c,k}^n} \left(1 + \lambda_\mu \left(\frac{r_{c,k}}{r_y}\right)^{2n}\right) \cos n\varphi_{c,k} \,,$$

$$\frac{\partial B_n(r_0)}{\partial \varphi_c} = -\frac{\mu_0 I_k}{2\pi} \frac{n r_0^{n-1}}{r_c^n} \left(1 + \left(\frac{r_c}{r_y}\right)^{2n}\right) \sin n\varphi_c,$$

$$\frac{\partial B_n(r_0)}{\partial r_c} = \frac{\mu_0 I_k}{2\pi} \frac{n r_0^{n-1}}{r_c^{n+1}} \left(1 - \left(\frac{r_c}{r_y}\right)^{2n}\right) \cos n\varphi_c.$$

Increase of the azimuthal coil size by 0.1 mm produces (in units of 10^{-4}):

$$b_1 = -14, \qquad b_3 = 1.2 \qquad b_5 = 0.03$$

Specified tolerances on coils: ± 0.025 mm

Coil Winding and Curing

Field of a Ring Current

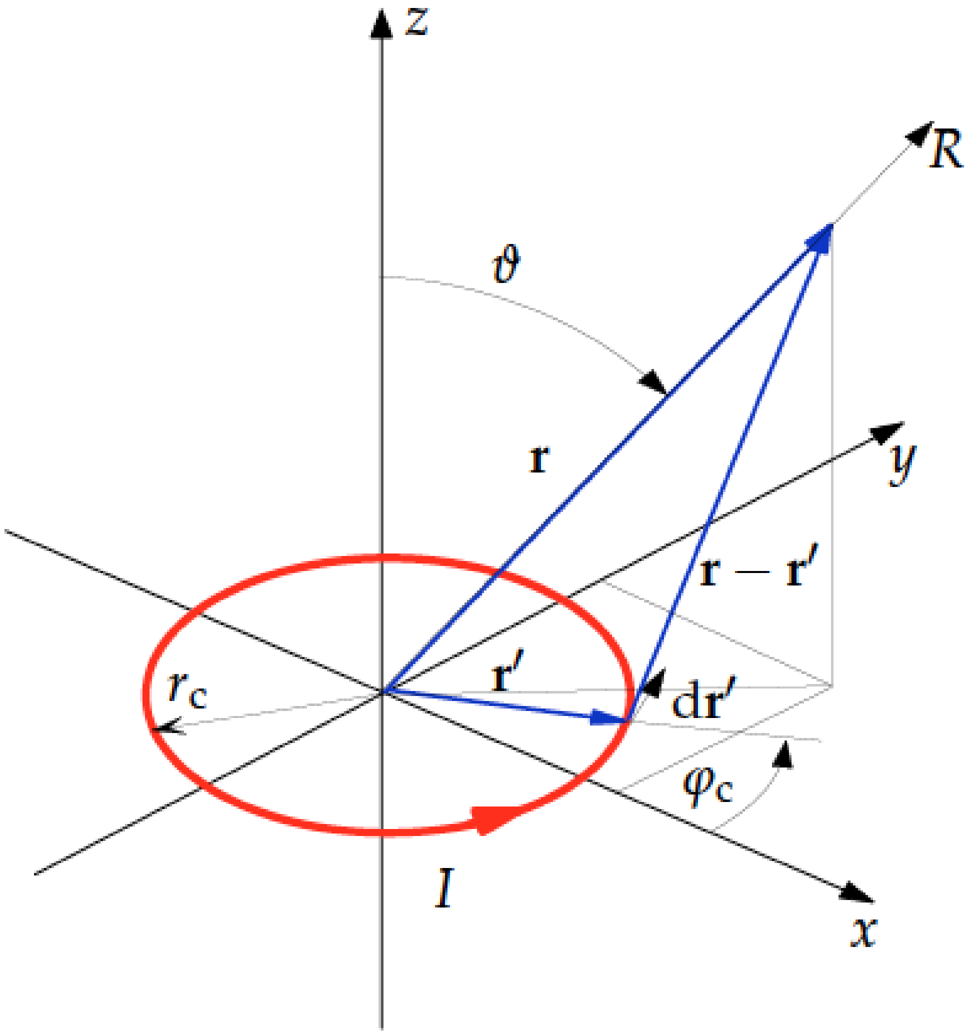
On axis:

$$A_\varphi(r,z) = \frac{\mu_0 I r_c^2}{4} \frac{r}{(r_c^2+z^2)^{\frac{3}{2}}},$$

$$B_z(z) = \frac{\mu_0 I}{2} \frac{r_c^2}{(r_c^2+z^2)^{\frac{3}{2}}}.$$

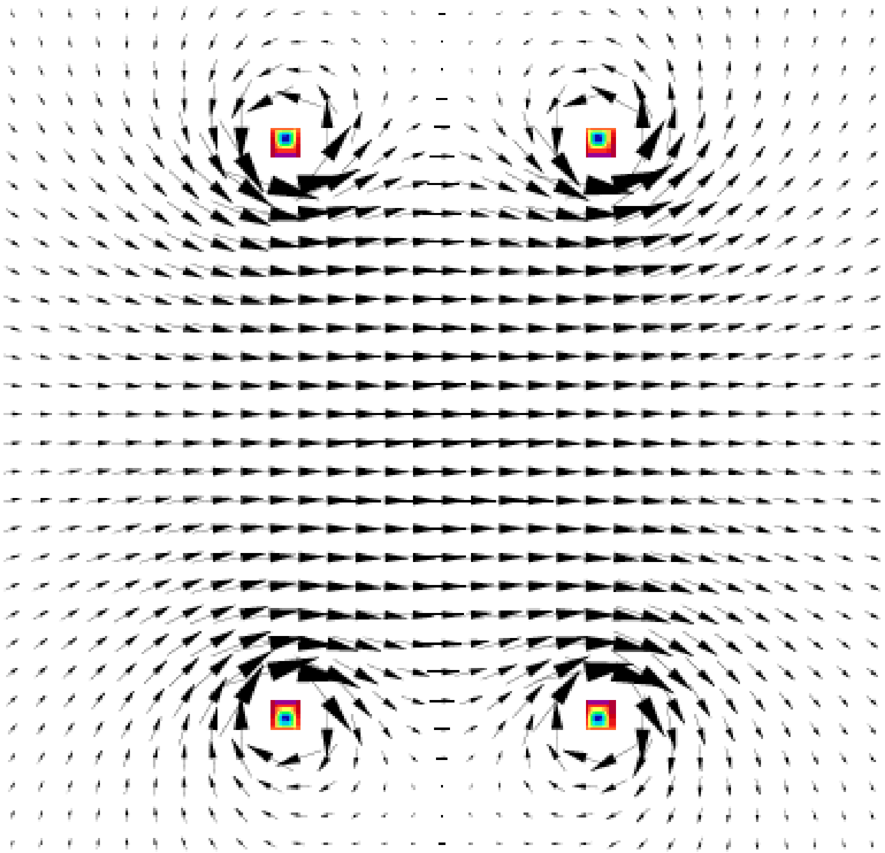
In the center:

$$B_z(z=0) = \frac{\mu_0 I}{2 r_c}.$$



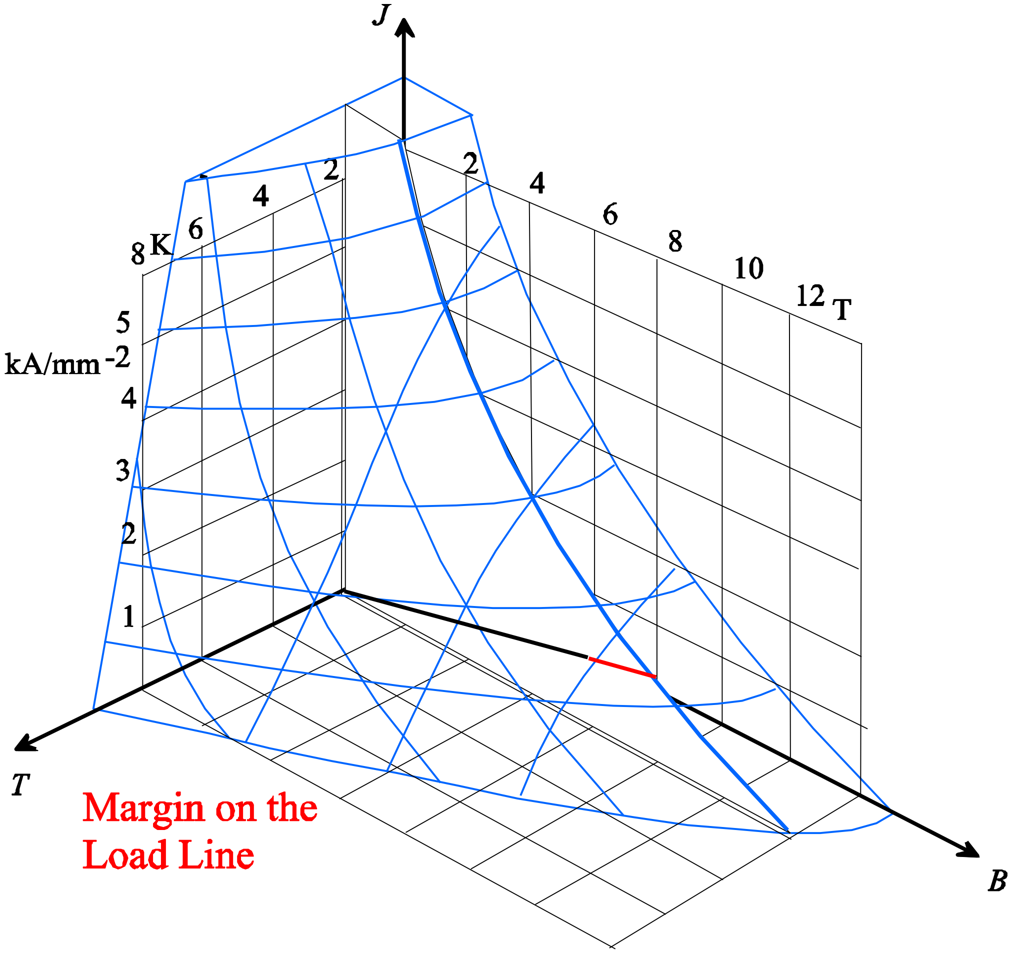
Helmholtz Coils

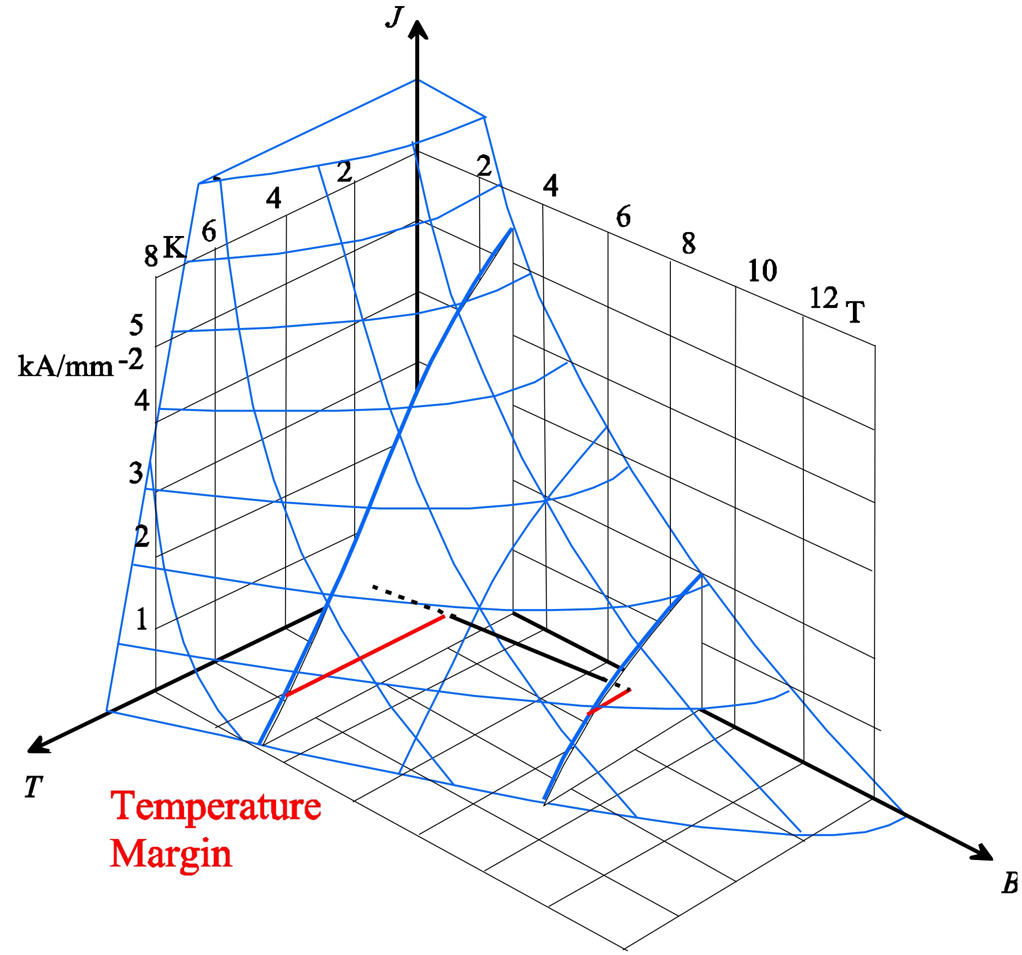
$$\frac{\mathrm{d}B_z}{\mathrm{d}z} = \frac{-3\mu_0 I r_c^2}{2}\left(\frac{z+z_c}{\sqrt{r_c^2+(z+z_c)^2}^5} + \frac{z-z_c}{\sqrt{r_c^2+(z-z_c)^2}^5}\right),$$

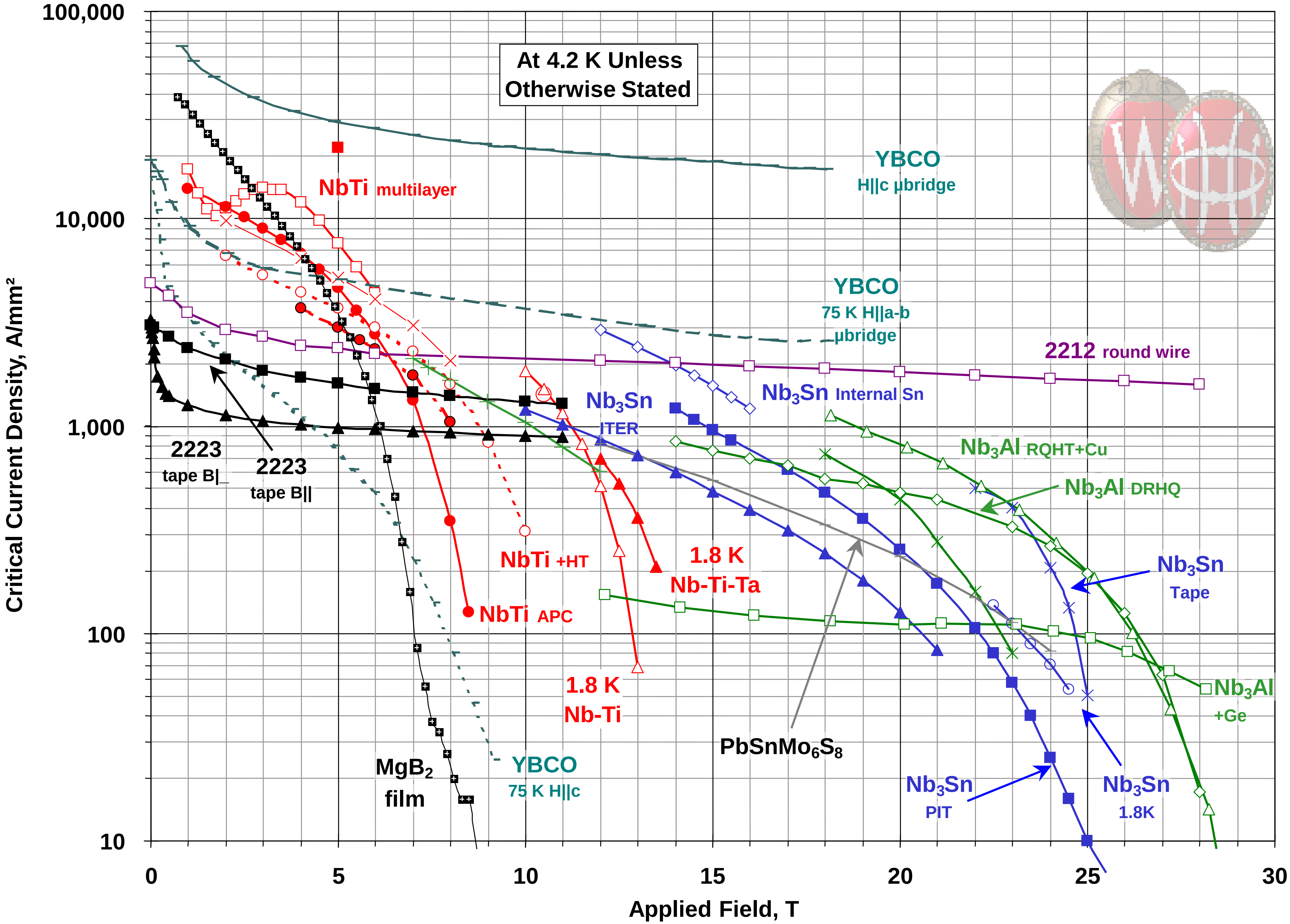


zero at $z = 0$ only if $z_c = r_c/2$.

Critical Surface of NbTi

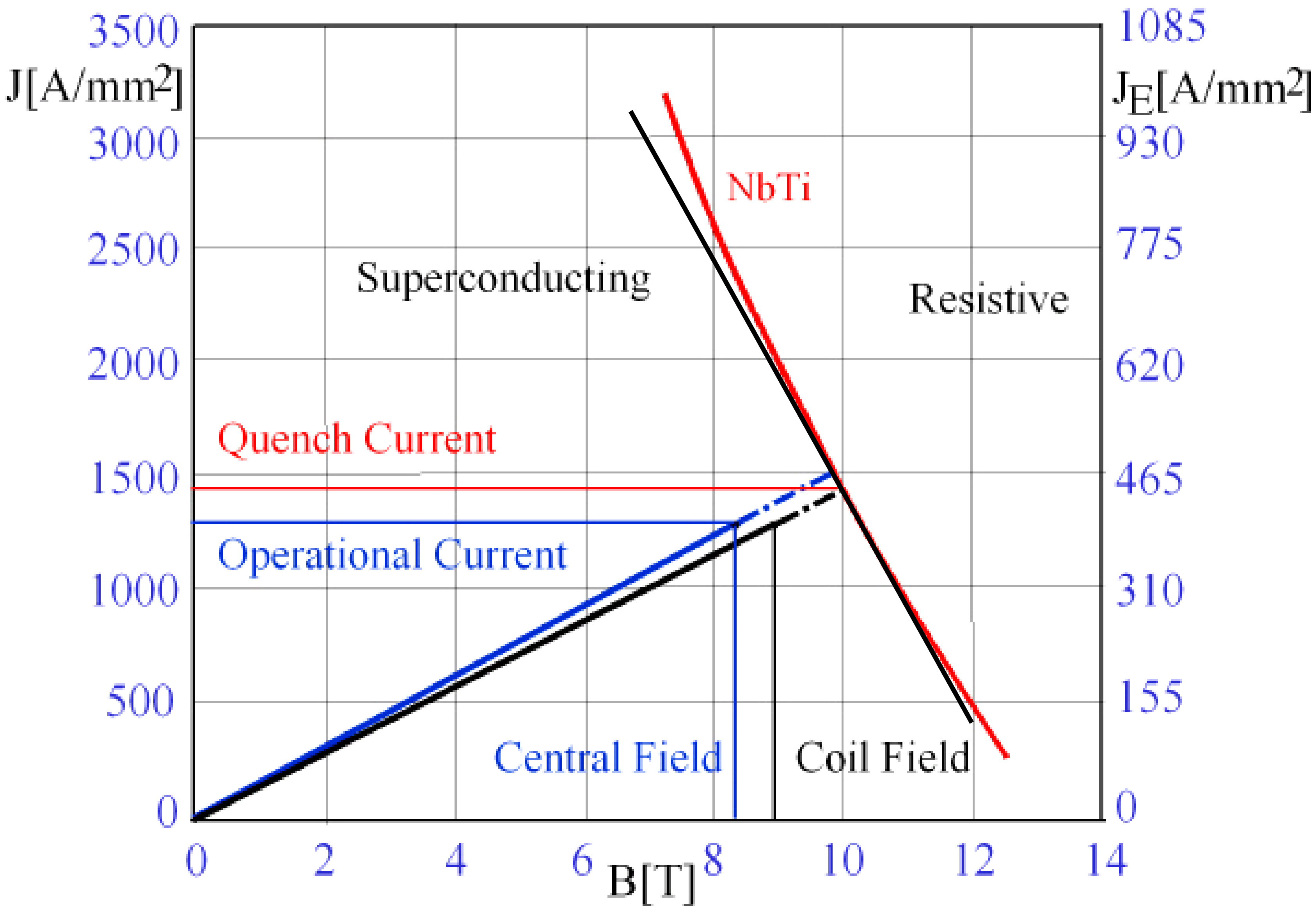




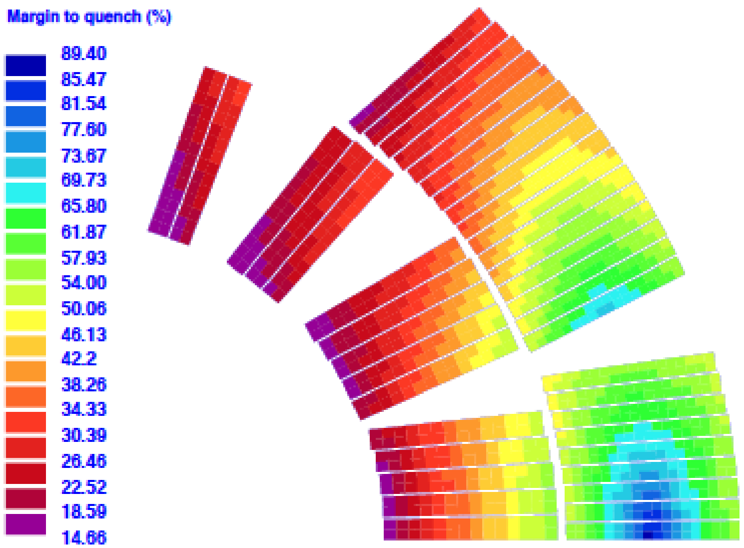


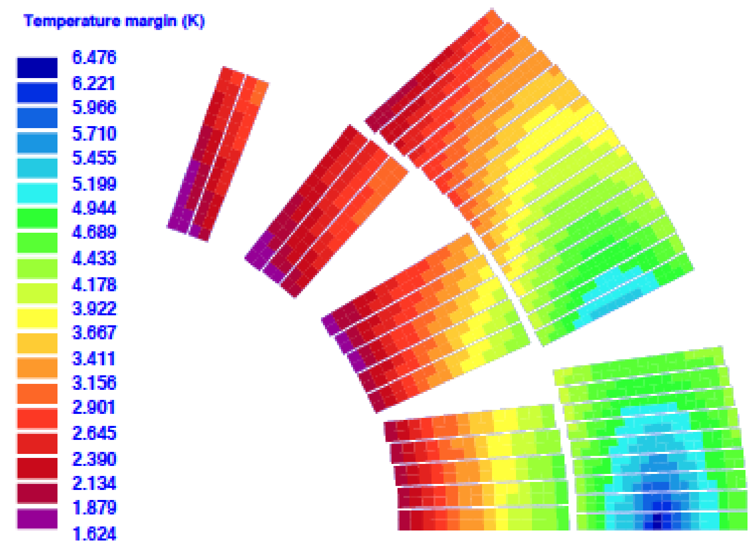
At 4.2 K Unless Otherwise Stated
100,000
10,000
1,000
100
10
0
5
10
15
20
25
30
Applied Field, T
Critical Current Density, A/mm²
YBCO
H||c μbridge
NbTi multilayer
YBCO
75 K H||a-b
μbridge
2212 round wire
Nb3Sn Internal Sn
Nb3Sn
ITER
Nb3Al RQHT+Cu
Nb3Al DRHQ
2223
tape B|_
2223
tape B||
NbTi +HT
1.8 K
Nb-Ti-Ta
NbTi APC
Nb3Sn
Tape
1.8 K
Nb-Ti
PbSnMo6S8
Nb3Al
+Ge
MgB2
film
YBCO
75 K H||c
Nb3Sn
PIT
Nb3Sn
1.8K

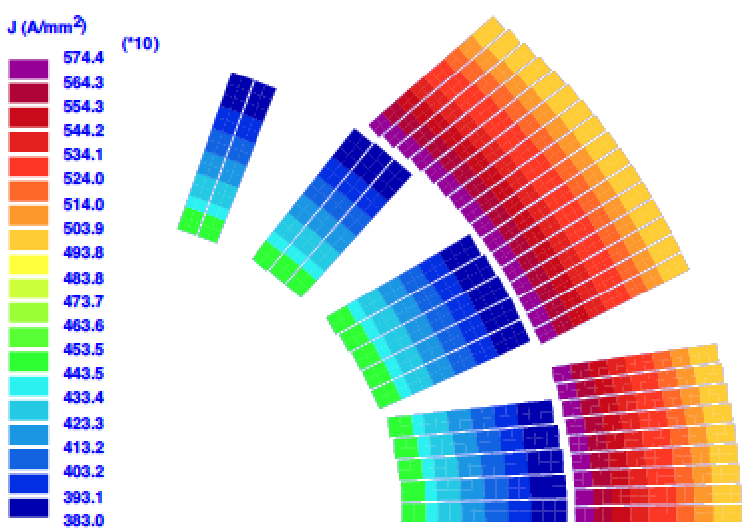
Working Point of LHC Dipole Magnets

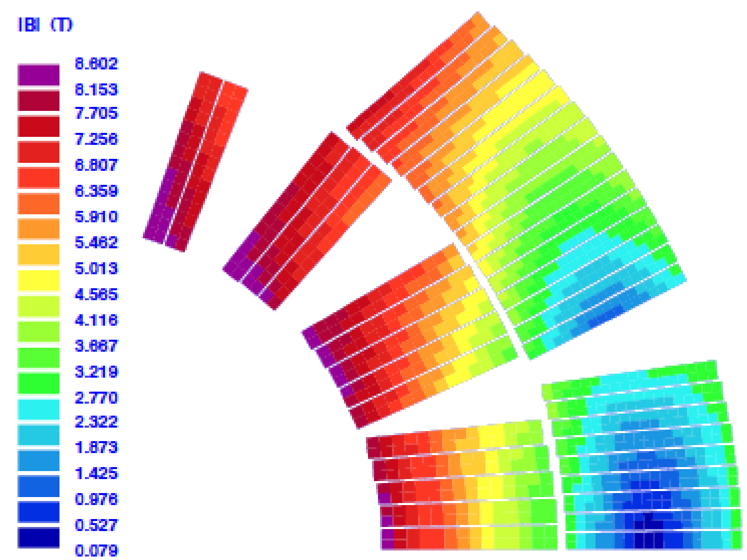


Margins


Margin to quench (%)
89.40
85.47
81.54
77.60
73.67
69.73
65.80
61.87
57.93
54.00
50.06
46.13
42.2
38.26
34.33
30.39
26.46
22.52
18.59
14.66

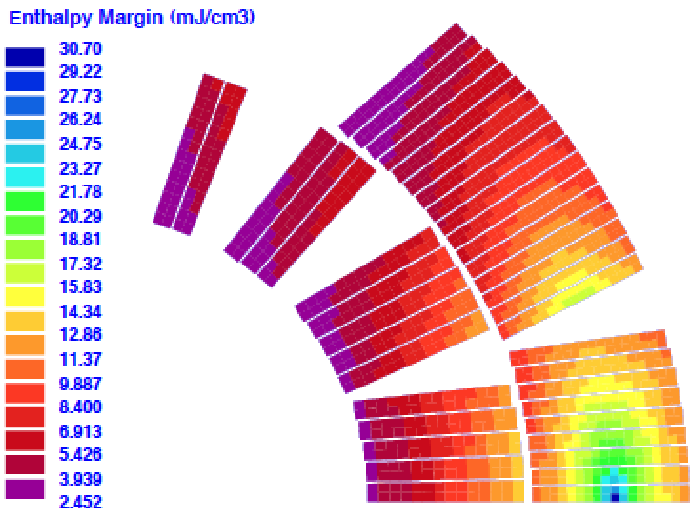

Temperature margin (K)
6.476
6.221
5.966
5.710
5.455
5.199
4.944
4.689
4.433
4.178
3.922
3.667
3.411
3.156
2.901
2.645
2.390
2.134
1.879
1.624

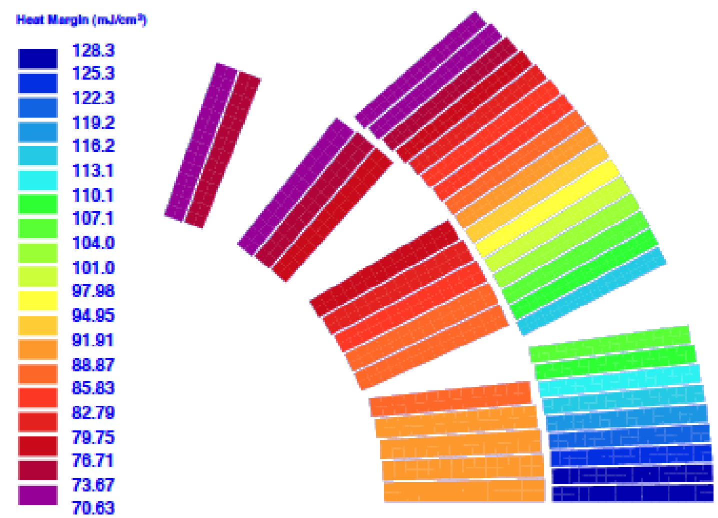

J (A/mm^2)
(*10)
574.4
564.3
554.3
544.2
534.1
524.0
514.0
503.9
493.8
483.8
473.7
463.6
453.5
443.5
433.4
423.3
413.2
403.2
393.1
383.0


|B| (T)
8.602
8.153
7.705
7.256
6.807
6.359
5.910
5.462
5.013
4.565
4.116
3.667
3.219
2.770
2.322
1.873
1.425
0.976
0.527
0.079

Margins

$$\Delta h := \int_{T_b}^{T_c(J,B)} \rho\, c_p(T)\, \mathrm{d}T\,,$$

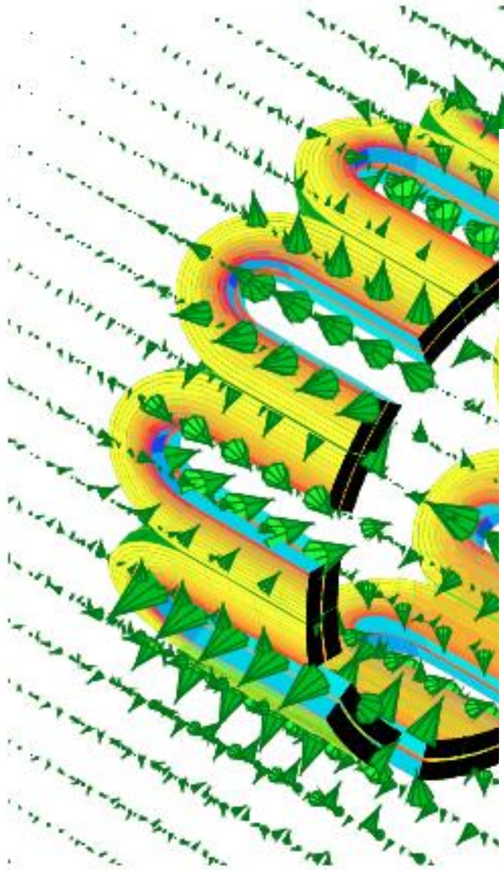


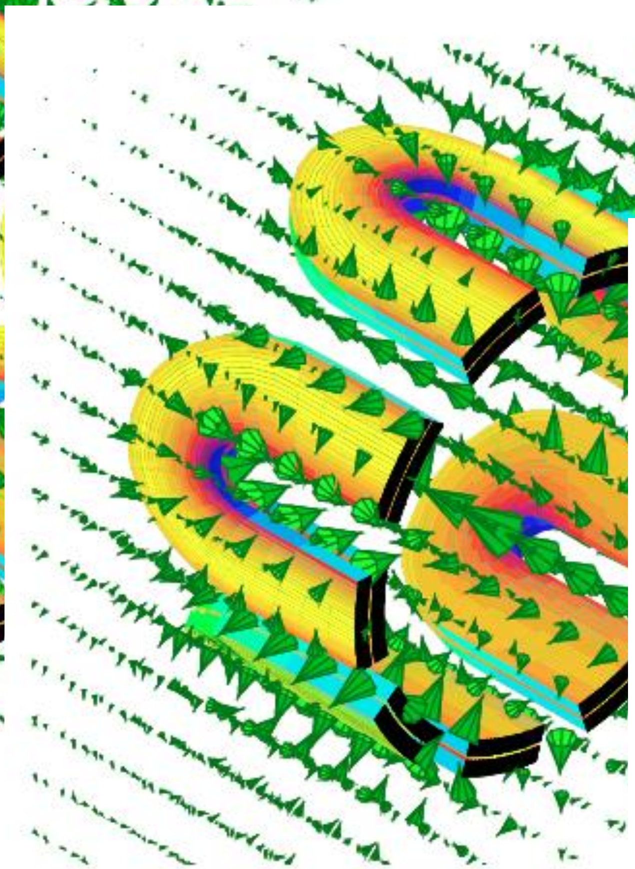


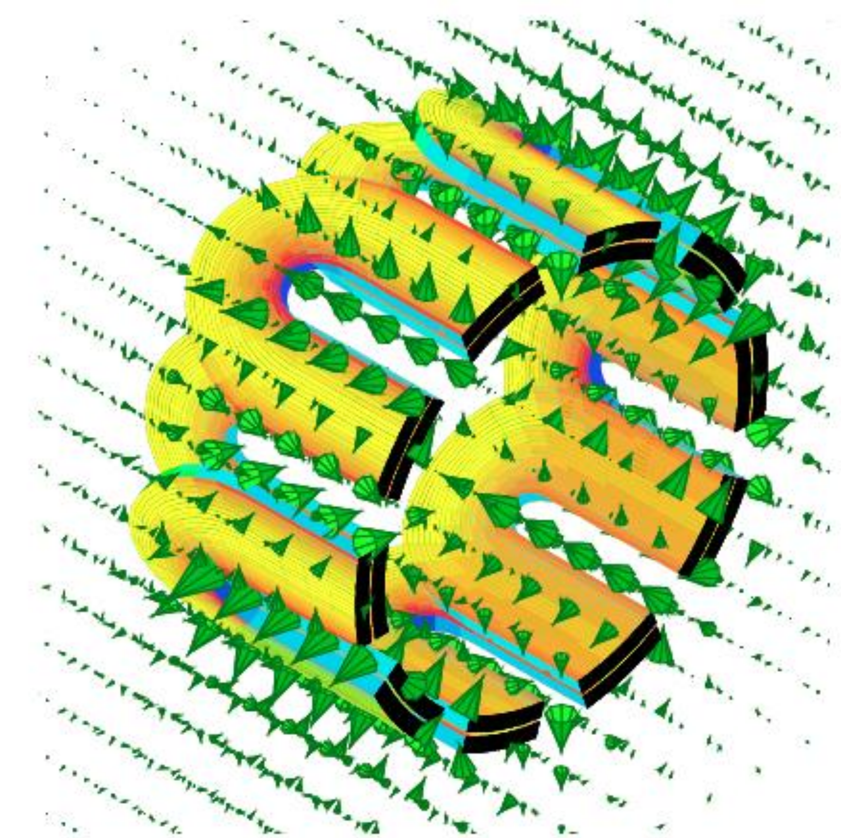
Enthalpy margin

Heat reserve (copper, SC and helium)

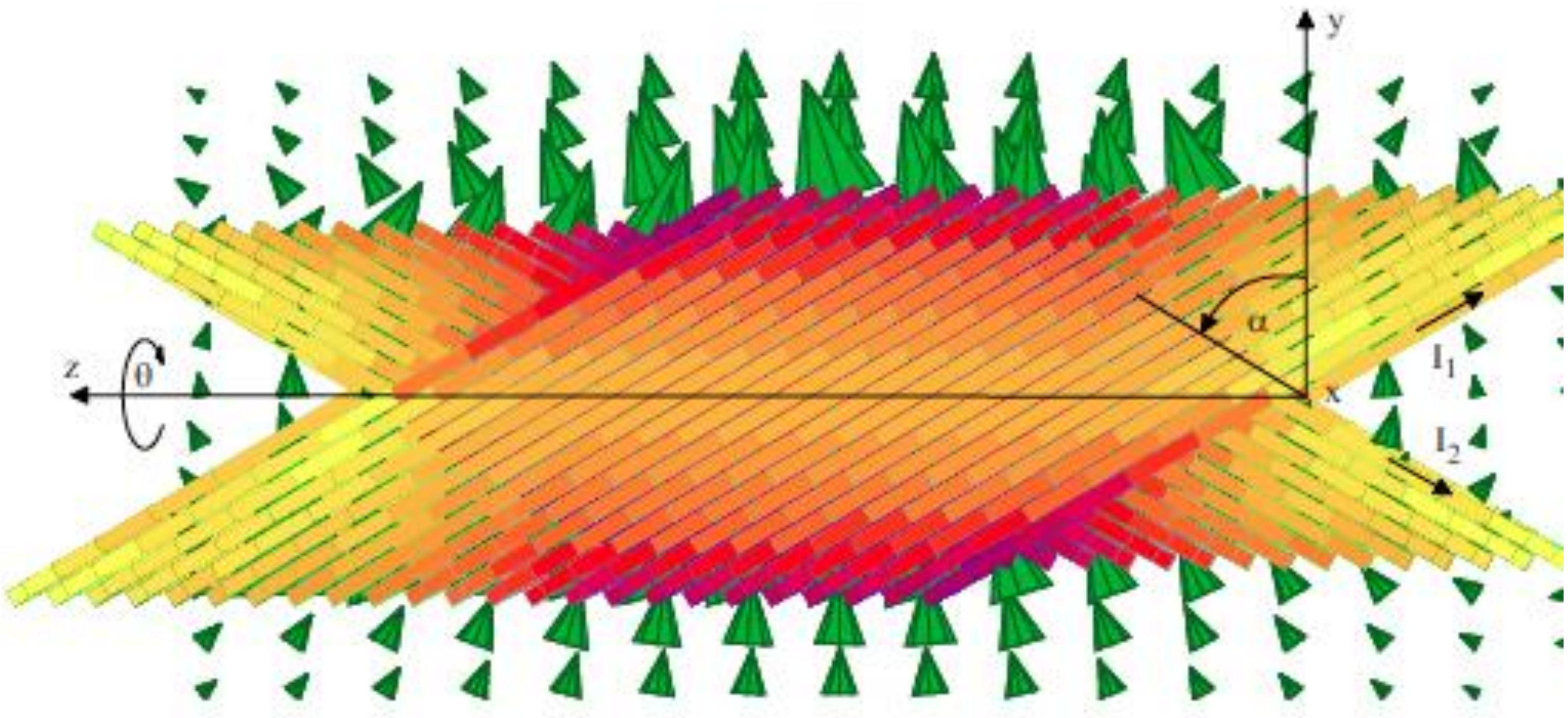
Margins





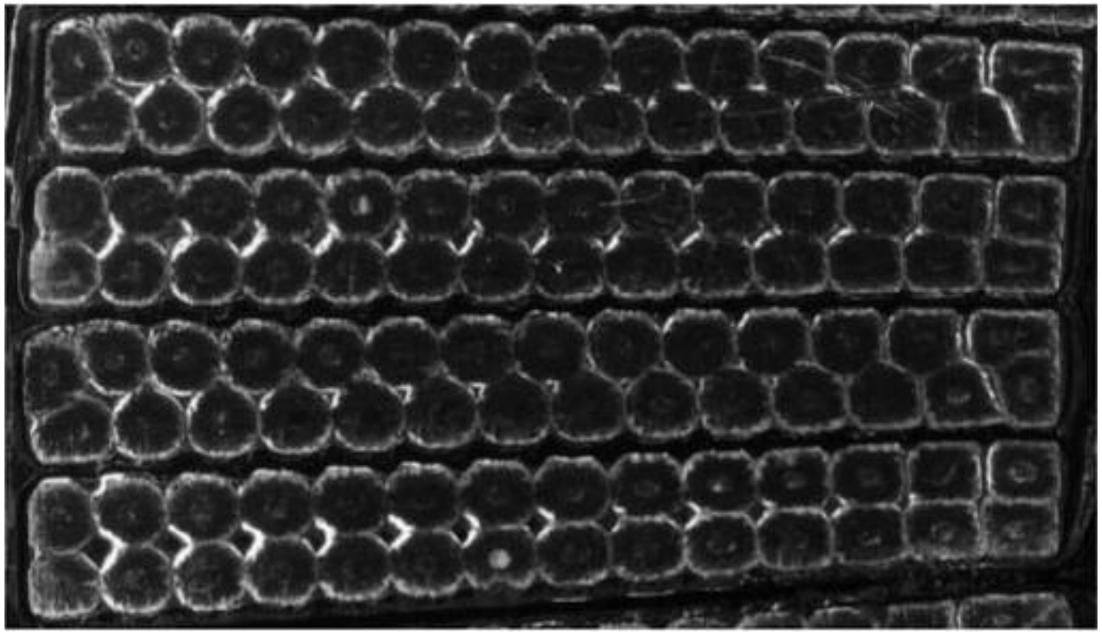


Nested Helices



Coil Block, Pinning Centers in Filaments

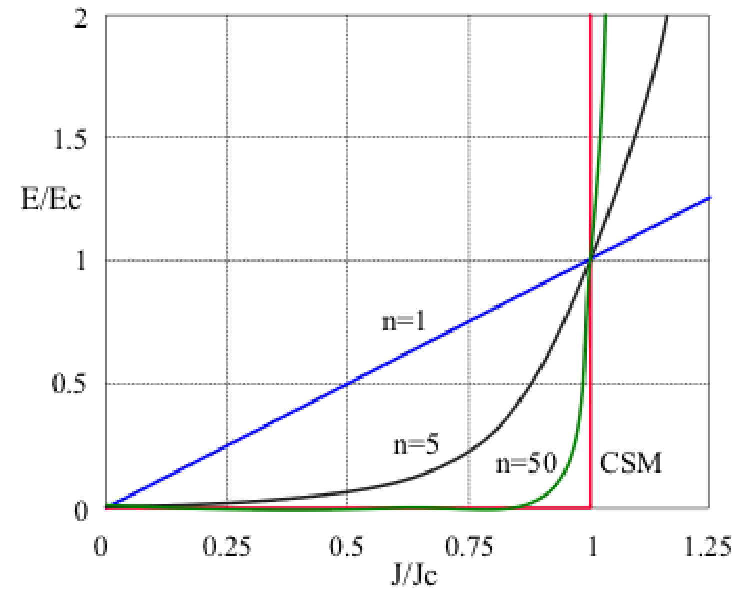
Magnetization

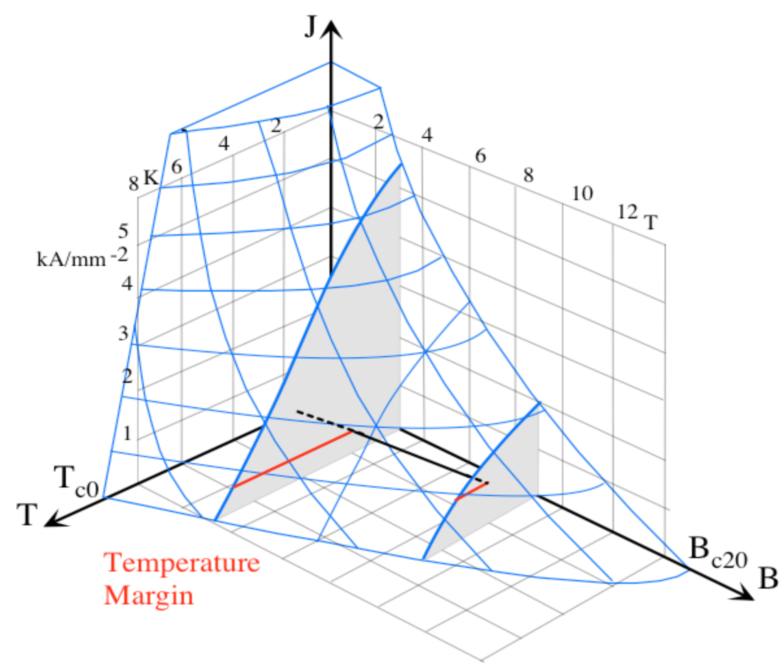


Grading of current density

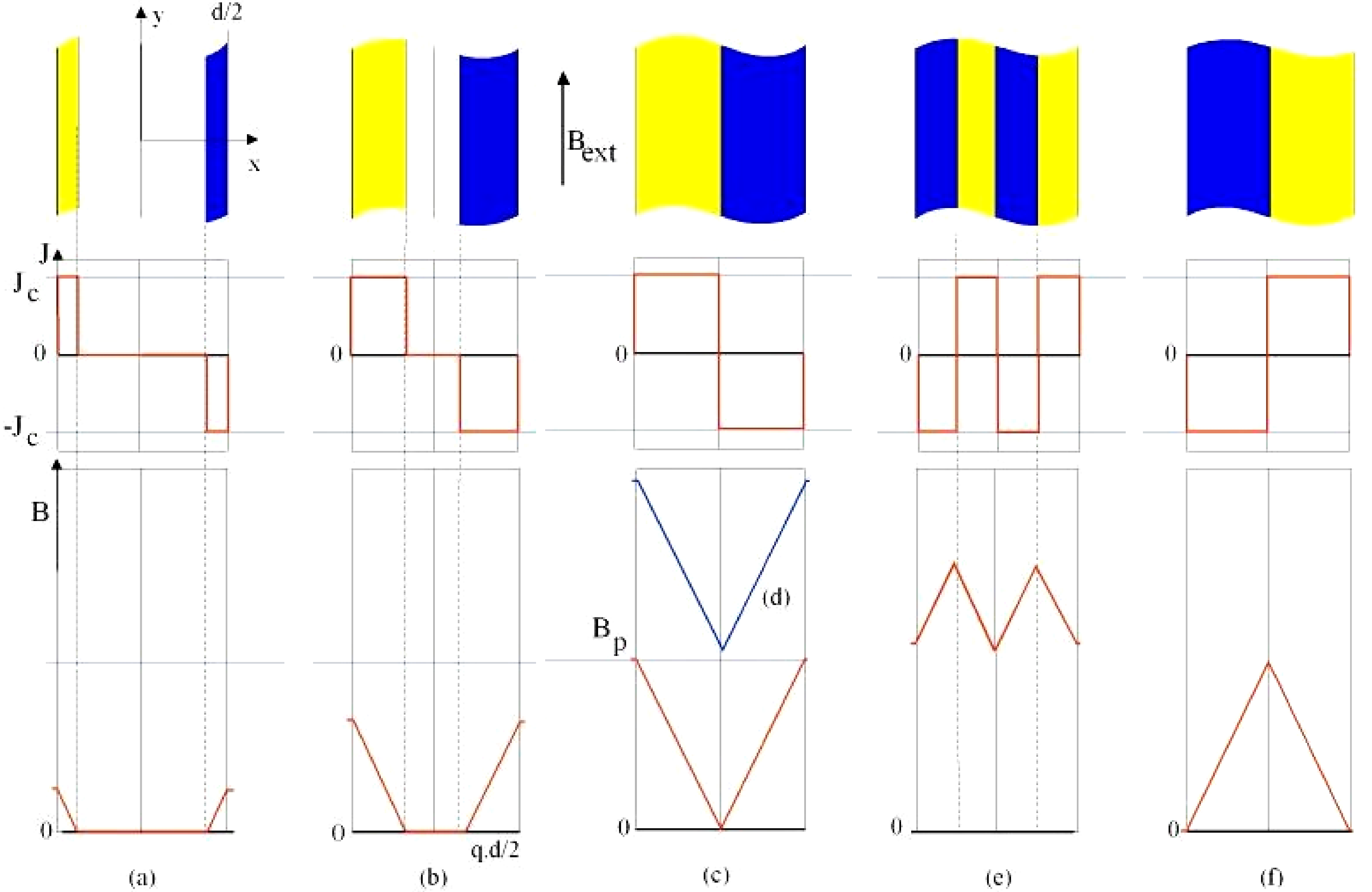
Superconductor Properties

- Hard Superconductors (Type 2)
  - Magnetic field can penetrate
  - Magnetization with hysteresis
- Critical current density J_c
  - Current density at spec. electric field (E_c = 1 µV/cm)
- Critical surface
  - Dependence of J_c on T and B

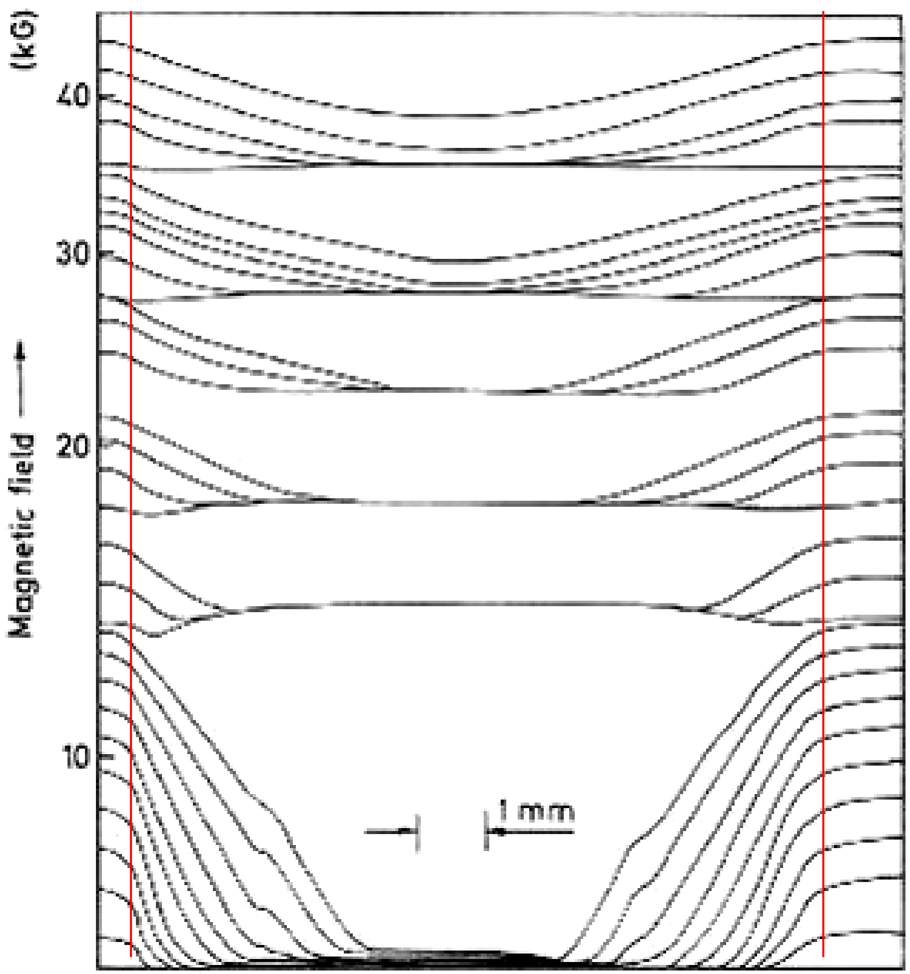




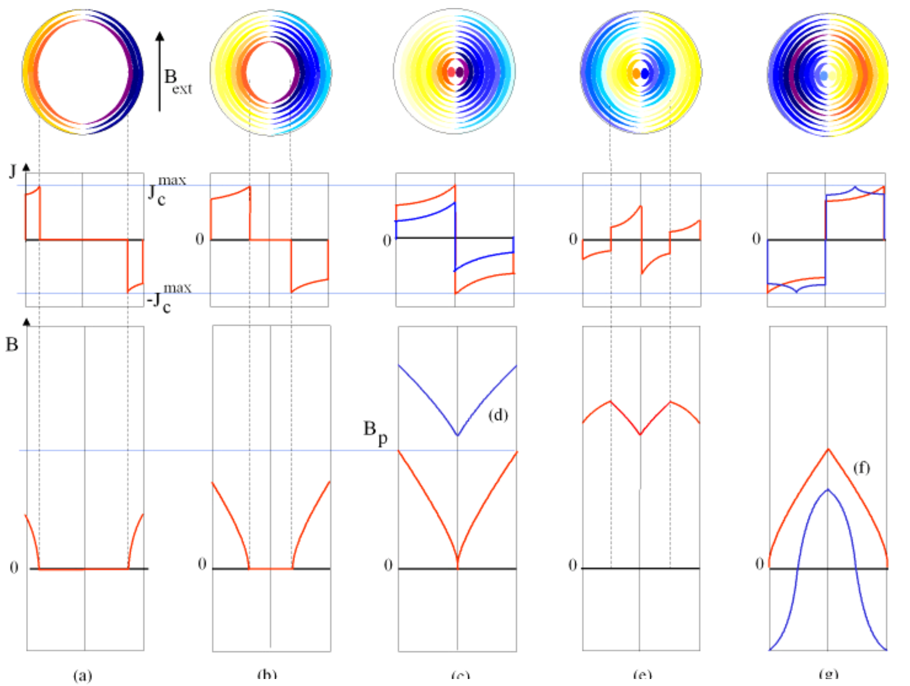
Bean's Critical State Model (CSM)

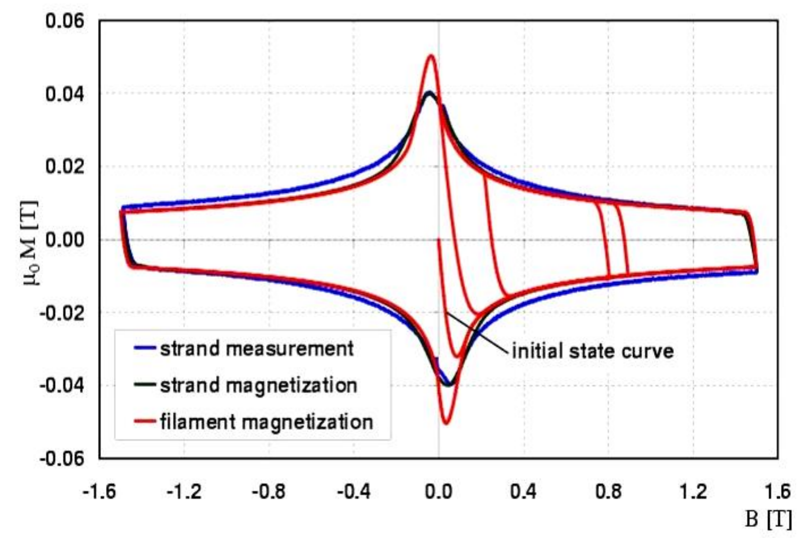


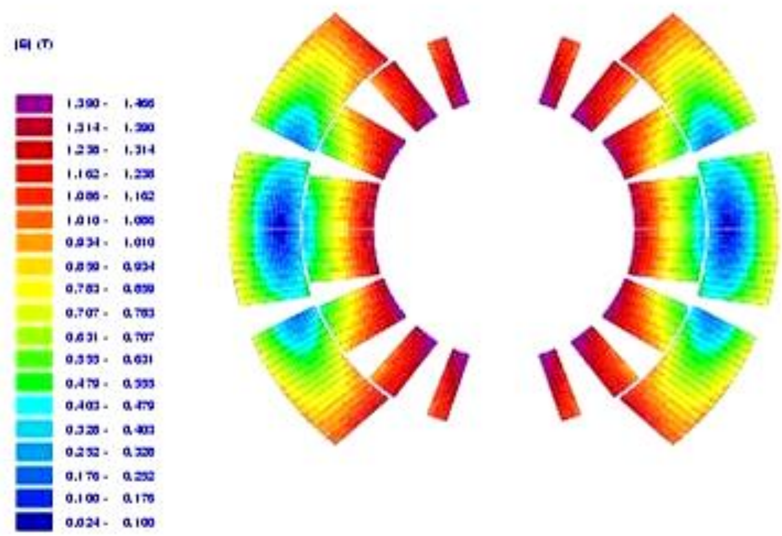
Screening Field in a Slab

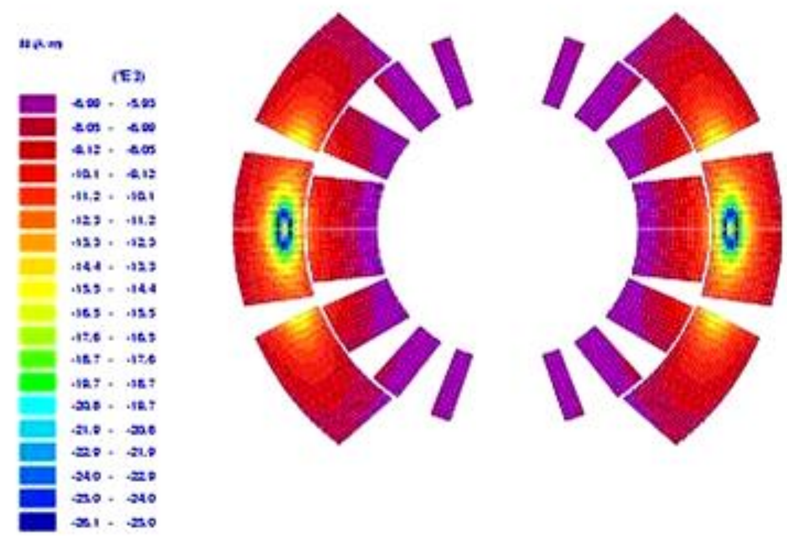


Superconducting Magnetization (Hysteresis Model)





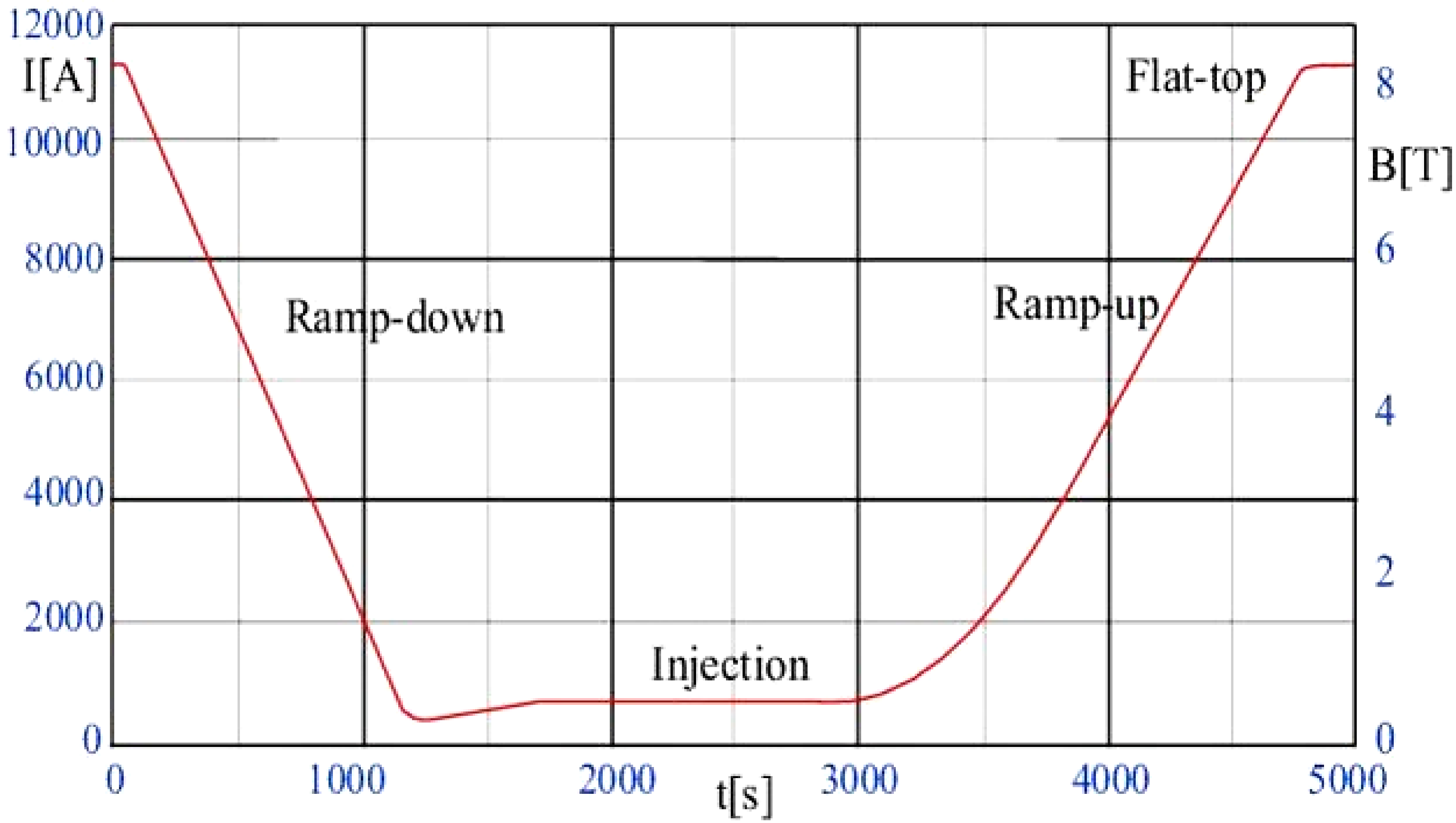




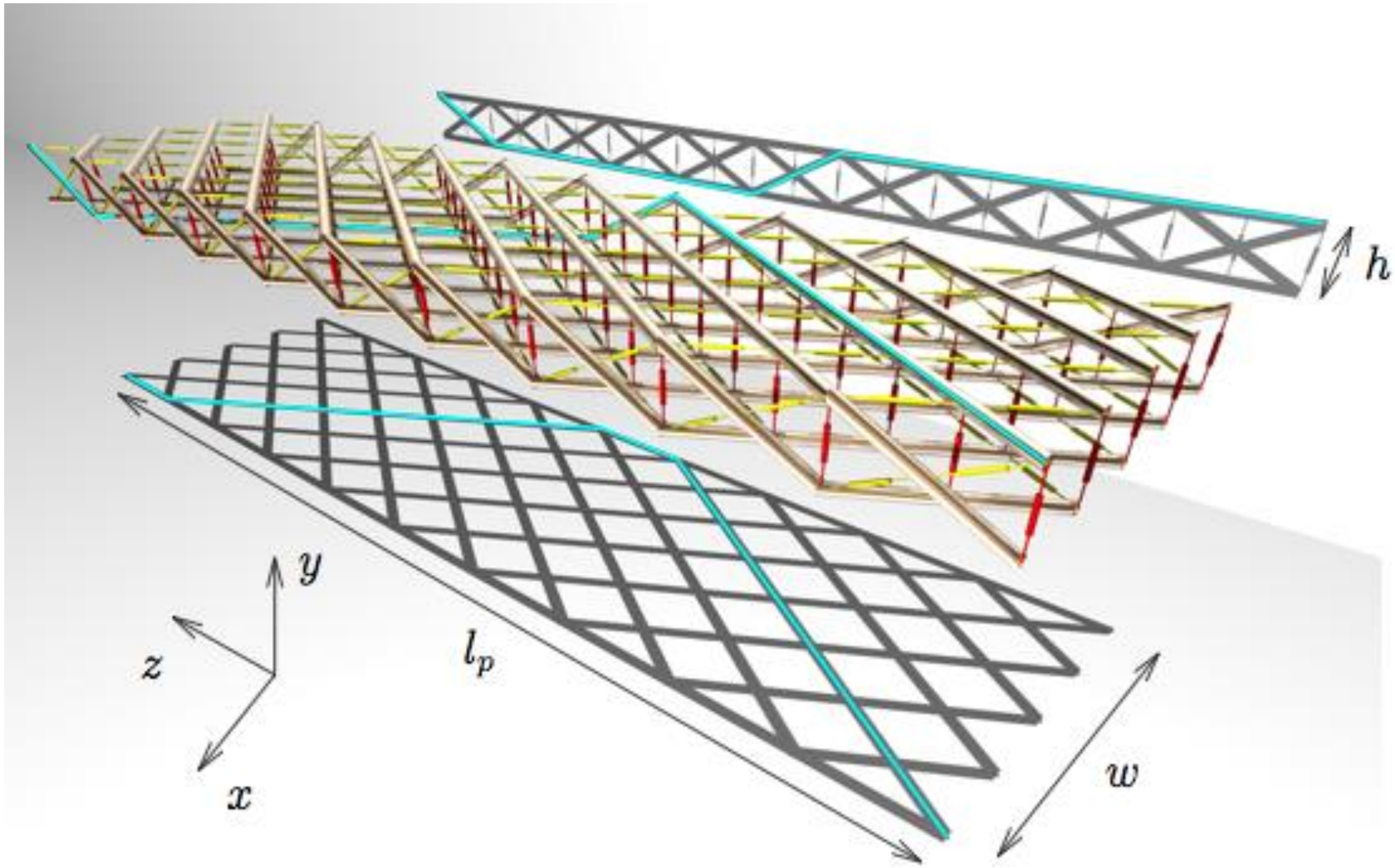
The LHC Excitation Cycle

$$V \approx 2\,E / I\,t$$

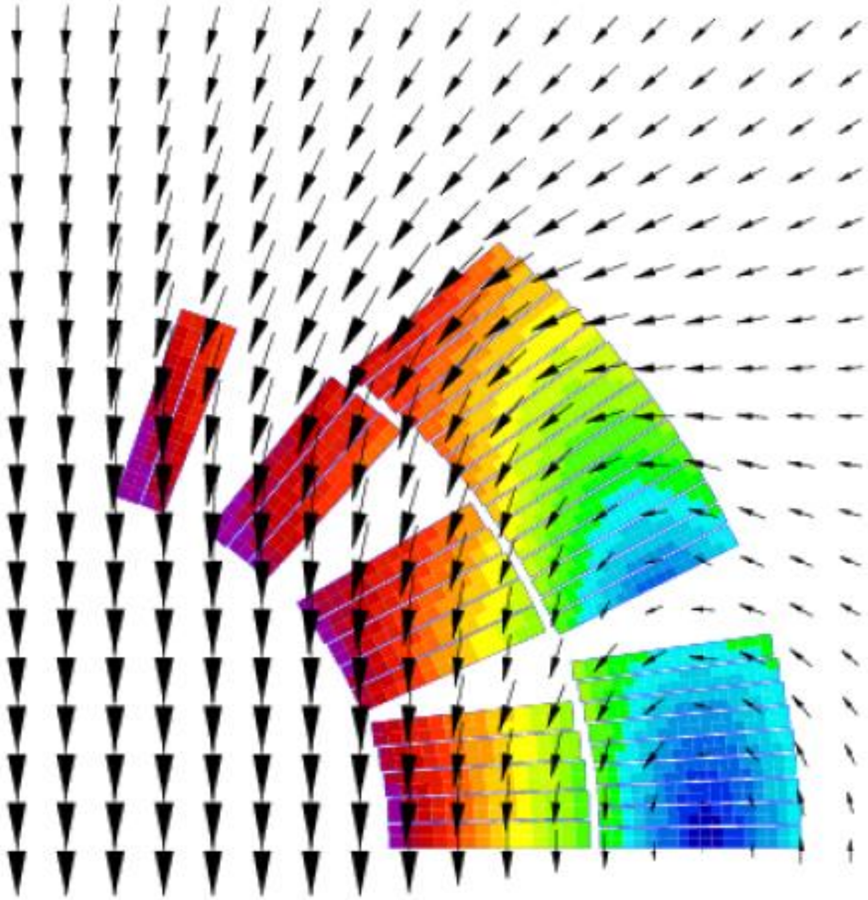
E = 1.15 TJ (320 kWh), I = 11800 A, Ramp rate 10 A/s, 155 V

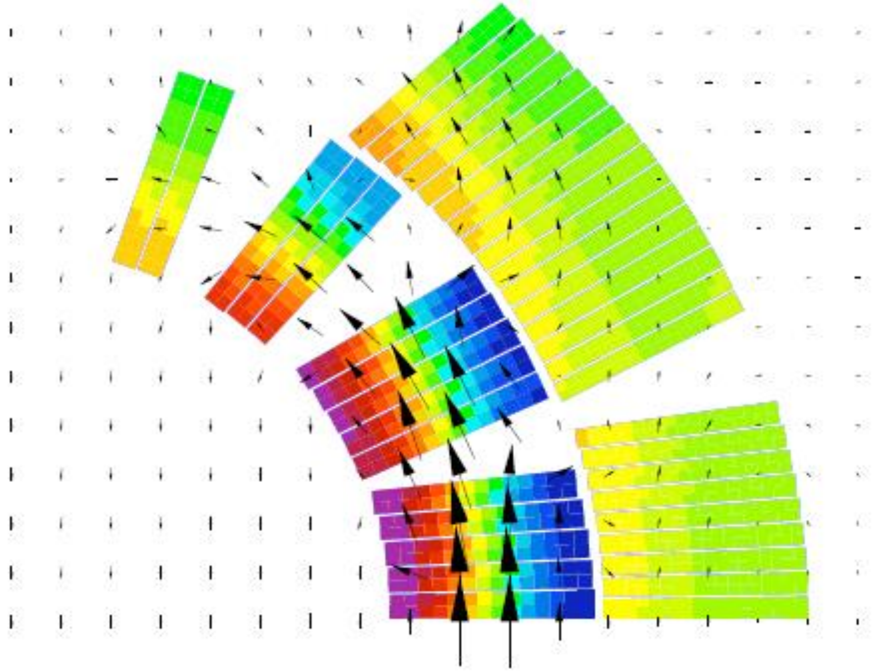


Eddy Currents in Rutherford Cables

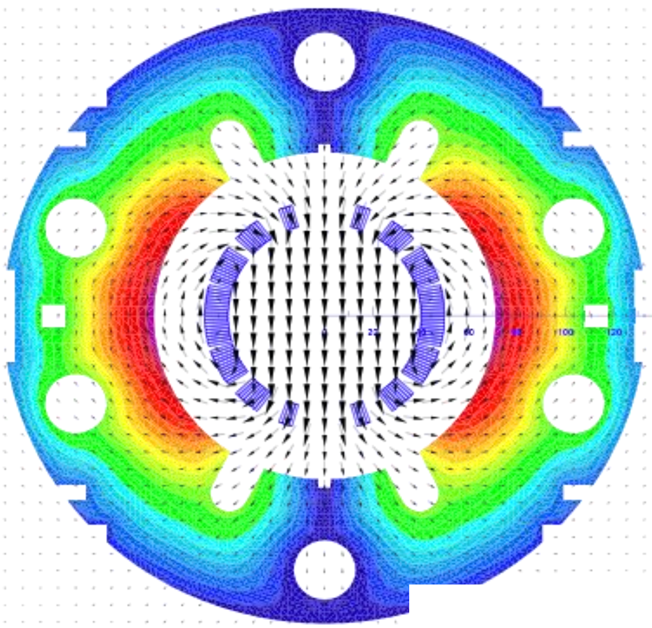


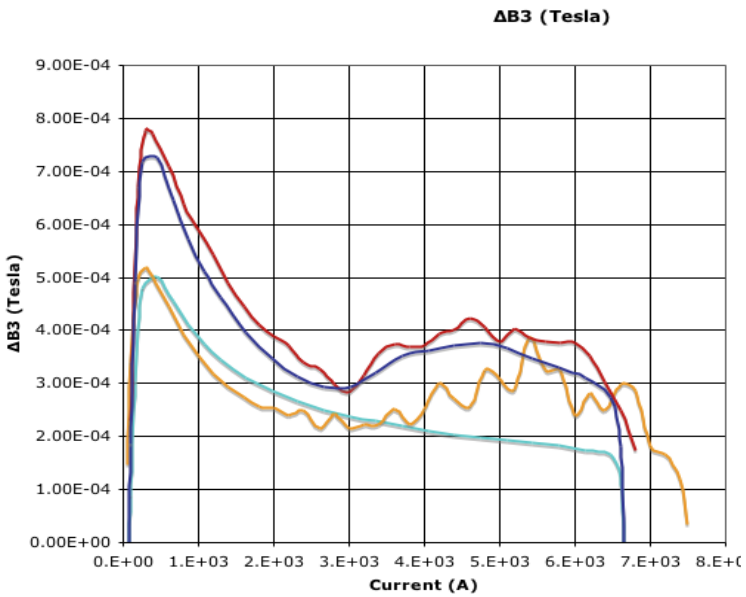
Field Generated by ISCC

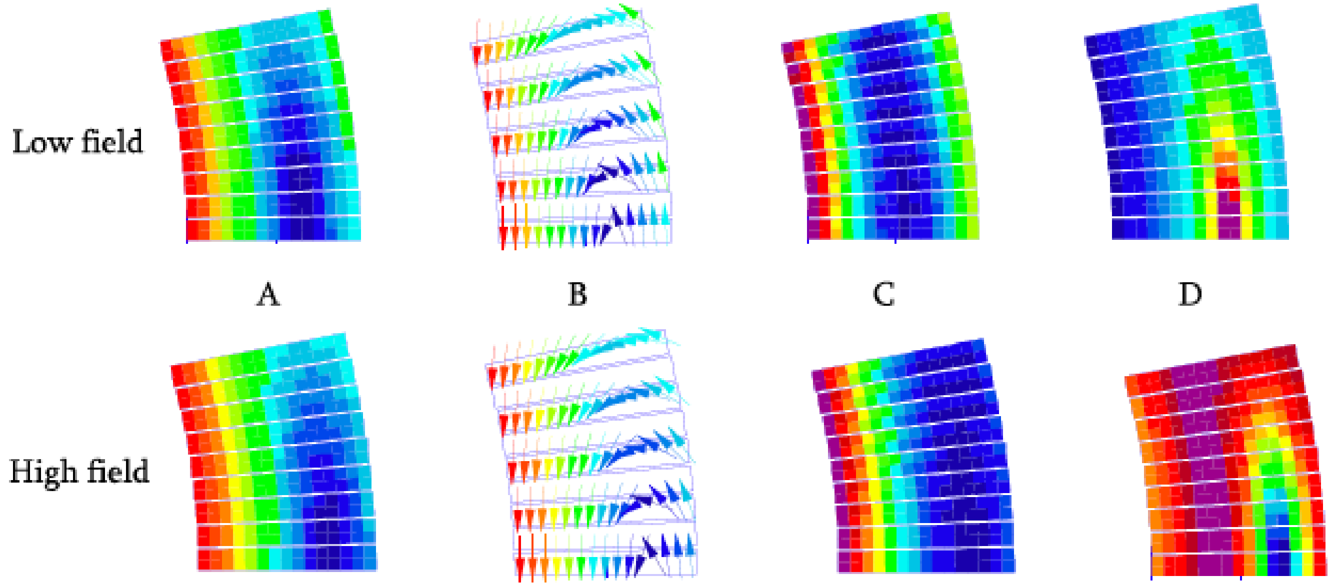




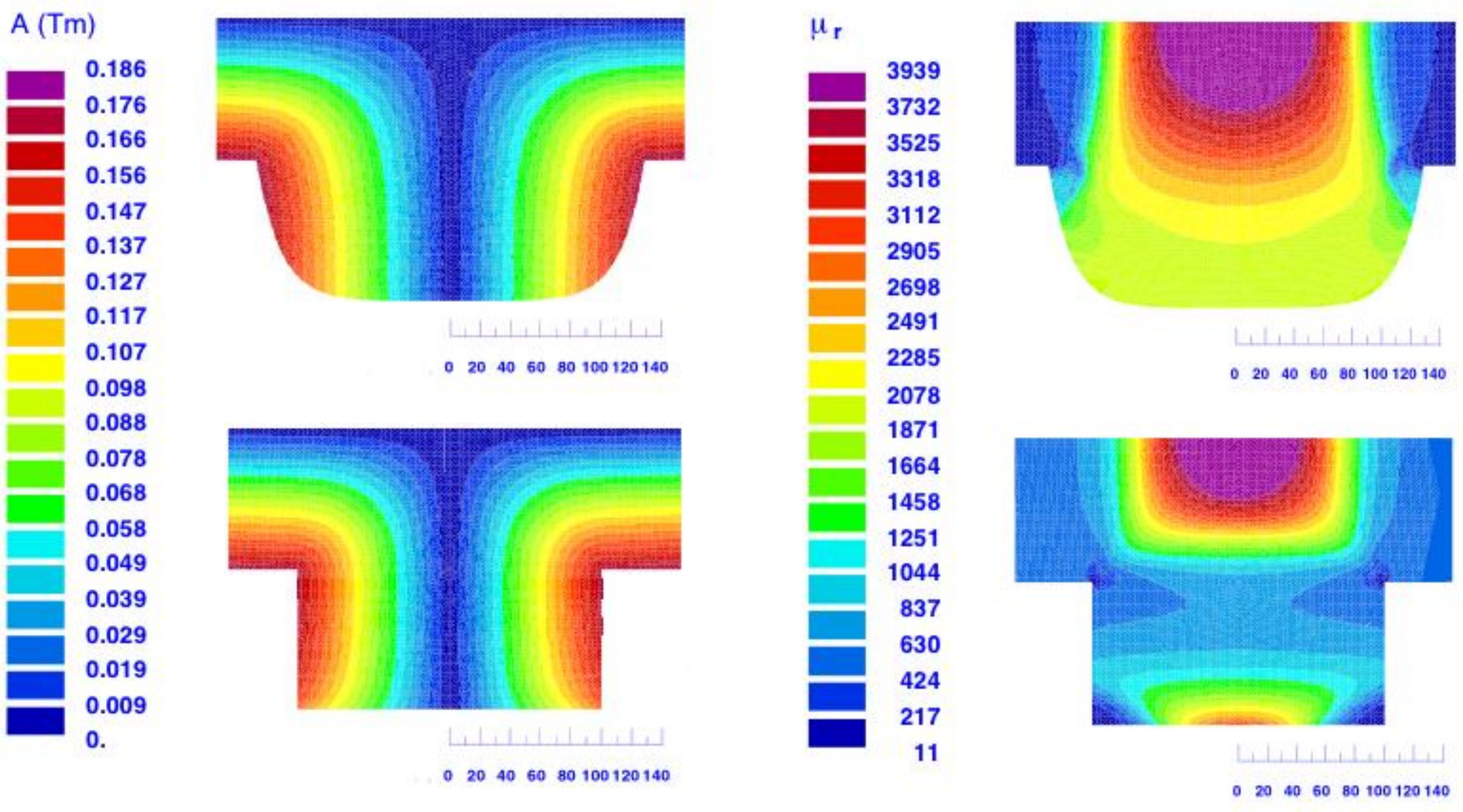
2-D Transient Field Computation for GSI-001



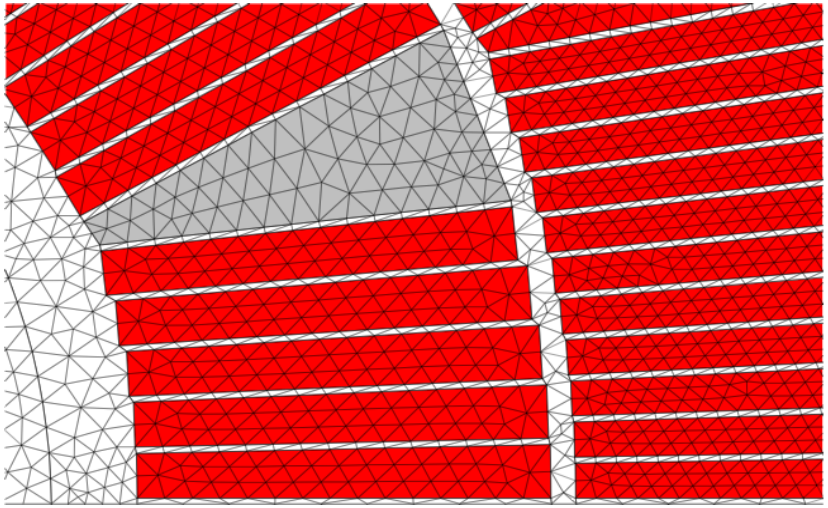


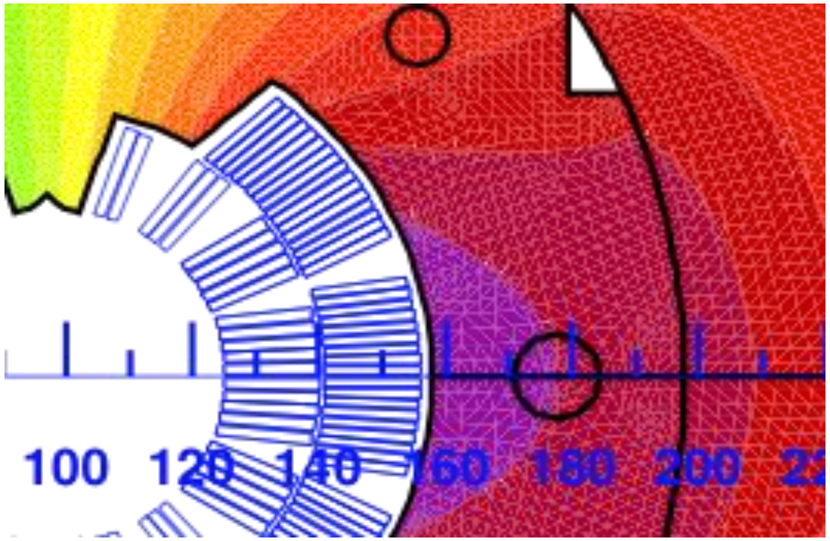


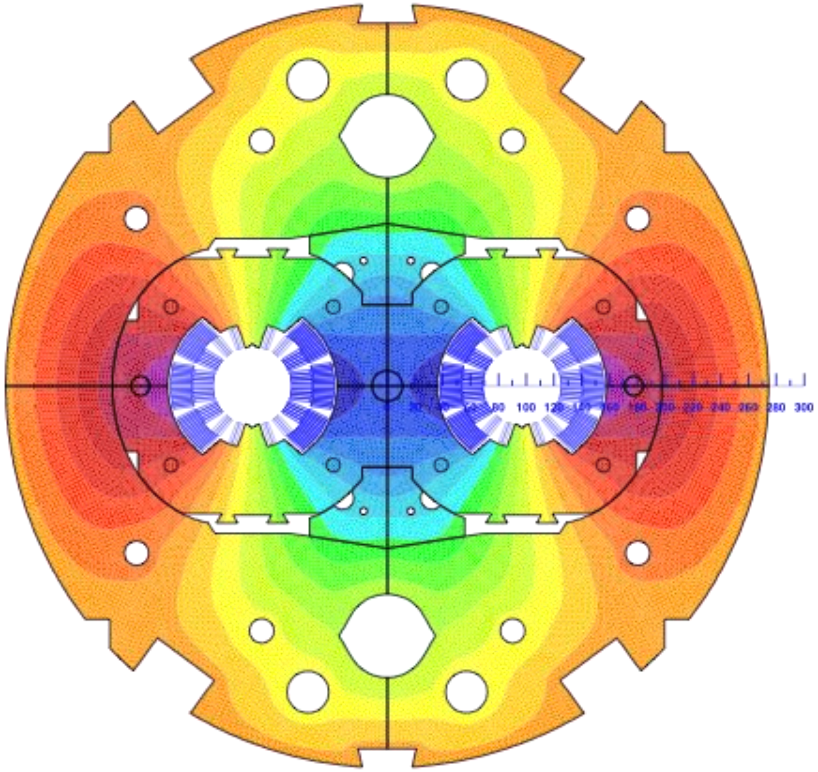
Rogoswki Profiles



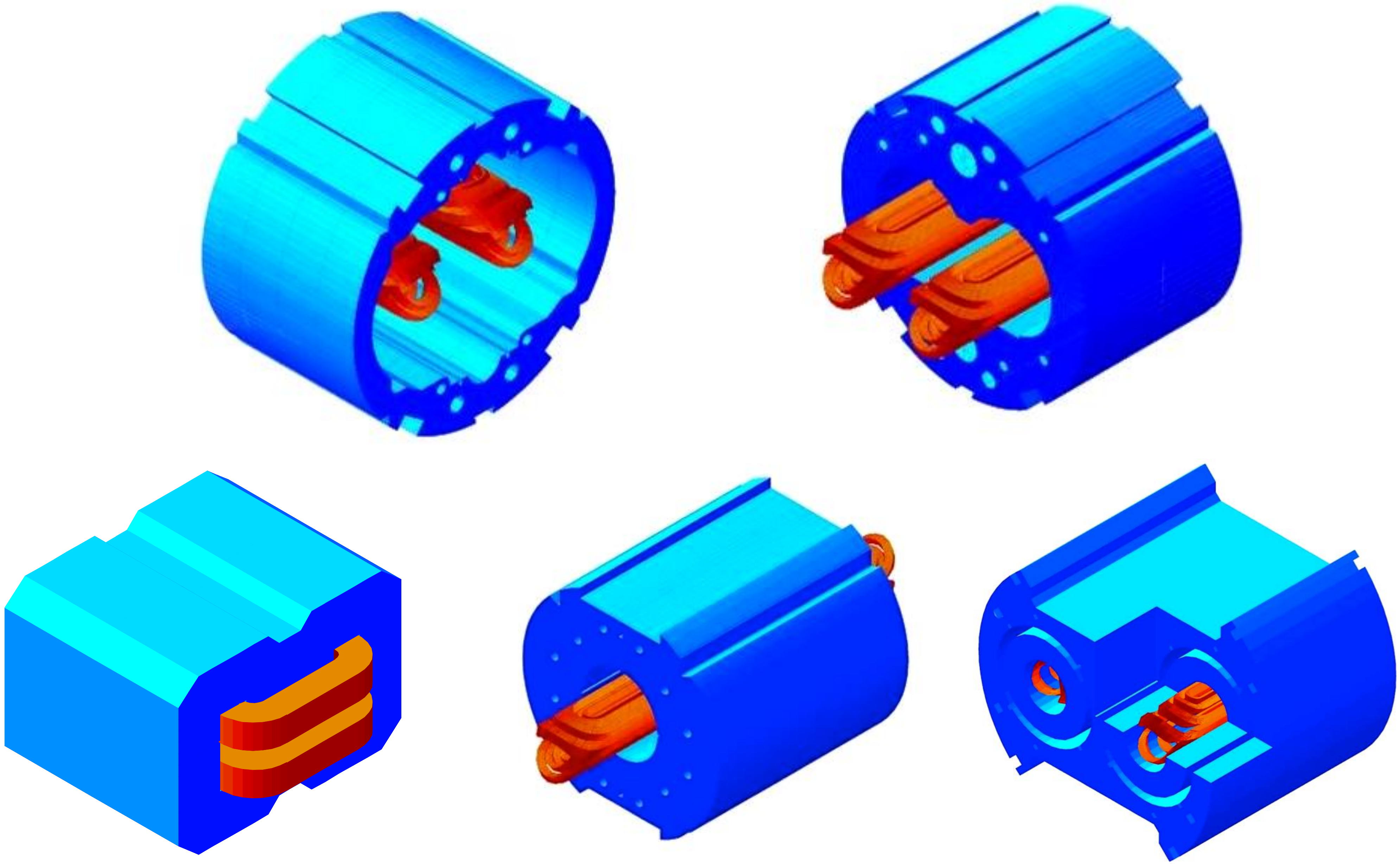
Finite-Element / Boundary-Element Coupling



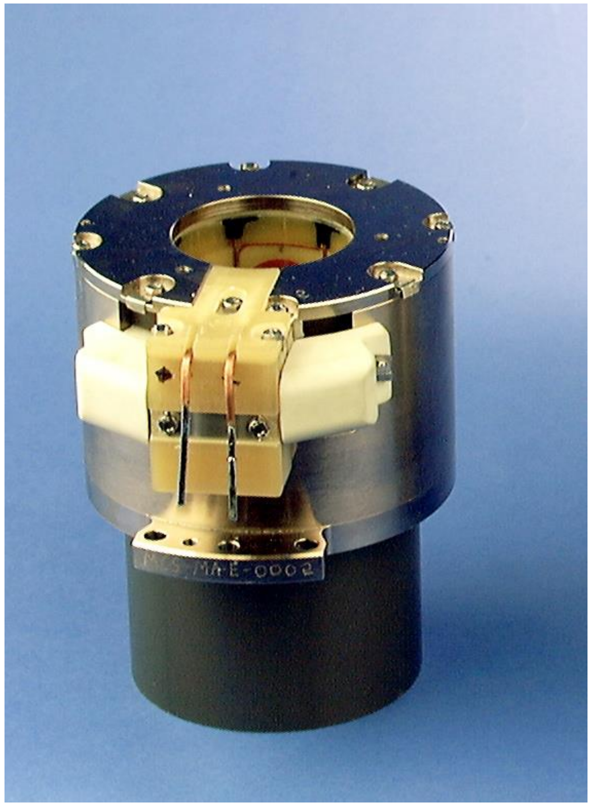




Magnet Extremities

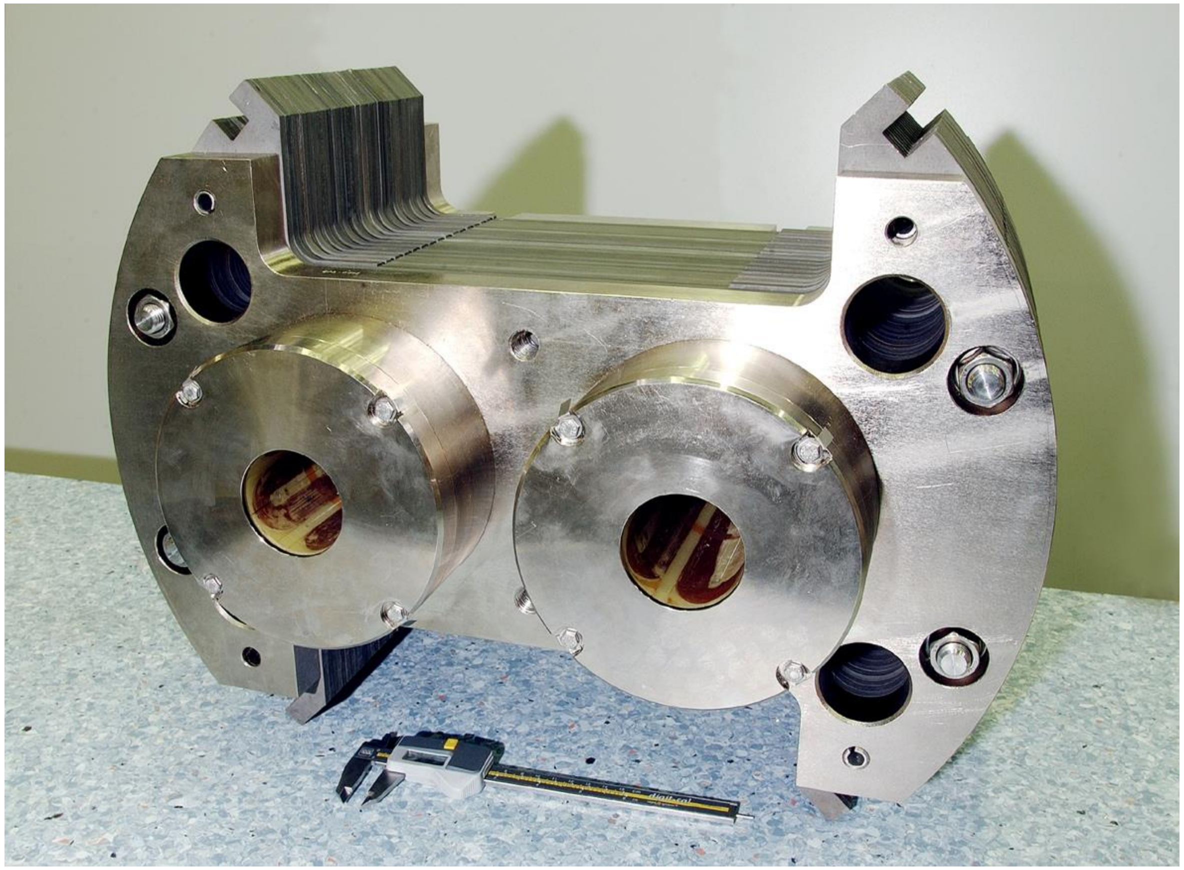


Corrector Magnets

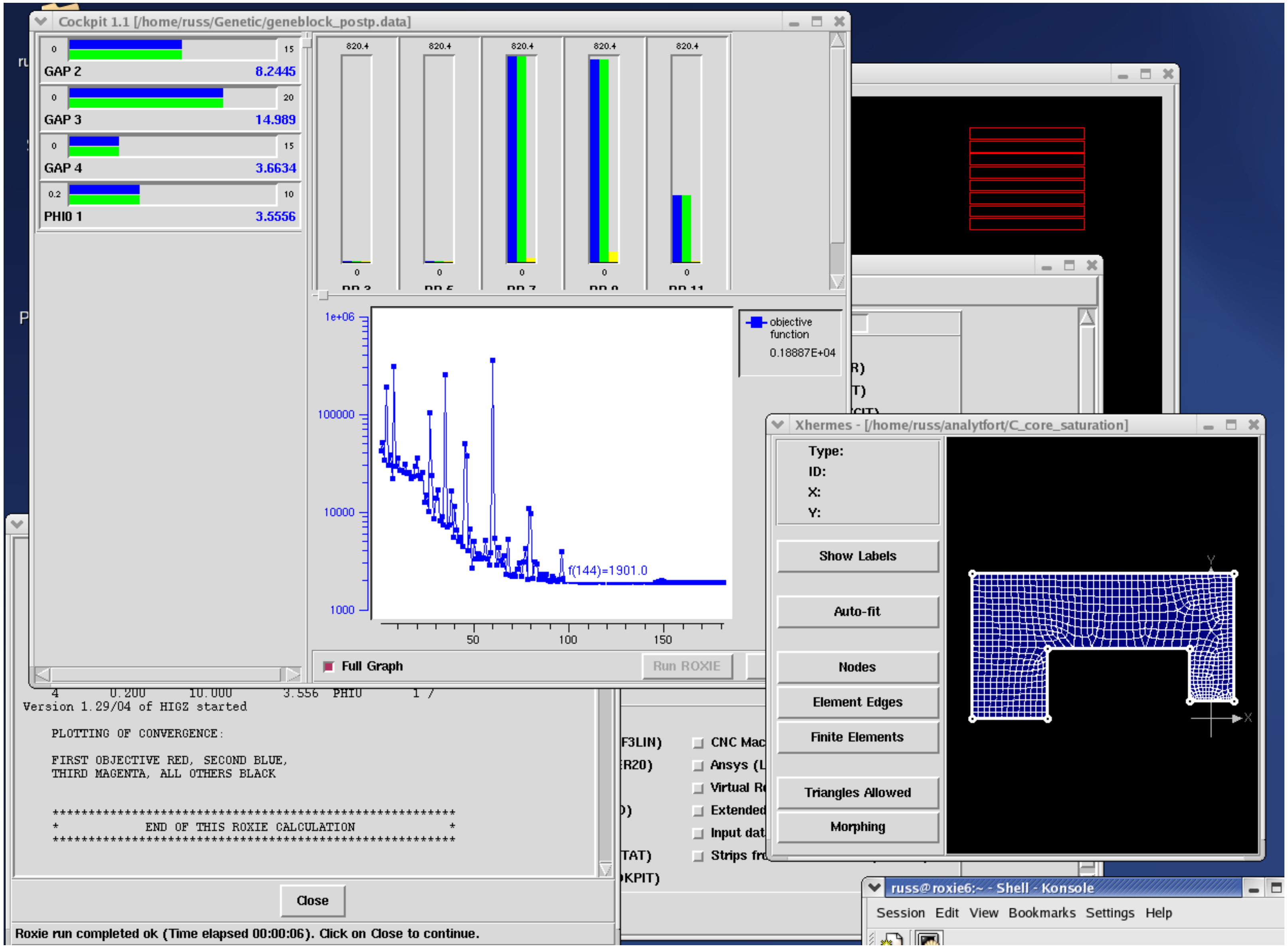


Sextupole-spool pieces

Octupole



The CERN Field Computation Program ROXIE



Objectives for the ROXIE Development

- ➔ Automatic generation of coil and yoke geometries
  - Features: Layers, coil-blocks, conductors, strands, holes, keys
- ➔ Field computation specially suited for magnet design (Ar, BEM-FEM)
  - No meshing of the coil
  - No artificial boundary conditions
  - Higher order quadrilateral meshes, Parametric mesh generator
  - Modeling of superconductor magnetization
- ➔ Mathematical optimization techniques
  - Genetic optimization, Pareto optimization, Search algorithms
- ➔ CAD/CAM interfaces
  - Drawings, End-spacer design and manufacture

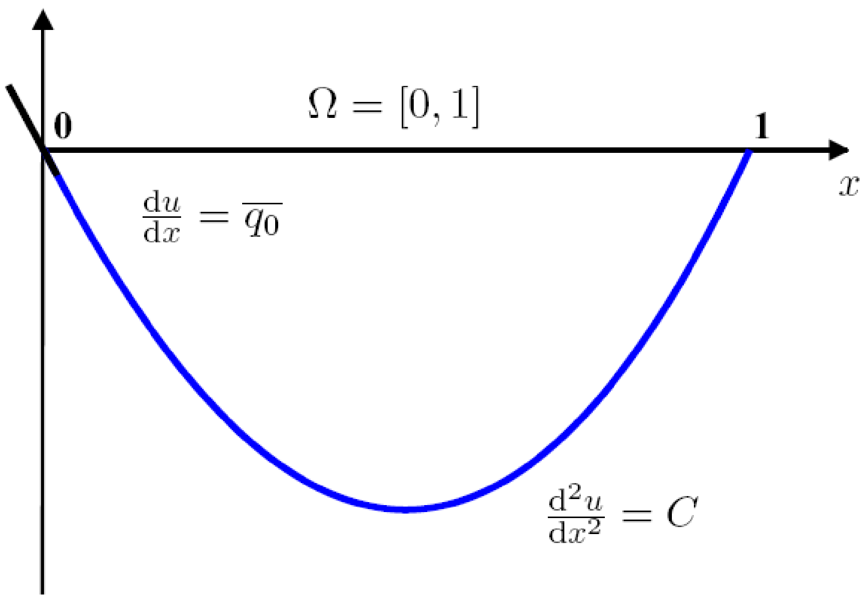
Integrated Design

- Feature based geometry modeling
- Conceptual design using genetic optimization
- Minimization of iron saturation effects (BEM-FEM)
- Calculation of superconductor magnetization
- Eddy-currents in Rutherford cables
- Quenchsimulation
- 3D-Coil geometry and yoke optimization
- Sensitivity analysis
- Making of drawings, rapid prototyping
- Inverse field computation

The Model Problem (1-D)

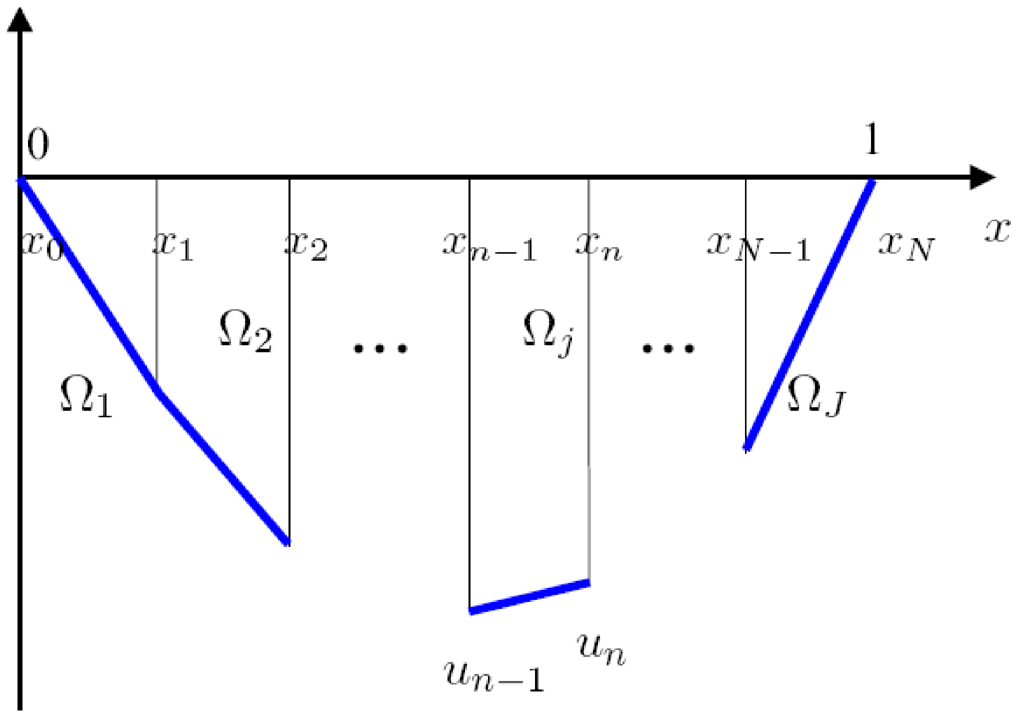
$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} = f(x), \qquad x \in \Omega$$

$$u(x)|_{x=0} = \overline{u}_0 \qquad u(x)|_{x=1} = \overline{u}_1 \quad \text{or} \quad \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=1} = \overline{q}_1$$



$$u(x) = \frac{C}{2}\left(x^2 - x\right)$$

Shape Functions



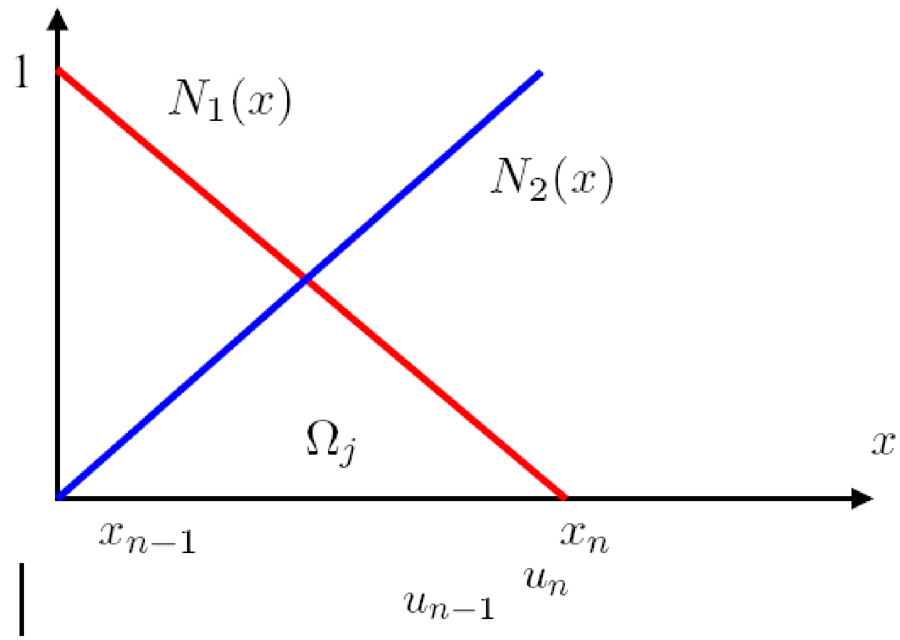
$$\Omega = \bigcup_{j=1}^{J} \Omega_j$$

$$\Omega_j = [x_{n-1}, x_n]$$

$$u_j(x) = \alpha_{j1} + \alpha_{j2} x \qquad x \in \Omega_j$$

$$u_{n-1} = \alpha_{j1} + \alpha_{j2} x_{n-1} \qquad u_n = \alpha_{j1} + \alpha_{j2} x_n$$

Shape Functions



Cramer's rule

$$\alpha_{j1} = \frac{\begin{vmatrix} u_{n-1} & x_{n-1} \\ u_n & x_n \end{vmatrix}}{\begin{vmatrix} 1 & x_{n-1} \\ 1 & x_n \end{vmatrix}}$$

$$\alpha_{j1} = \frac{x_n u_{n-1} - x_{n-1} u_n}{x_n - x_{n-1}}$$

$$\alpha_{j2} = \frac{u_n - u_{n-1}}{x_n - x_{n-1}}$$

$$u_j(x) = \alpha_{j1} + \alpha_{j2} x = \frac{x_n - x}{x_n - x_{n-1}} u_{n-1} + \frac{-x_{n-1} + x}{x_n - x_{n-1}} u_n$$

What have we won? We can express the field in the element as a function of the node potentials using known polynomials in the spatial coordinates

The Weighted Residual

$$R(x) := \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} - f(x)$$

$$\int_\Omega w(x) R(x)\, \mathrm{d}\Omega = \int_\Omega w(x) \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2}\, \mathrm{d}\Omega - \int_\Omega w(x) f(x)\, \mathrm{d}\Omega = 0$$

$$\int_a^b \phi \psi' \, \mathrm{d}x = [\phi\psi]_a^b - \int_a^b \phi' \psi \, \mathrm{d}x \qquad w(x) = \phi \qquad \frac{\mathrm{d}u(x)}{\mathrm{d}x} = \psi$$

$$-\int_\Omega \frac{\mathrm{d}w(x)}{\mathrm{d}x} \frac{\mathrm{d}u(x)}{\mathrm{d}x}\, \mathrm{d}\Omega + \left[w(x) \frac{\mathrm{d}u(x)}{\mathrm{d}x} \right]_0^1 - \int_\Omega w(x) f(x)\, \mathrm{d}\Omega = 0$$

What have we won? Removal of the second derivative, a way to incorporate Neumann boundary conditions

Galerkin's Method

$$\int_\Omega \frac{\mathrm{d}w(x)}{\mathrm{d}x}\frac{\mathrm{d}u(x)}{\mathrm{d}x}\,\mathrm{d}\Omega = -\int_\Omega w(x)f(x)\,\mathrm{d}\Omega$$

$$\int_{\Omega_j} \frac{\mathrm{d}w_l(x)}{\mathrm{d}x}\sum_{k=1,2}\frac{\mathrm{d}N_{jk}(x)}{\mathrm{d}x}u^{(k)}\,\mathrm{d}\Omega_j = -\int_{\Omega_j} w_l(x)f(x)\,\mathrm{d}\Omega_j\,, \qquad l = 1,2.$$

$$\int_{\Omega_j} \frac{\mathrm{d}N_{jl}(x)}{\mathrm{d}x}\sum_{k=1,2}\frac{\mathrm{d}N_{jk}(x)}{\mathrm{d}x}u^{(k)}\,\mathrm{d}\Omega_j = -\int_{\Omega_j} N_{jk}(x)f(x)\,\mathrm{d}\Omega_j\,, \qquad l = 1,2$$

$$\int_{x_{n-1}}^{x_n}\left(\frac{\mathrm{d}N_{j1}}{\mathrm{d}x}\frac{\mathrm{d}N_{j1}}{\mathrm{d}x}u_{n-1} + \frac{\mathrm{d}N_{j1}}{\mathrm{d}x}\frac{\mathrm{d}N_{j2}}{\mathrm{d}x}u_n\right)\mathrm{d}x = -\int_{x_{n-1}}^{x_n} N_{j1}f(x)\,\mathrm{d}x$$

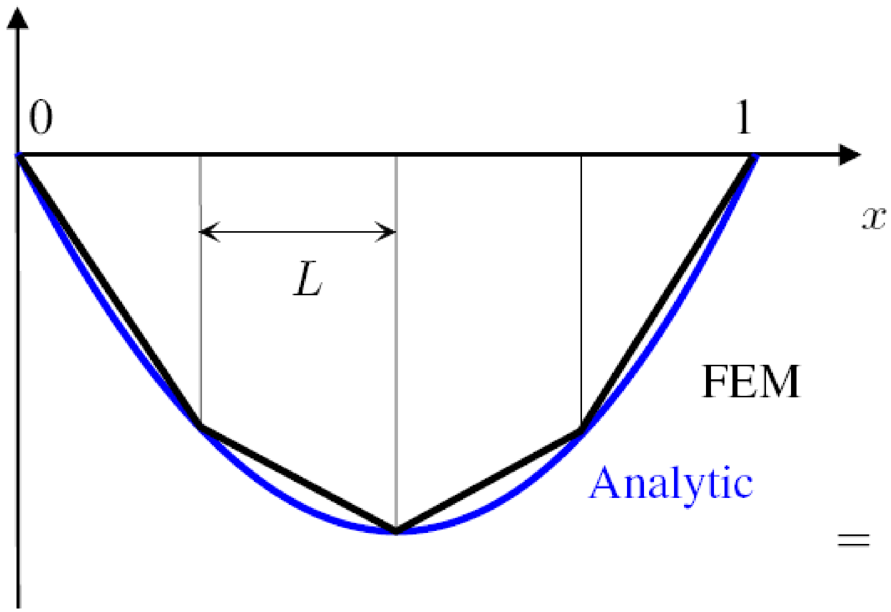
$$\int_{x_{n-1}}^{x_n}\left(\frac{\mathrm{d}N_{j2}}{\mathrm{d}x}\frac{\mathrm{d}N_{j1}}{\mathrm{d}x}u_{n-1} + \frac{\mathrm{d}N_{j2}}{\mathrm{d}x}\frac{\mathrm{d}N_{j2}}{\mathrm{d}x}u_n\right)\mathrm{d}x = -\int_{x_{n-1}}^{x_n} N_{j2}f(x)\,\mathrm{d}x$$

$$\boxed{[k_j]\{u_j\} = \{f_j\}}$$

Linear equation system for the node potentials

Numerical Example

4 finite elements $\Omega_j, j = 1, ..., 4$ of equidistant length L



$$[k_j] = \int_{x_{n-1}}^{x_n} \begin{pmatrix} \frac{dN_{j1}}{dx}\frac{dN_{j1}}{dx} & \frac{dN_{j1}}{dx}\frac{dN_{j2}}{dx} \\ \frac{dN_{j2}}{dx}\frac{dN_{j1}}{dx} & \frac{dN_{j2}}{dx}\frac{dN_{j2}}{dx} \end{pmatrix} dx$$

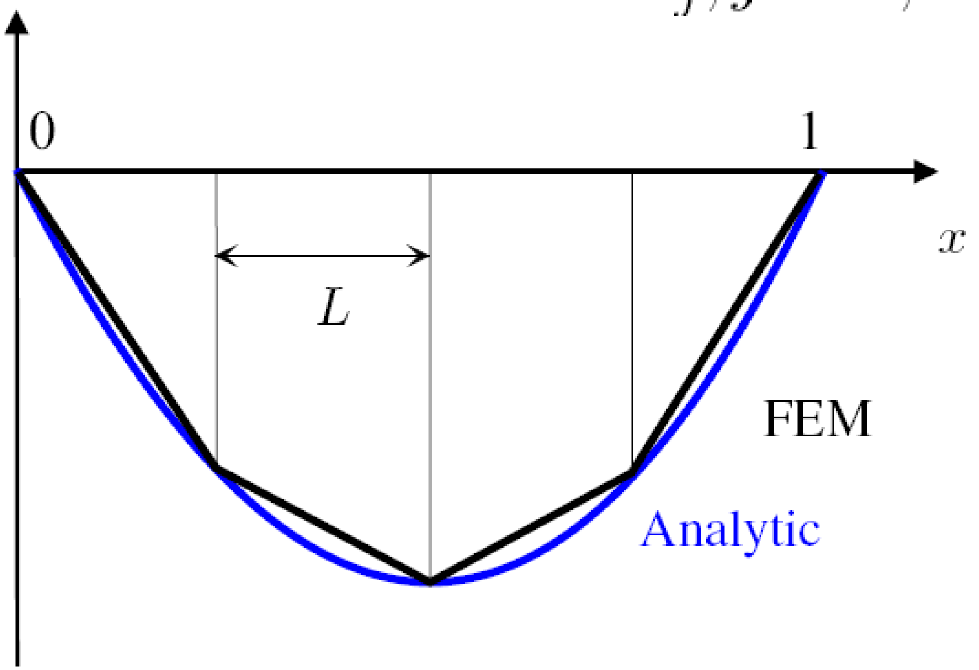
$$= \int_{x_{n-1}}^{x_n} \begin{pmatrix} \frac{1}{(x_n - x_{n-1})^2} & \frac{-1}{(x_n - x_{n-1})^2} \\ \frac{-1}{(x_n - x_{n-1})^2} & \frac{1}{(x_n - x_{n-1})^2} \end{pmatrix} dx = \begin{pmatrix} \frac{1}{L} & \frac{-1}{L} \\ \frac{-1}{L} & \frac{1}{L} \end{pmatrix}$$

$$\{f_j\} = -\int_{x_{n-1}}^{x_n} \begin{pmatrix} N_{j1} \\ N_{j2} \end{pmatrix} C\,dx = -C \int_{x_{n-1}}^{x_n} \begin{pmatrix} \frac{x_n - x}{x_n - x_{n-1}} \\ \frac{-x_{n-1} + x}{x_n - x_{n-1}} \end{pmatrix} dx$$

$$= -\frac{C}{2L} \begin{pmatrix} 2x_n x - x^2 \\ -2x_{n-1}x + x^2 \end{pmatrix} \Bigg|_{x_{n-1}}^{x_n} = -\frac{C}{2L} \begin{pmatrix} (x_n - x_{n-1})^2 \\ (x_{n-1} - x_n)^2 \end{pmatrix} = -\begin{pmatrix} 0.5\,CL \\ 0.5\,CL \end{pmatrix}$$

Numerical Example

4 finite elements $\Omega_j, j = 1, ..., 4$ of equidistant length L



$$\begin{pmatrix} \frac{1}{L} & \frac{-1}{L} & 0 & 0 & 0 \\ \frac{1}{L} & \frac{2}{L} & \frac{-1}{L} & 0 & 0 \\ 0 & \frac{-1}{L} & \frac{2}{L} & \frac{-1}{L} & 0 \\ 0 & 0 & \frac{-1}{L} & \frac{2}{L} & \frac{-1}{L} \\ 0 & 0 & 0 & \frac{-1}{L} & \frac{1}{L} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} = - \begin{pmatrix} 0.5CL \\ CL \\ CL \\ CL \\ 0.5CL \end{pmatrix}$$

$$\begin{pmatrix} \frac{2}{L} & \frac{-1}{L} & 0 \\ \frac{-1}{L} & \frac{2}{L} & \frac{-1}{L} \\ 0 & \frac{-1}{L} & \frac{2}{L} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} CL \\ CL \\ CL \end{pmatrix}$$

Essential boundary conditions (Dirichlet)

$$\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} \frac{3L}{4} & \frac{L}{2} & \frac{L}{4} \\ \frac{L}{2} & L & \frac{L}{2} \\ \frac{L}{4} & \frac{L}{2} & \frac{2L}{4} \end{pmatrix} \begin{pmatrix} CL \\ CL \\ CL \end{pmatrix} = \begin{pmatrix} -0.375 \\ -0.5 \\ -0.375 \end{pmatrix}$$

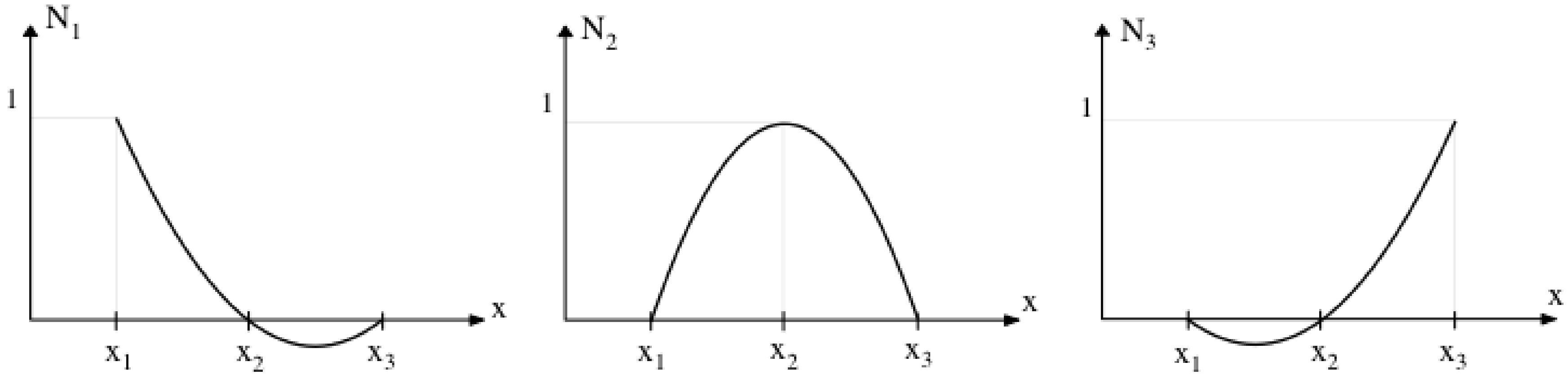
Higher order elements

$$u^{(1)} = \alpha_{j1} + \alpha_{j2}x_1 + \alpha_{j3}x_1^2$$
$$u^{(2)} = \alpha_{j1} + \alpha_{j2}x_2 + \alpha_{j3}x_2^2$$
$$u^{(3)} = \alpha_{j1} + \alpha_{j2}x_3 + \alpha_{j3}x_3^2$$

$$u_j(x) = \sum_{k=1}^{3} N_{jk}(x)u^{(k)}$$

$$N_{j1}(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}, \qquad N_{j2}(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$N_{j3}(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}.$$

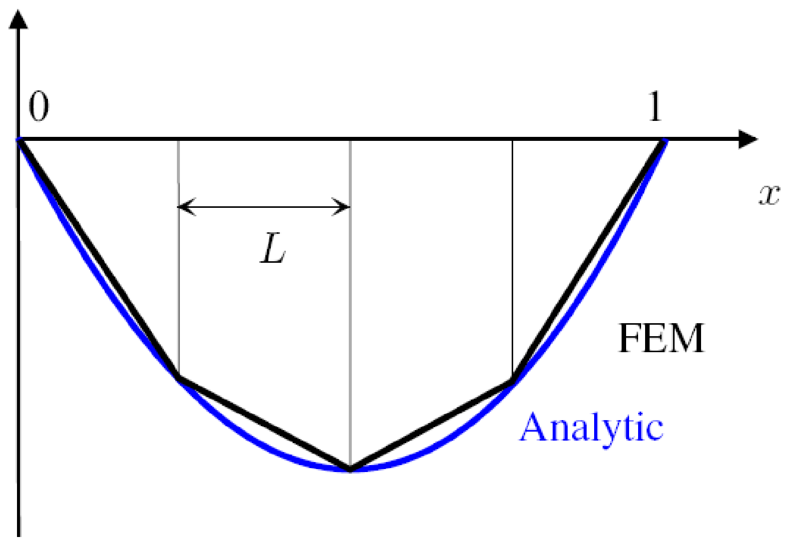


Two Quadratic Elements (l = 2 L)

$$[k_j] = \begin{pmatrix} \frac{7}{6l} & \frac{-8}{6l} & \frac{1}{6l} \\ \frac{-8}{6l} & \frac{16}{6l} & \frac{-8}{6l} \\ \frac{1}{6l} & \frac{-8}{6l} & \frac{7}{6l} \end{pmatrix}$$

$$\{f_j\} = -\frac{1}{3}c \begin{pmatrix} l \\ 4l \\ l \end{pmatrix}$$

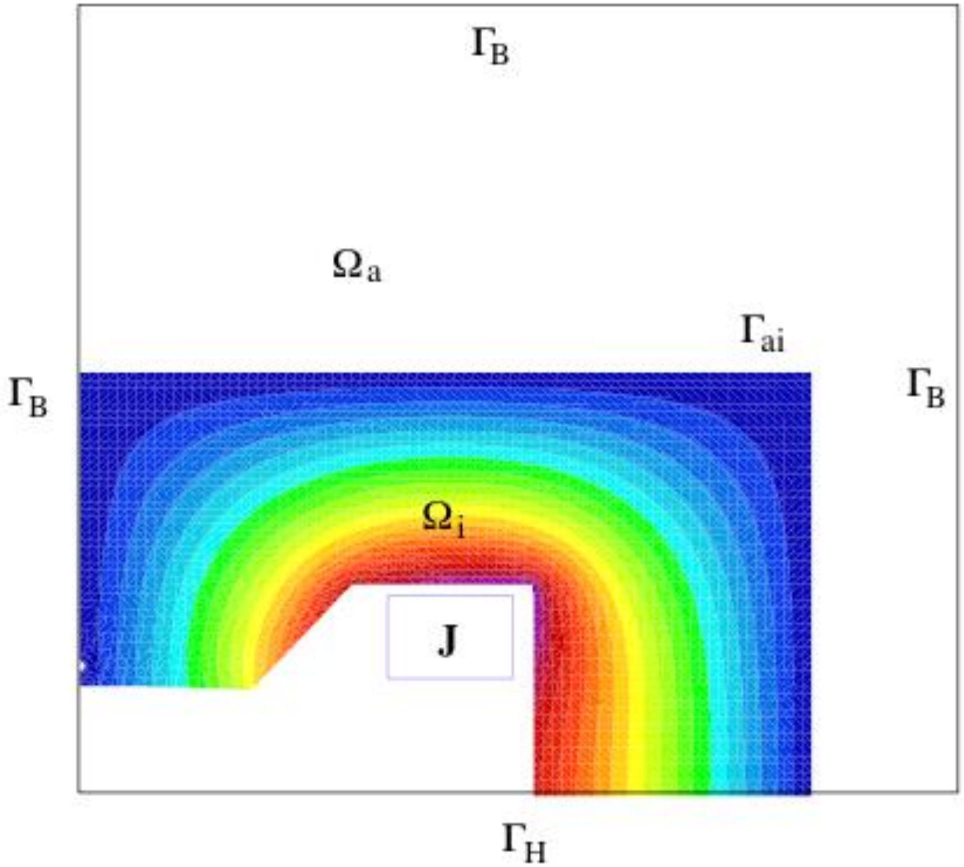
$$\begin{pmatrix} \frac{2}{l} & \frac{-1}{l} & 0 \\ \frac{-1}{l} & \frac{2}{l} & \frac{-1}{l} \\ 0 & \frac{-1}{l} & \frac{2}{l} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} cl \\ cl \\ cl \end{pmatrix}$$



$$\begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} \frac{3l}{4} & \frac{l}{2} & \frac{l}{4} \\ \frac{l}{2} & l & \frac{l}{2} \\ \frac{l}{4} & \frac{l}{2} & \frac{3l}{4} \end{pmatrix} \begin{pmatrix} cl \\ cl \\ cl \end{pmatrix} = \begin{pmatrix} -0.375 \\ -0.5 \\ -0.375 \end{pmatrix}$$

Weak Form in the FEM Problem

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A} = \mathbf{J} \quad \text{in } \Omega$$



$$\begin{aligned}
\mathbf{A} \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_H, \\
\frac{1}{\mu} \operatorname{div} \mathbf{A} &= 0 \quad \text{on } \Gamma_B, \\
\mathbf{n} \times (\mathbf{A} \times \mathbf{n}) &= \mathbf{0} \quad \text{on } \Gamma_B, \\
\mathbf{n} \times \left(\frac{1}{\mu} (\operatorname{curl} \mathbf{A}) \times \mathbf{n} \right) &= \mathbf{0} \quad \text{on } \Gamma_H, \\
\left[\frac{1}{\mu} \operatorname{div} \mathbf{A} \right]_{ai} &= 0 \quad \text{on } \Gamma_{ai}, \\
\left[\frac{1}{\mu} (\operatorname{curl} \mathbf{A}) \times \mathbf{n} \right]_{ai} &= \mathbf{0} \quad \text{on } \Gamma_{ai}, \\
[\mathbf{A}]_{ai} &= \mathbf{0} \quad \text{on } \Gamma_{ai}.
\end{aligned}$$

Weak Form in the FEM Problem

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A} = \mathbf{J} \quad \text{in } \Omega$$

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A} - \mathbf{J} = \mathbf{R}$$

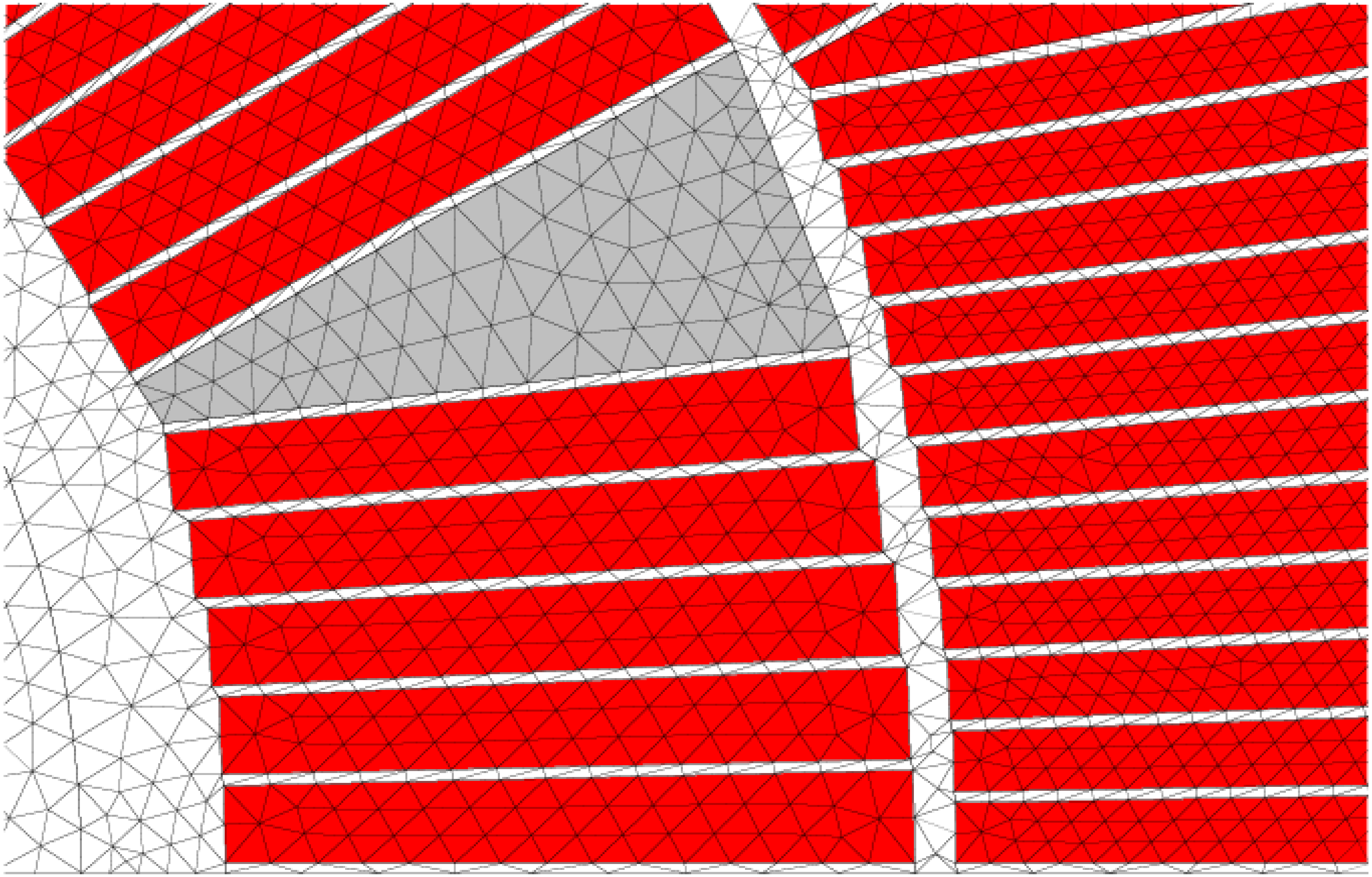
$$\int_\Omega \mathbf{w}_a \cdot \left(\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A} \right) \mathrm{d}\Omega = \int_\Omega \mathbf{w}_a \cdot \mathbf{J} \, \mathrm{d}\Omega \,, \qquad a = 1, 2, 3.$$

Integration by parts

$$\int_\Omega \frac{1}{\mu} \operatorname{curl} \mathbf{w}_a \cdot \operatorname{curl} \mathbf{A} \, \mathrm{d}\Omega + \int_\Omega \frac{1}{\mu} \operatorname{div} \mathbf{w}_a \operatorname{div} \mathbf{A} \, \mathrm{d}\Omega = \int_\Omega \mathbf{w}_a \cdot \mathbf{J} \, \mathrm{d}\Omega$$

Conclusion: 3-D is more complicated than addition just one dimension in space; it's a different mathematics, and thus often a separate software package

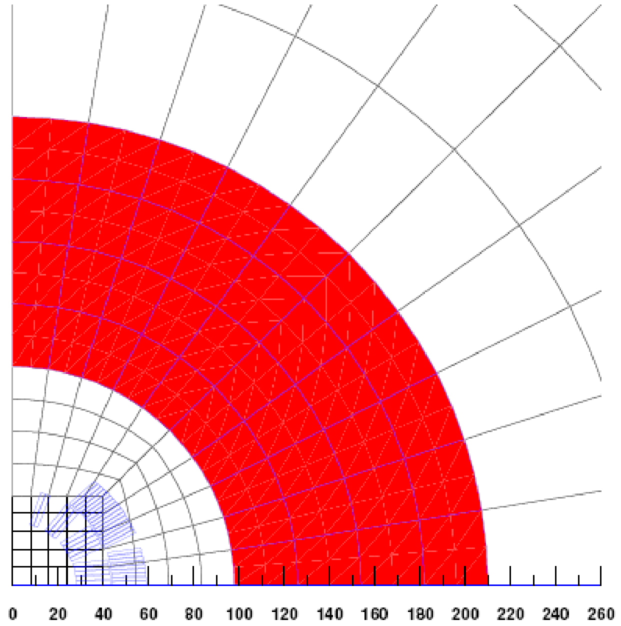
Meshing the Coil



Reduced Vector Potential Formulation

$$\mathbf{A} = \mathbf{A}_\mathrm{s} + \mathbf{A}_\mathrm{r} \qquad \mathbf{B} = \mu_0 \mathbf{H}_\mathrm{s} + \operatorname{curl} \mathbf{A}_\mathrm{r}$$

$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} (\mathbf{A}_\mathrm{r} + \mathbf{A}_\mathrm{s}) - \operatorname{grad} \frac{1}{\mu} \operatorname{div} (\mathbf{A}_\mathrm{r} + \mathbf{A}_\mathrm{s}) = \mathbf{J}$$



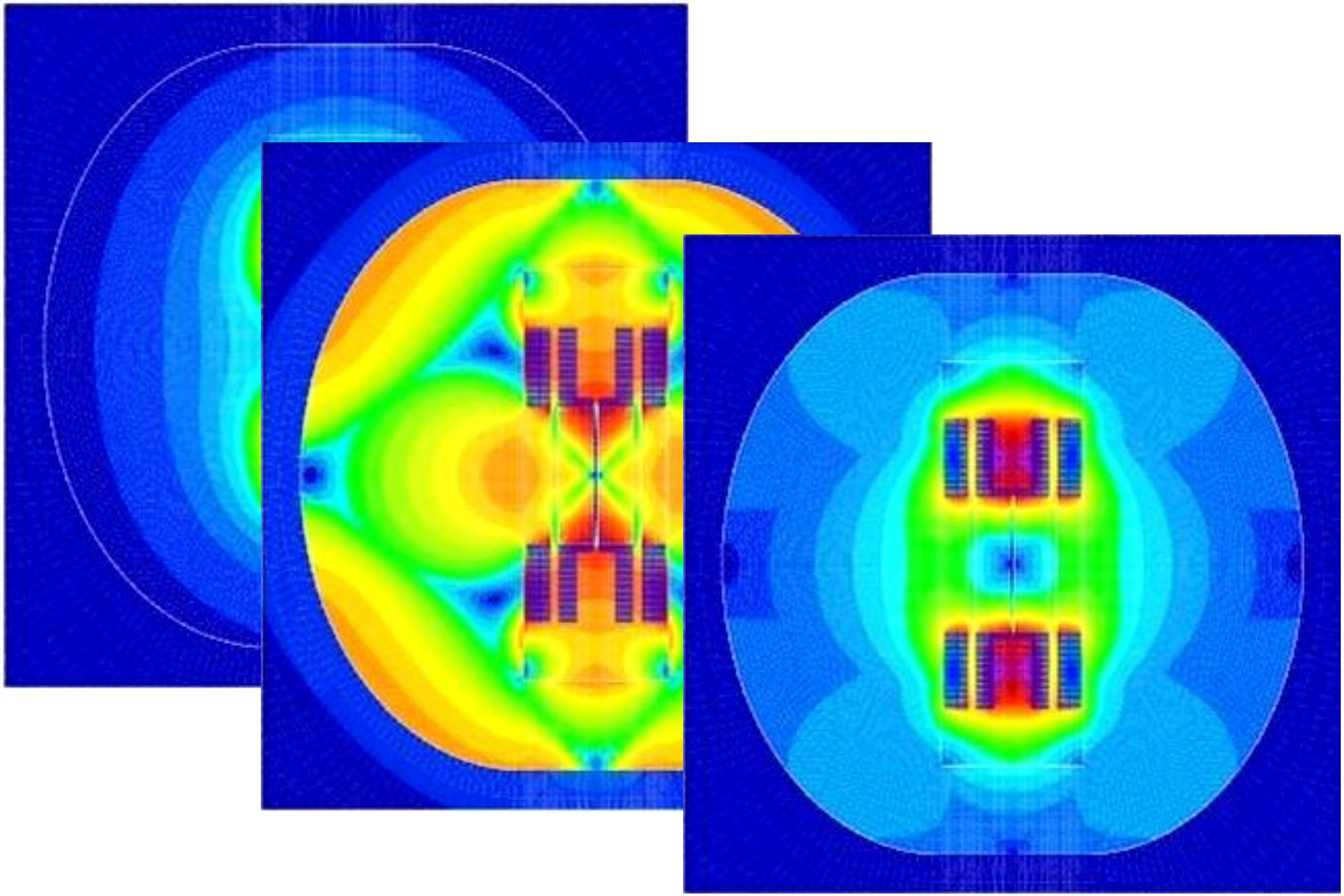
$$\operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A}_\mathrm{r} - \operatorname{grad} \frac{1}{\mu} \operatorname{div} \mathbf{A}_\mathrm{r} = \mathbf{J} - \operatorname{curl} \frac{1}{\mu} \operatorname{curl} \mathbf{A}_\mathrm{s}$$

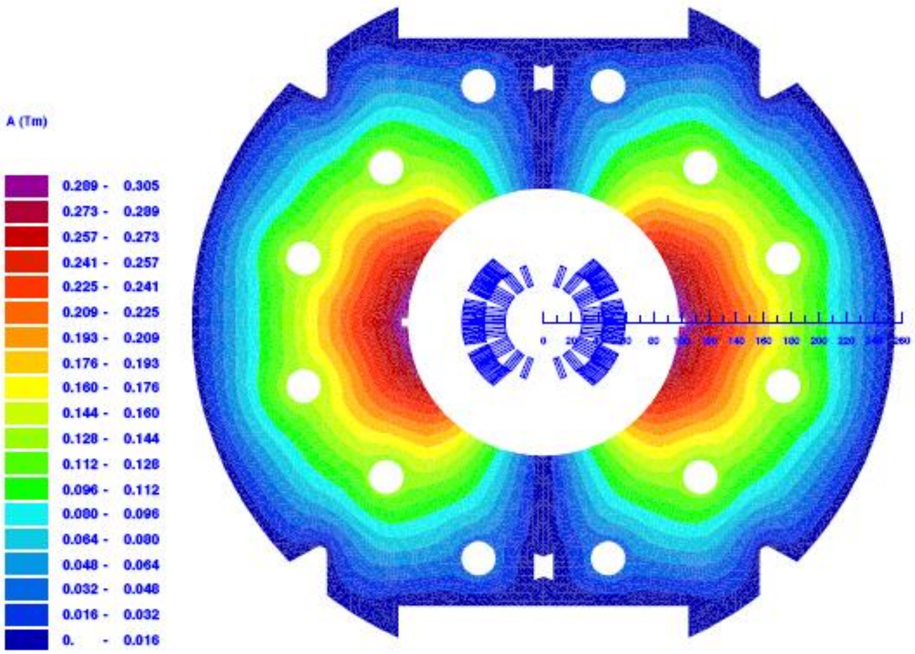
$$= \operatorname{curl} \mathbf{H}_\mathrm{s} - \operatorname{curl} \frac{\mu_0}{\mu} \mathbf{H}_\mathrm{s}$$

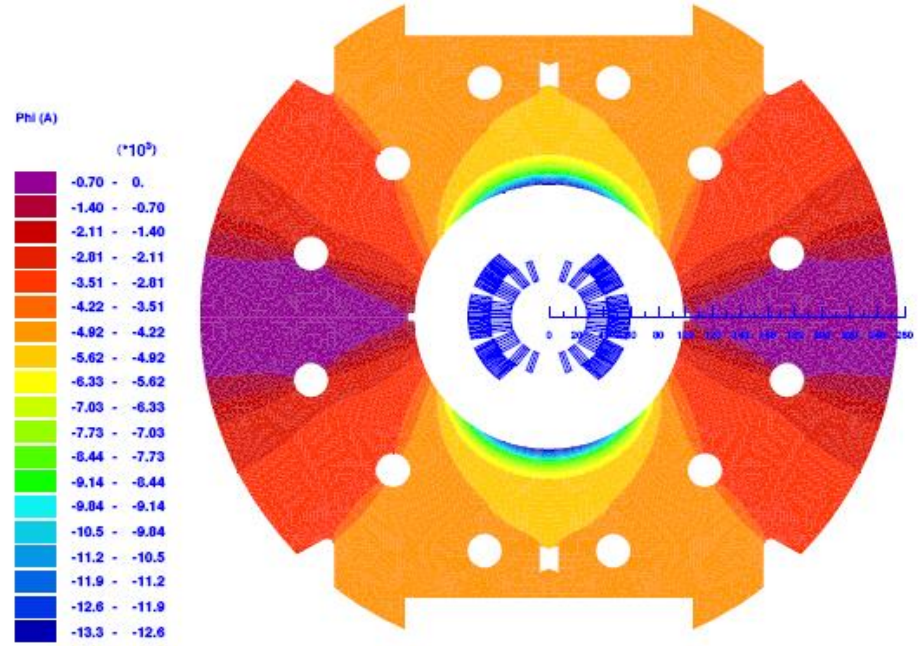
$$= \operatorname{curl} \left(\mathbf{H}_\mathrm{s} - \frac{\mu_0}{\mu} \mathbf{H}_\mathrm{s} \right)$$

Advantages: No meshing of the coil, no cancellation errors, distinction between source field and iron magnetization

Source, Reduced, Total Field

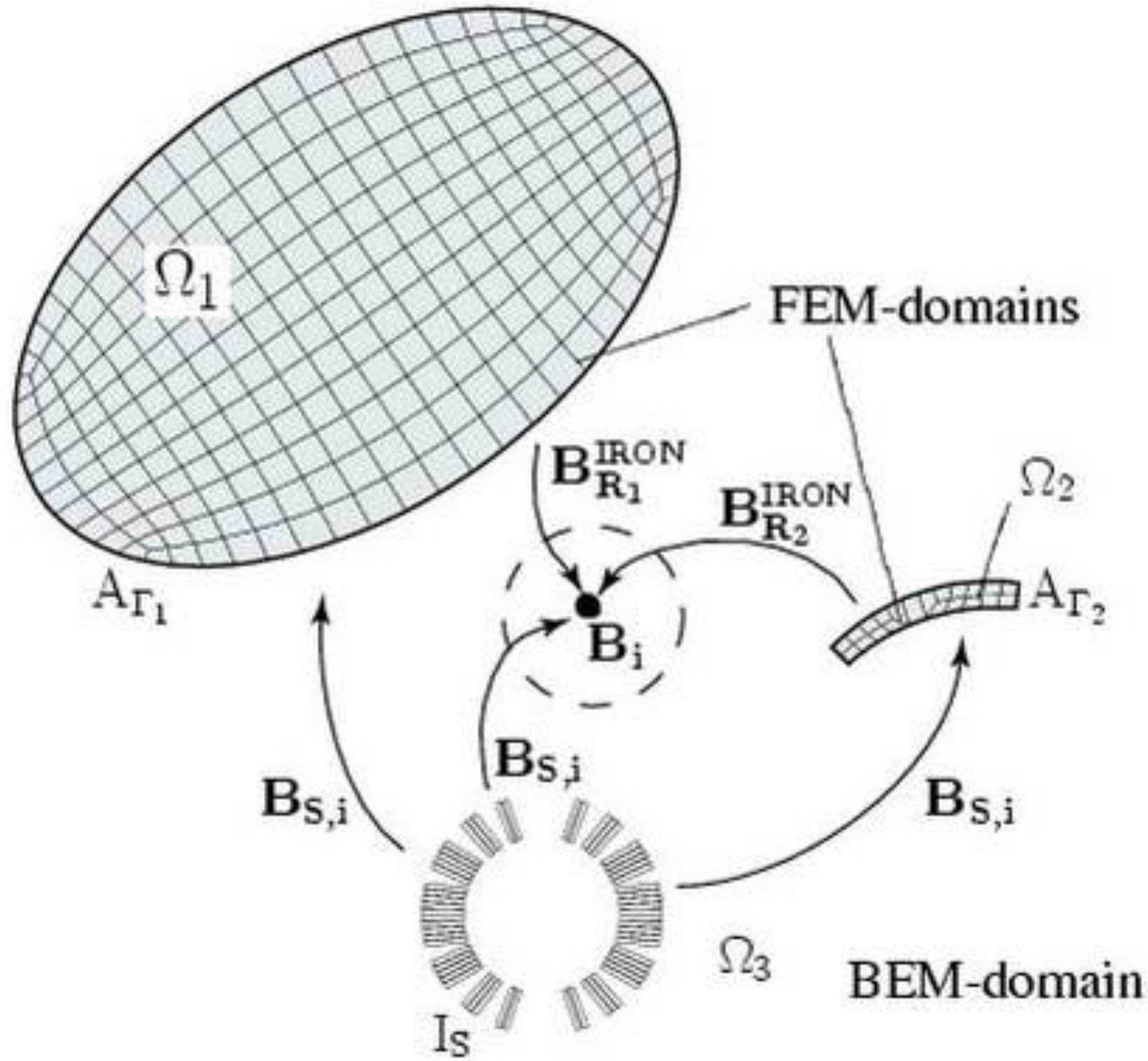




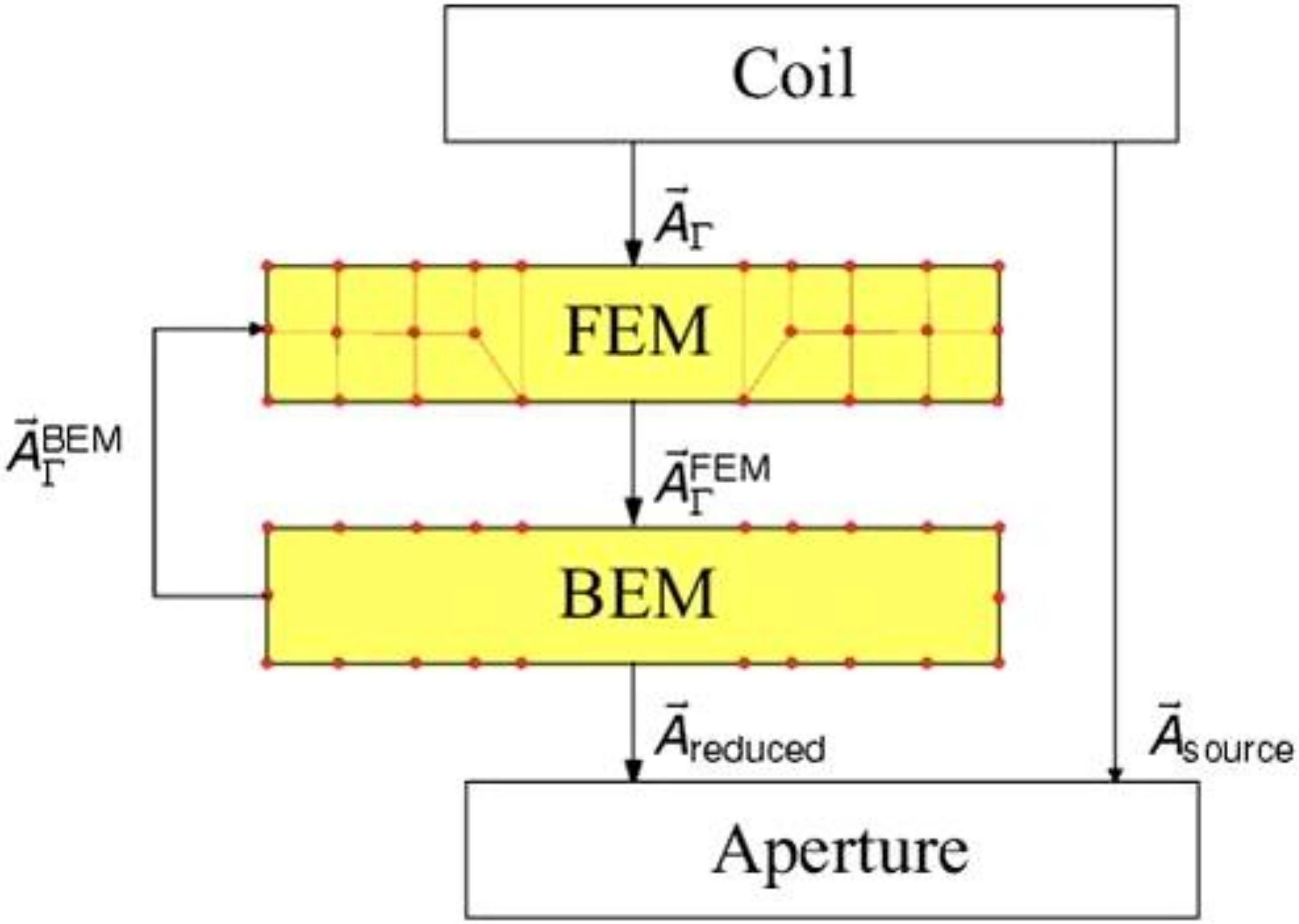


Number of finite elements	60	178	449	787	2799	6233
Total scalar potential	65.8	72.1	13.0	5.0	3.8	15.7
Vector potential	-40.5	-27.4	-7.4	-4.8	-3.8	25.0

BEM-FEM Coupling (Elementary Model Problem)

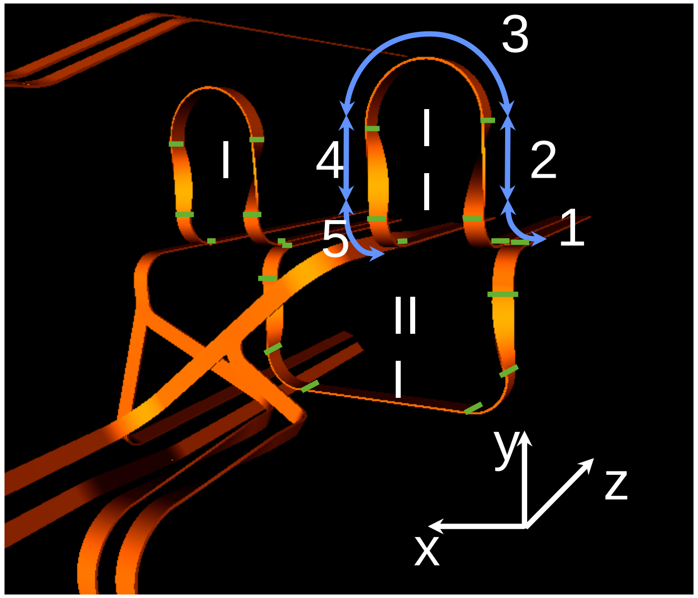


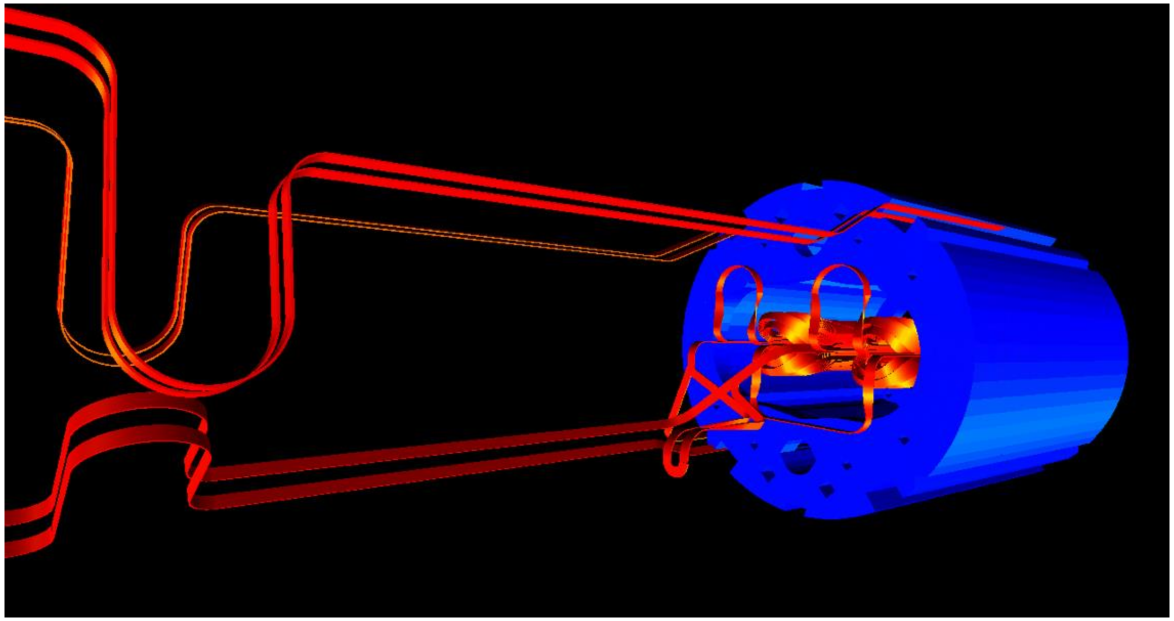
BEM-FEM Coupling



Forces (N) in the Connection Ends of the LHC Main Dipole

I	Fx	Fy	Fz
1	-39.7	-44.0	-45.4
2	-6.5	3.7	-41.7
3	-6.1	88.3	-38.2
4	1.25	3.9	-28.5
5	48.1	-46.7	-48.5
Sum	-2.95	5.2	-202.3



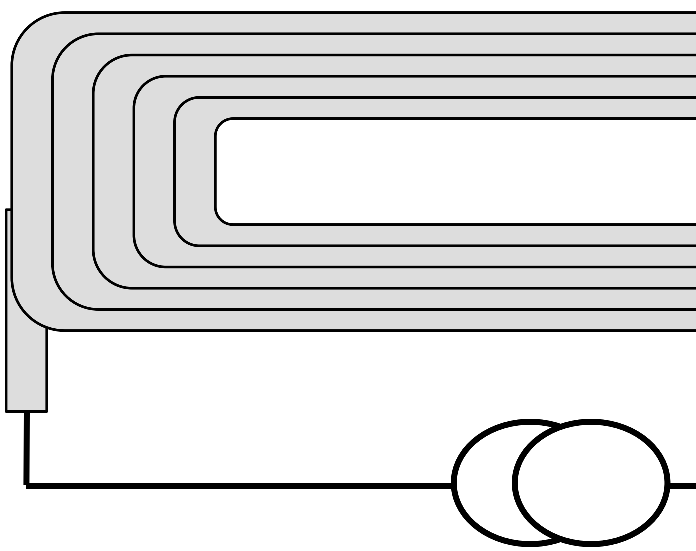


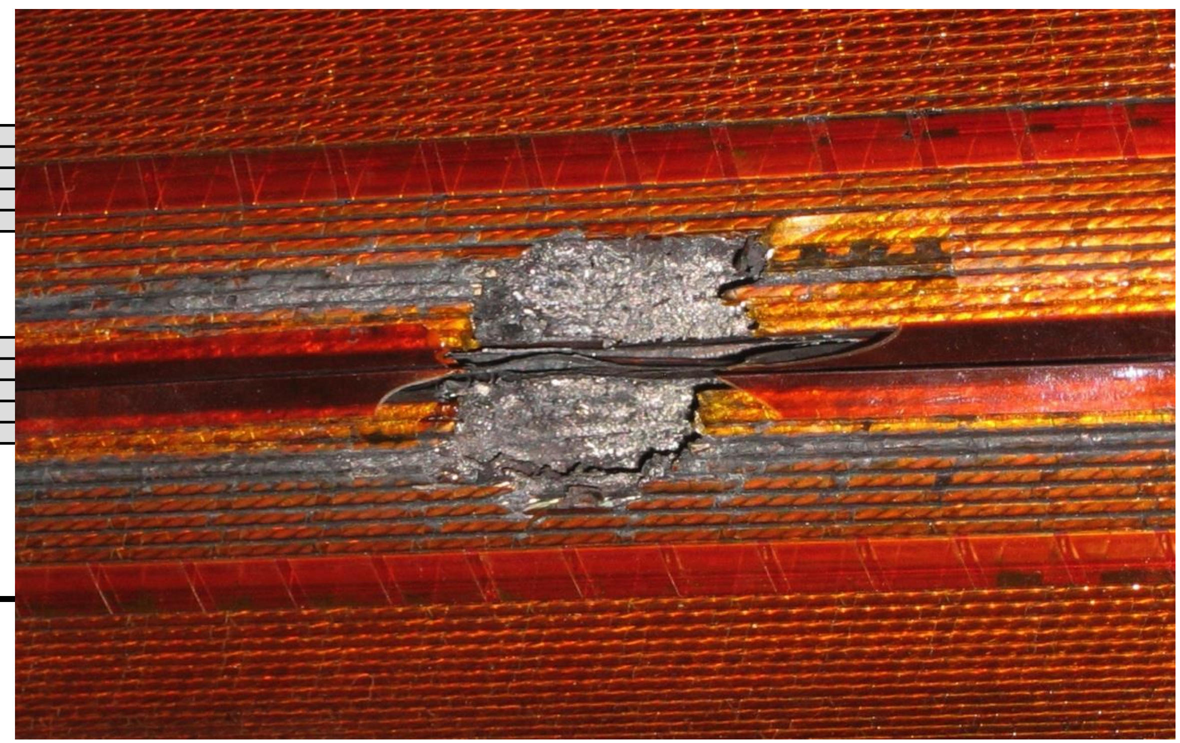
Quench

- Quench: Transition from SC to normal conducting state caused by

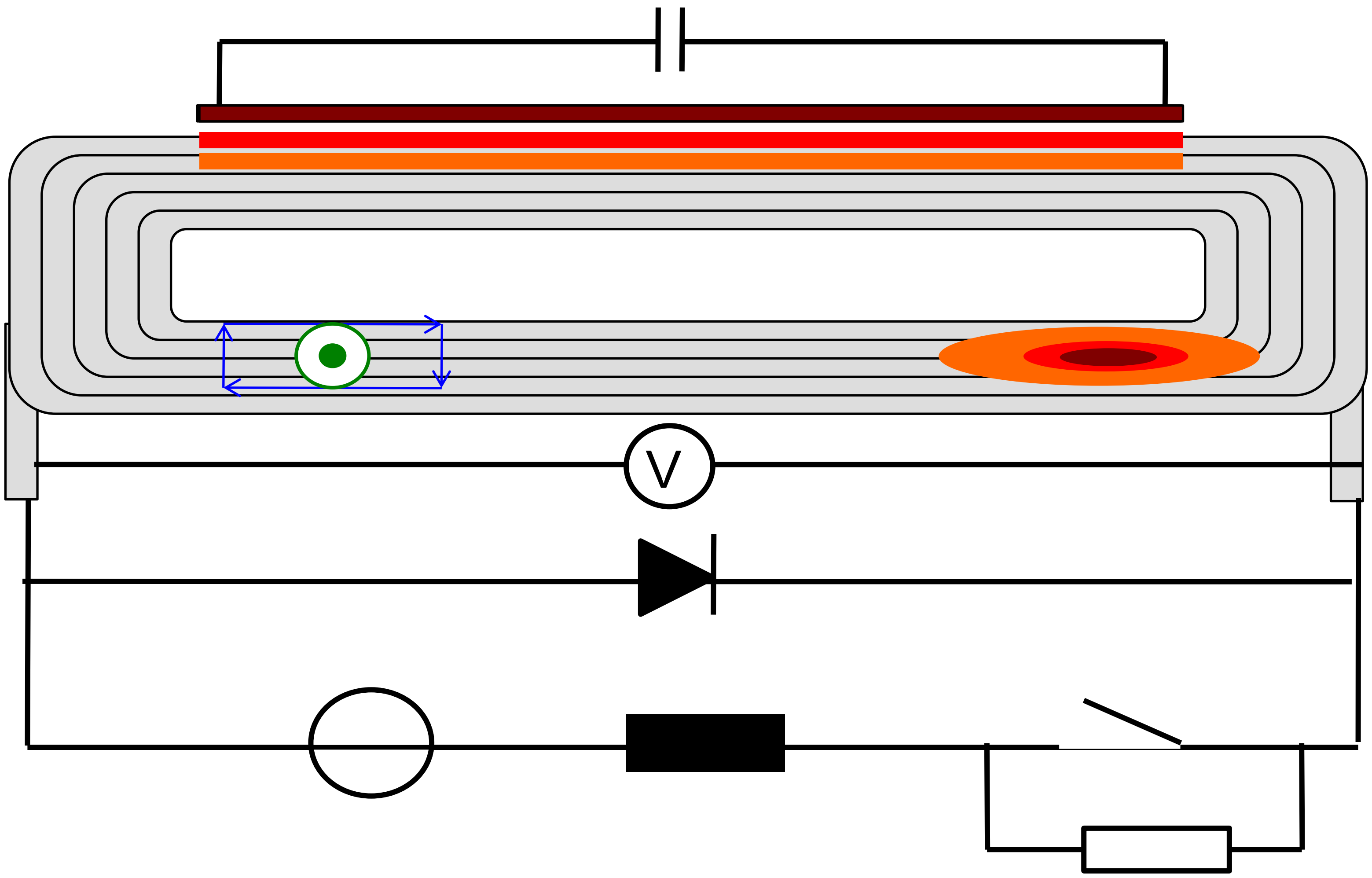
  beam losses, conductor movement, eddy currents etc.

- Propagation:
  - Normal conducting zone generates Ohmic heat
  - Quench- und temperature distribution determined by loss-mechanisms and cooling capacity

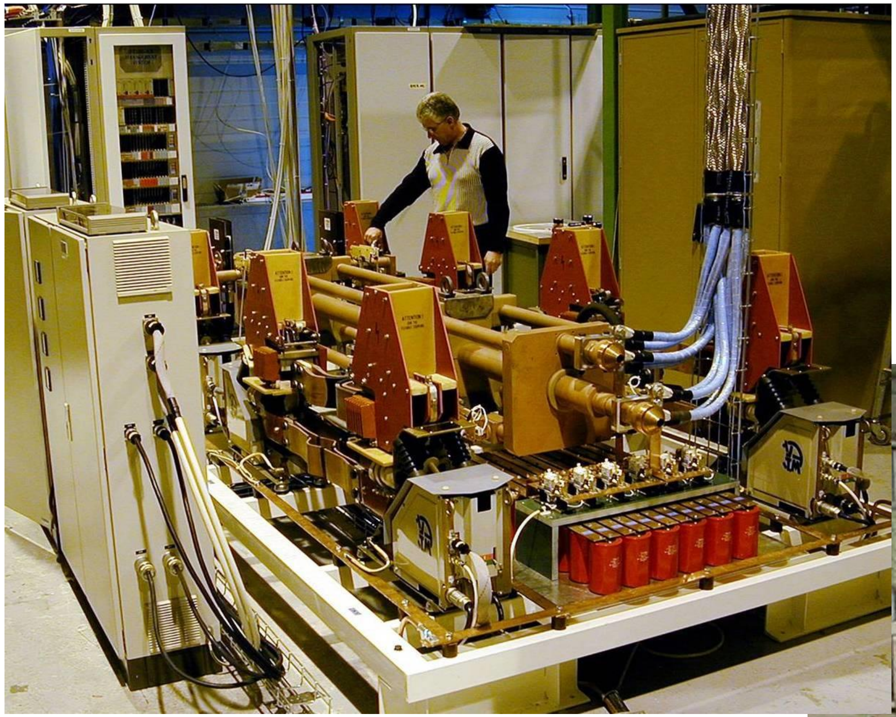


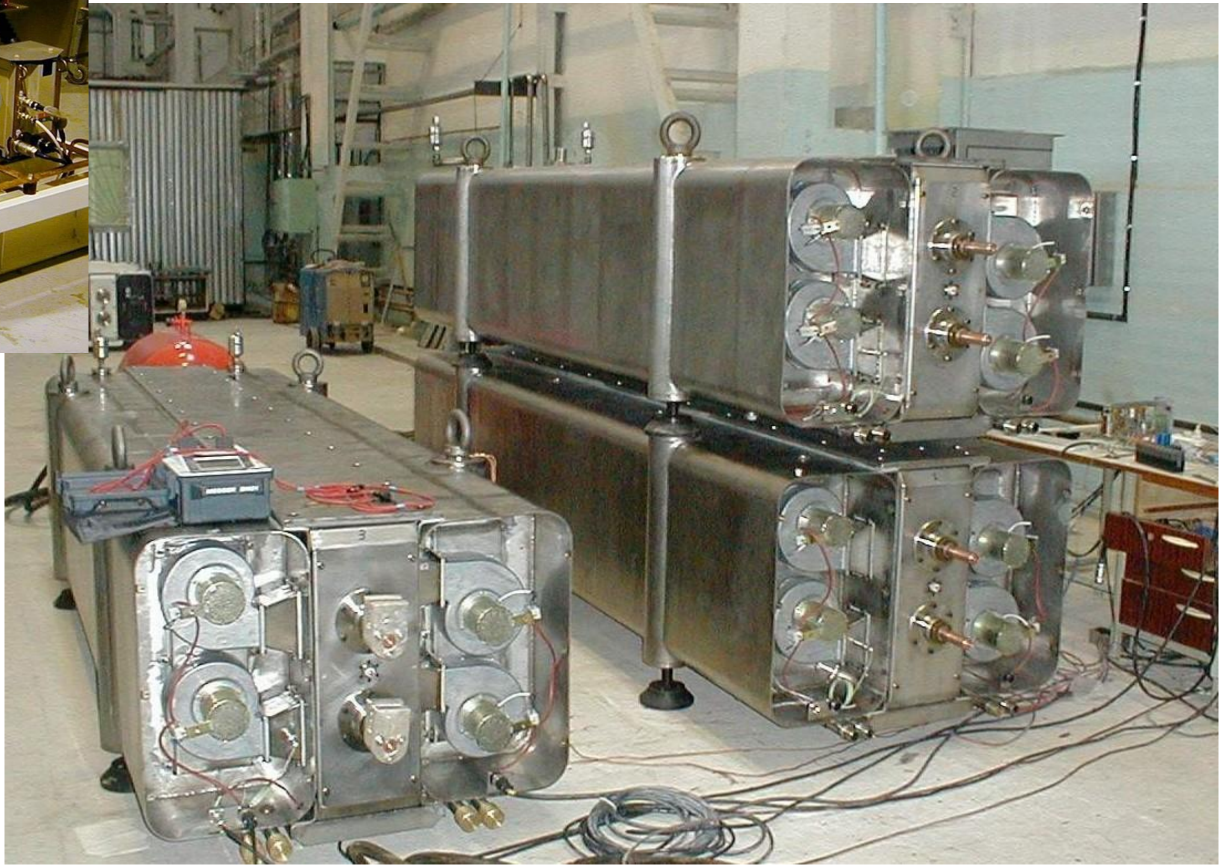


Quench Mechanism and Magnet Protection



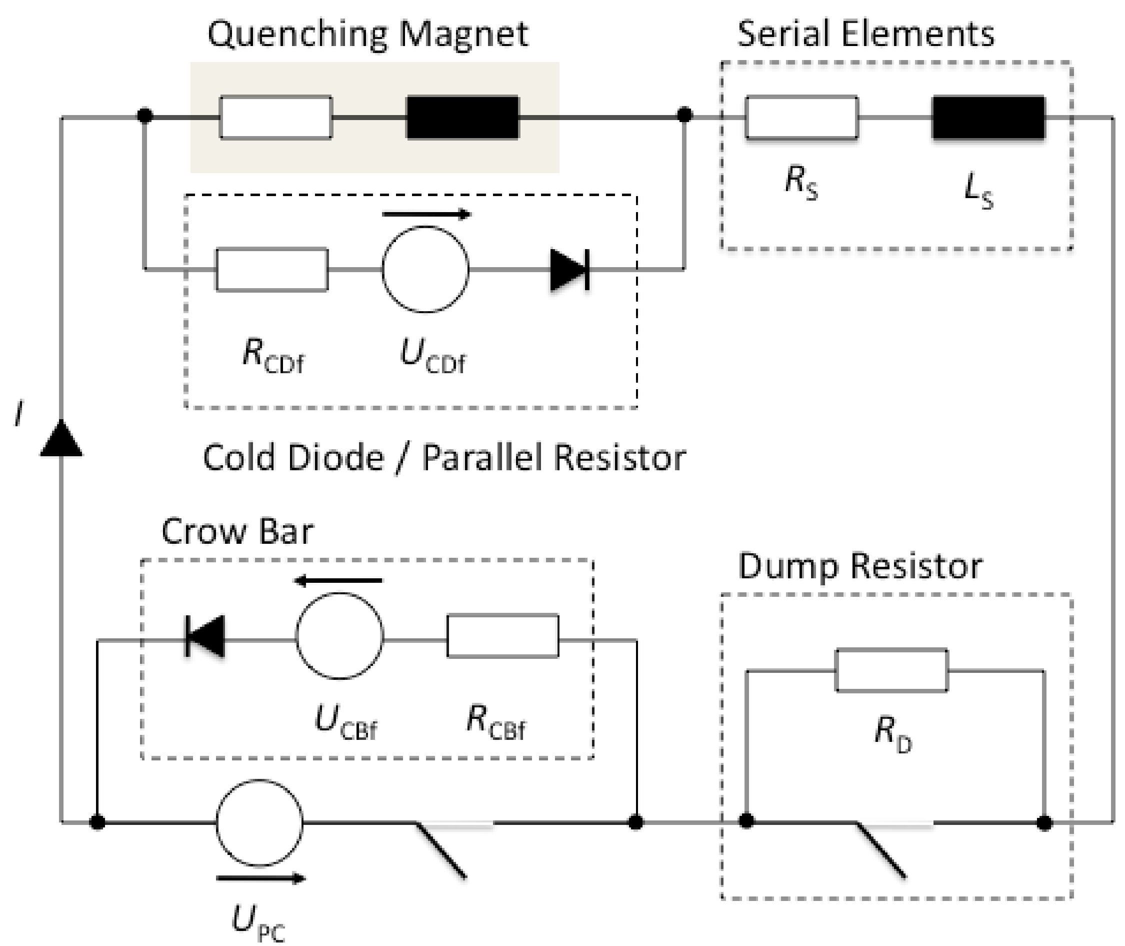
Switches and Dump Resistors





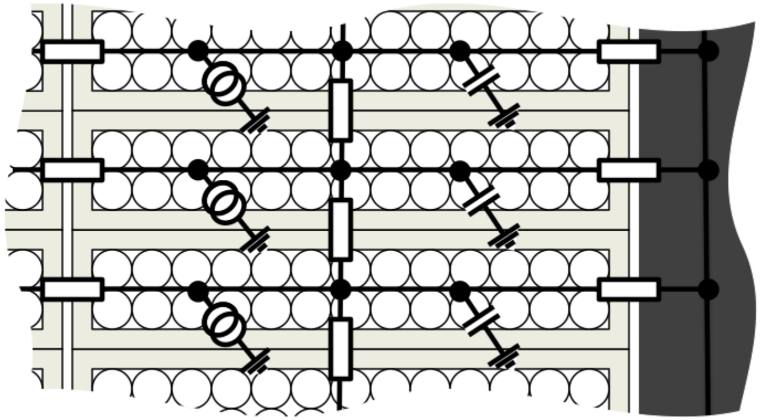
Electrical (External) Network

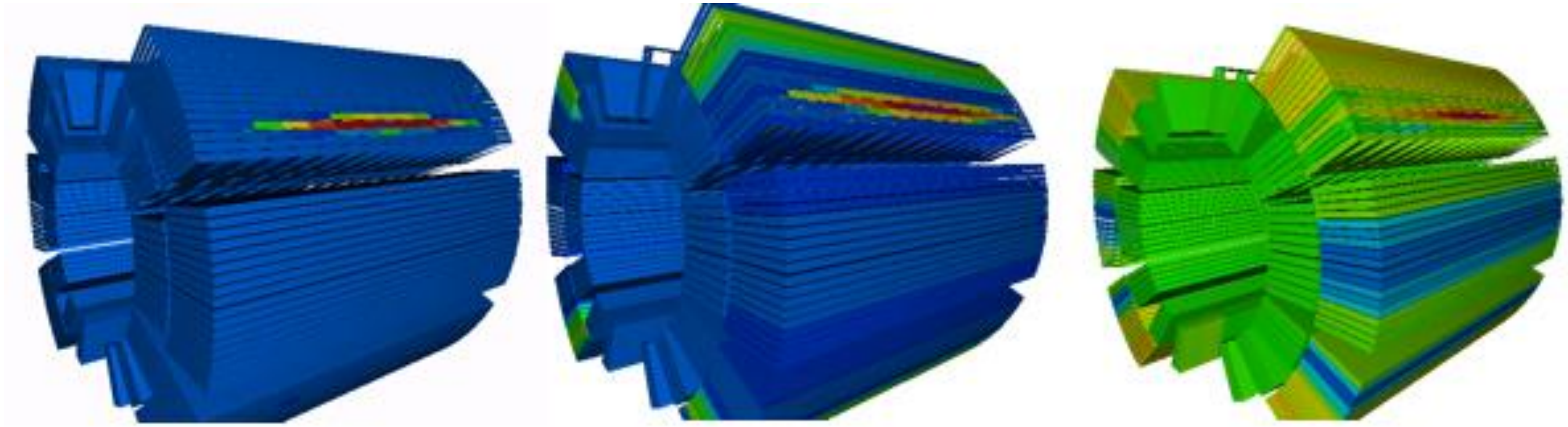
$$L_\mathrm{d}(B)\frac{\mathrm{d}I}{\mathrm{d}t} = U_\mathrm{Diode} - (R_\mathrm{Q}(B,T) + R_\mathrm{P}(t))I$$



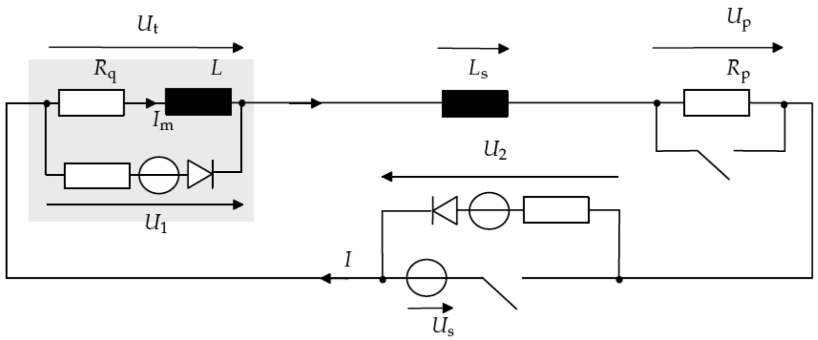
Thermal Model

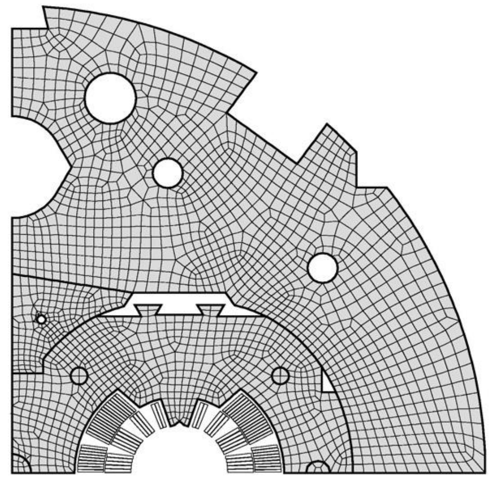
$$c(T)\frac{\mathrm{d}T}{\mathrm{d}t} = p - \vec{\nabla} \cdot \left(\kappa(T,B) \vec{\nabla} T \right)$$

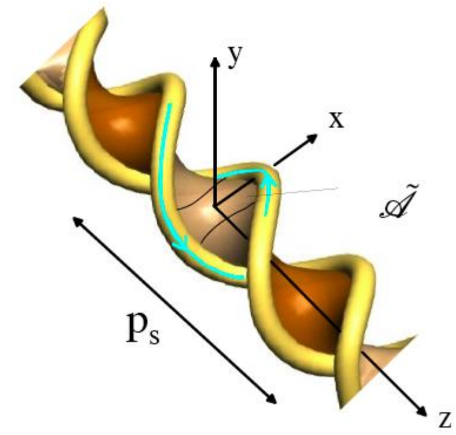


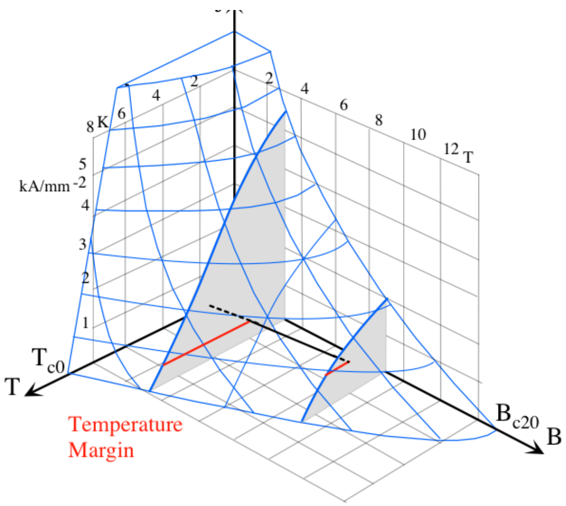


Quench Simulation (Multi-Physics, Multi-Scale)

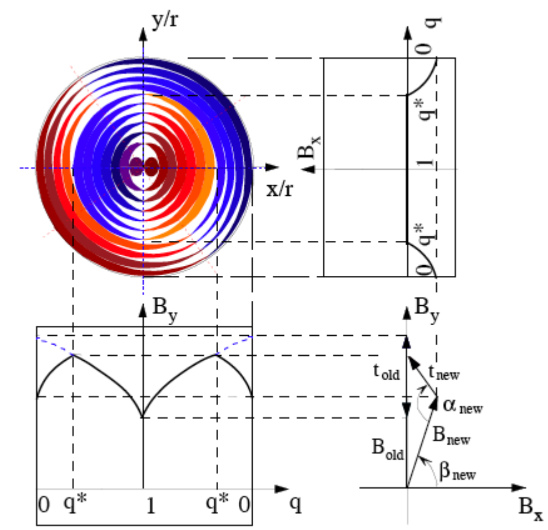


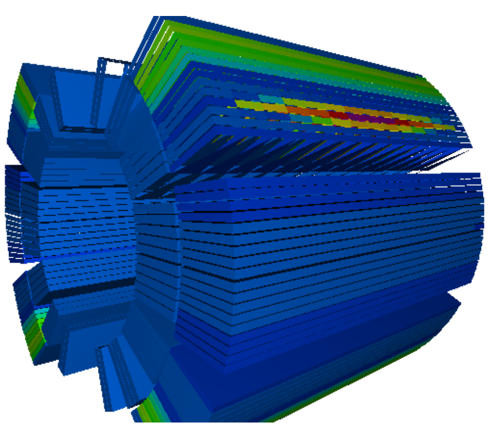


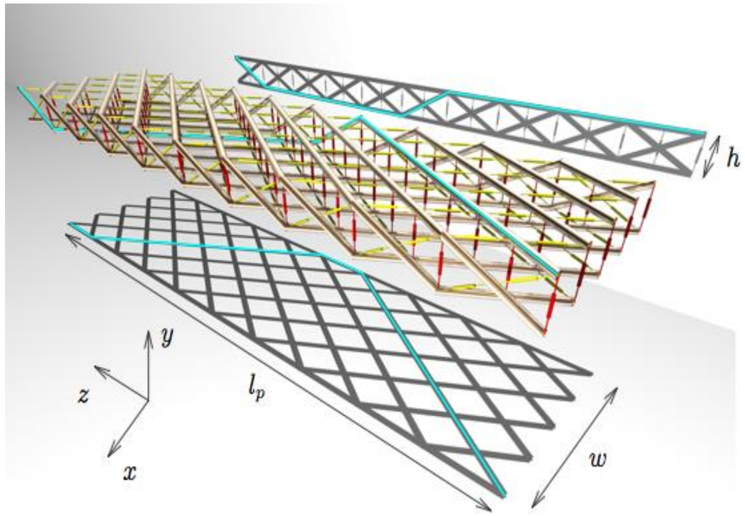




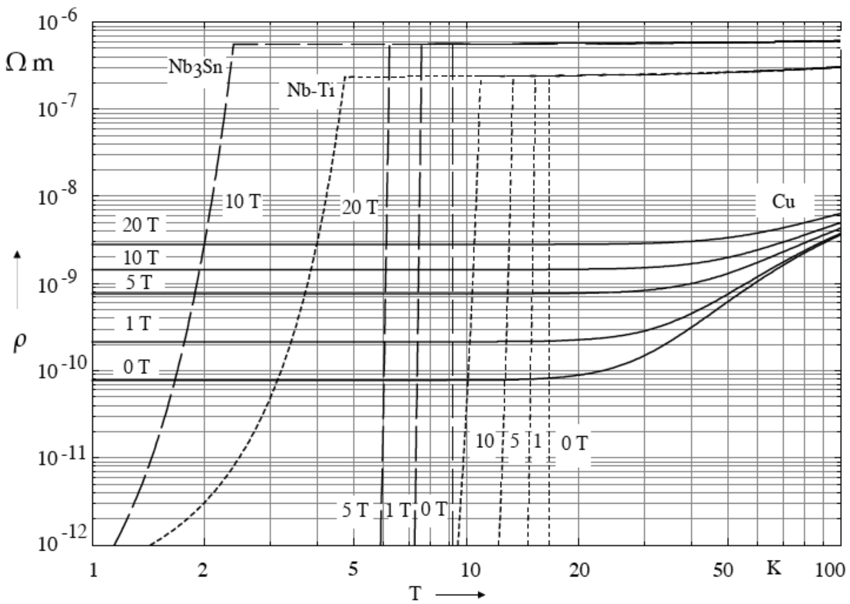
Quench Simulation in ROXIE

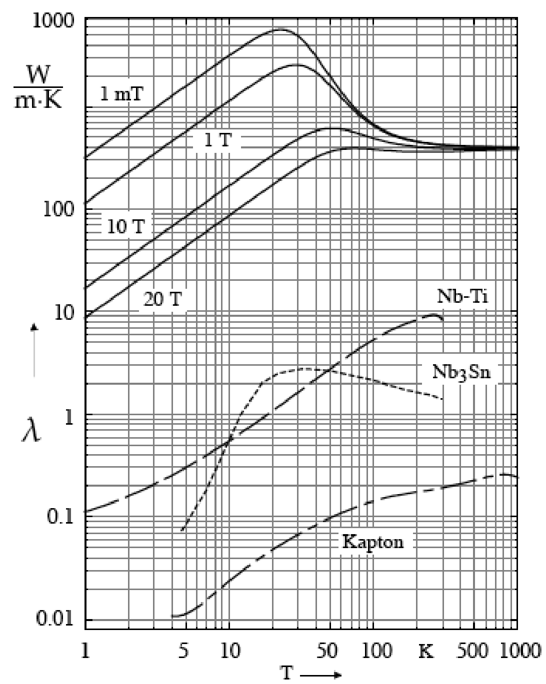


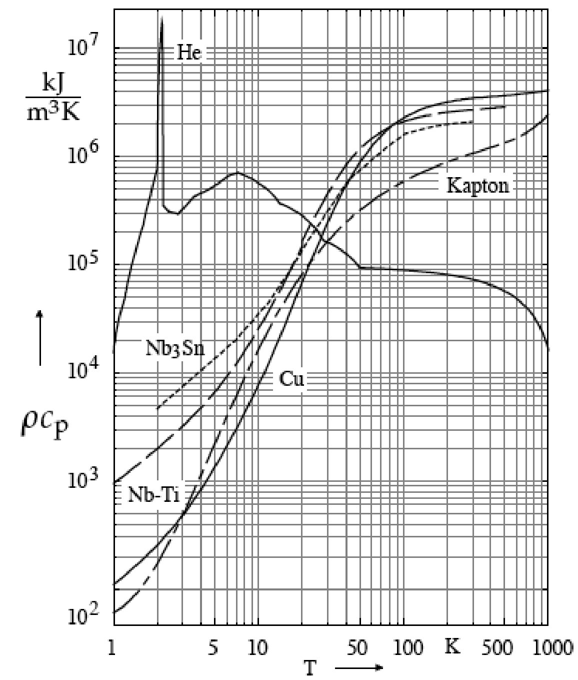




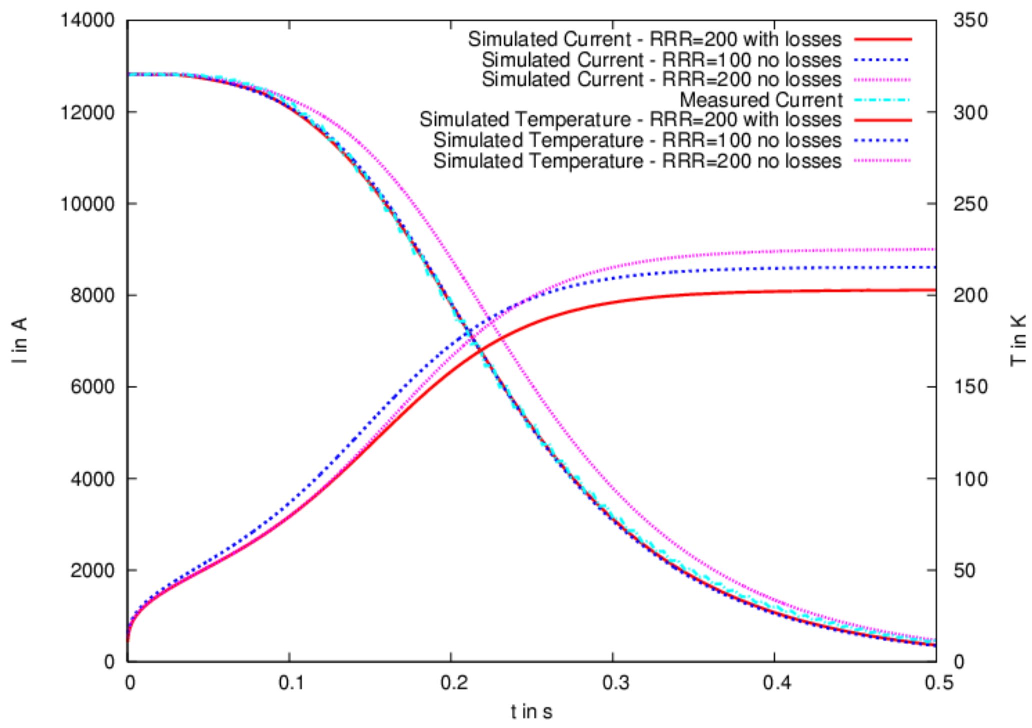
Material Parameters







Validation



Empirical parameters:

- RRR
- Ra/Rc
- IFCC effective res.
- heat conductivity
- heat capacity

➔ Different families of parameters yield exactly the same observable I(t).

➔ More than one solution exists.

➔ Great care must be taken to model

– all relevant phenomena,

– using realistic material parameters.

Quench Simulation

The challenge of quench simulation:

Model **all relevant physical phenomena** with **adequate accuracy** so that we can be confident to **simulate internal states** of a quenching magnet and understand its behavior.

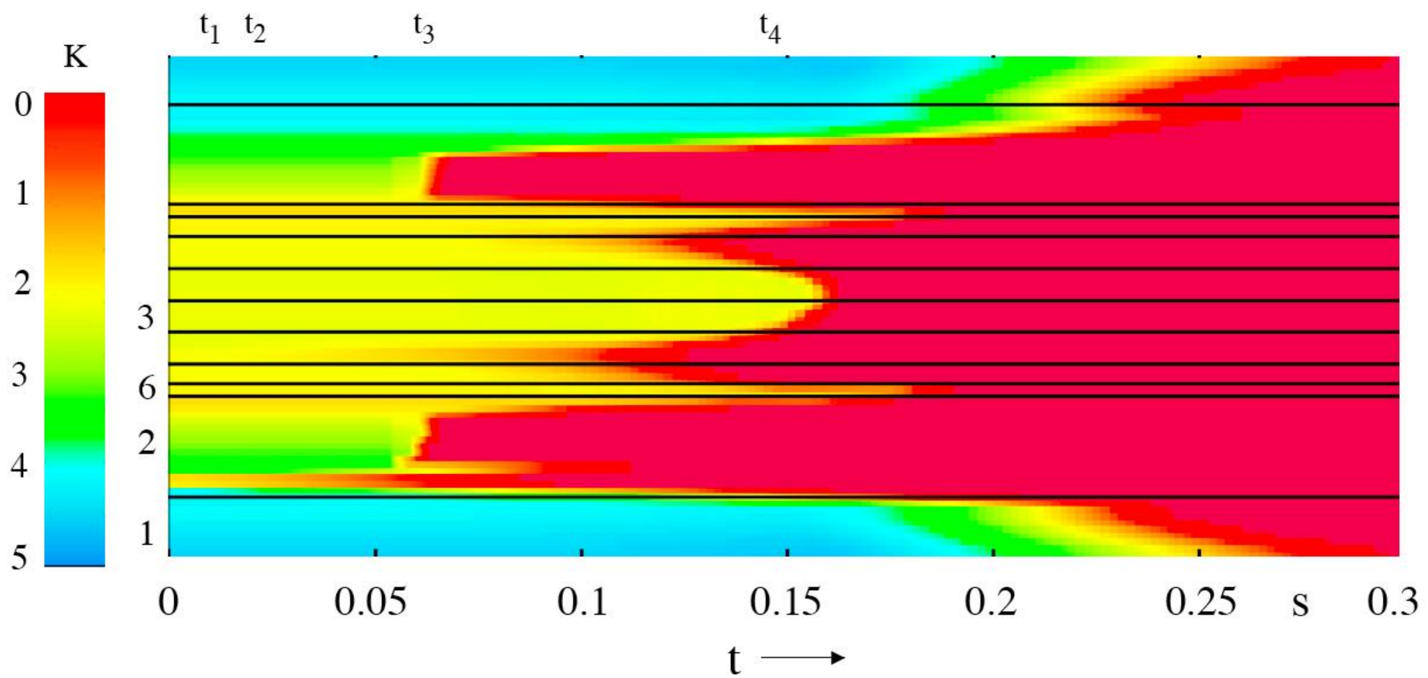
Validation:

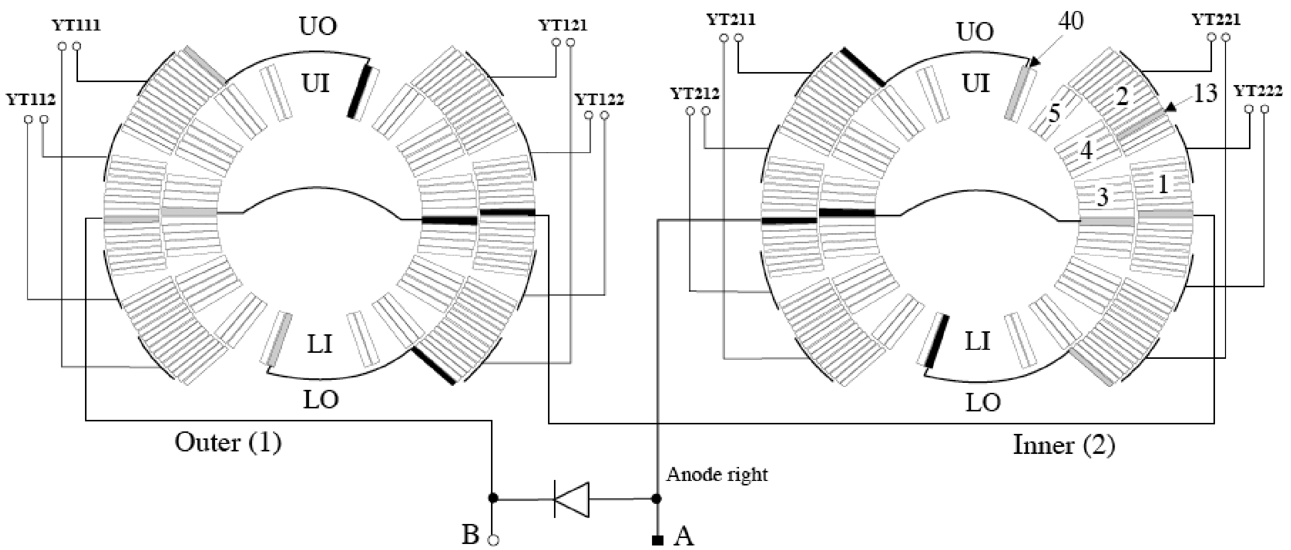
Measured quantities can be **reproduced** with all **material- and model-parameters within their range of uncertainty,**

Extrapolation and Introspection

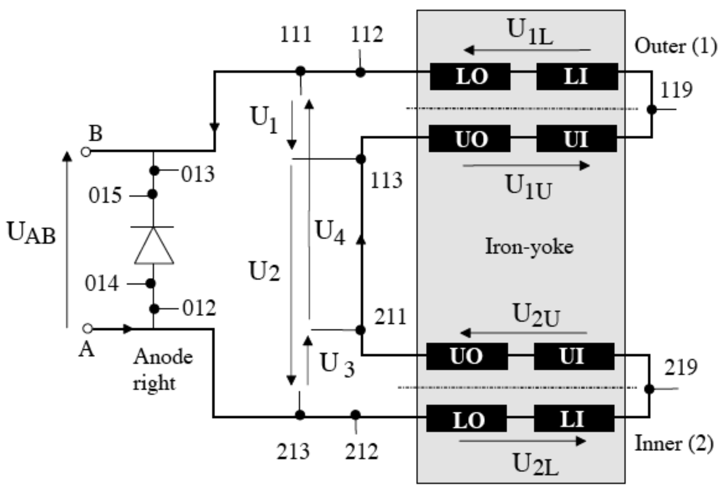
If the above criteria are reached, extrapolated results will match measurements **without adaptation of material- and model parameters**. It is then also possible to **simulate internal states** of the magnet that escape measurements.

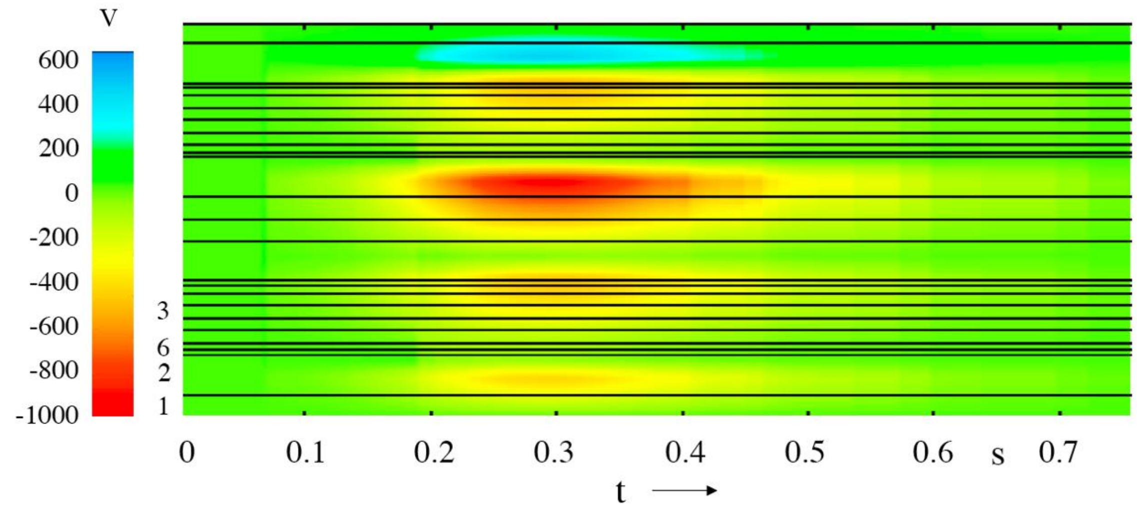
Introspection (Quench Margin)

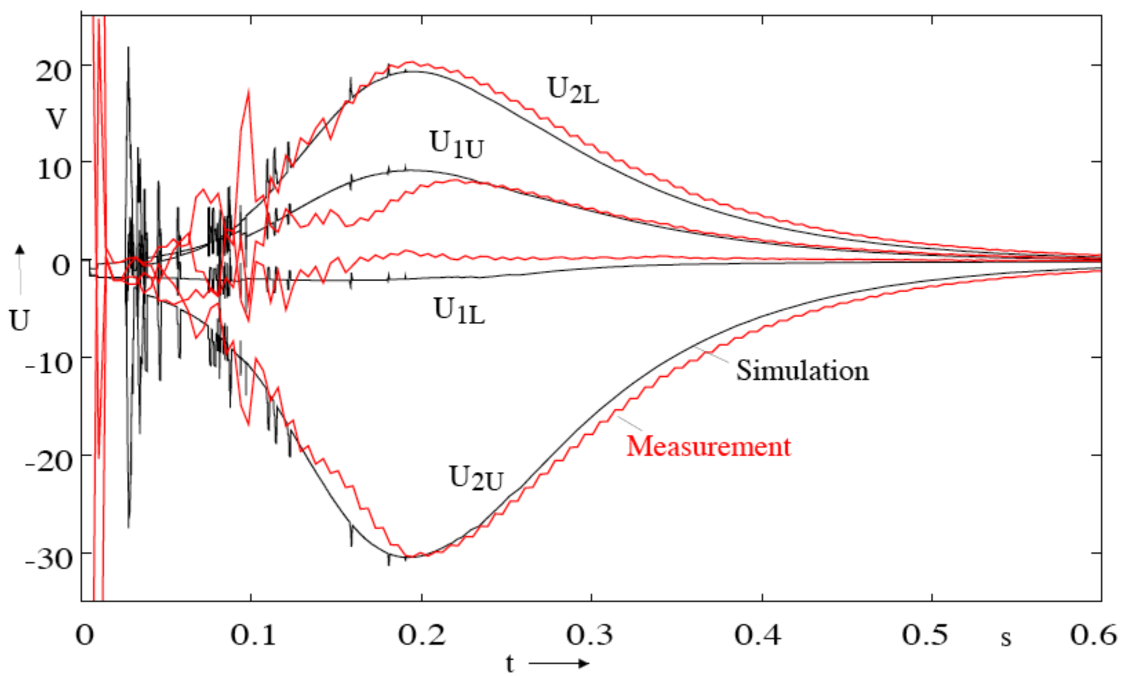




Introspection (Voltage Ripples)







Mathematical Formulation of Optimization Problems

$$X \subseteq \mathbb{R}^n \qquad (x_1, x_2, ..., x_n)^T \in X$$

$$\min\{f(\mathbf{x})\} \qquad f : X \to \mathbb{R}$$

Subject to

$$g_i(\mathbf{x}) \leq 0, \qquad i = 1, 2, ..., m,$$

$$h_j(\mathbf{x}) = 0, \qquad j = 1, 2, ..., p,$$

$$x_{l,\text{lower}} < x_l < x_{l,\text{upper}}, \qquad l = 1, 2, ..., n$$

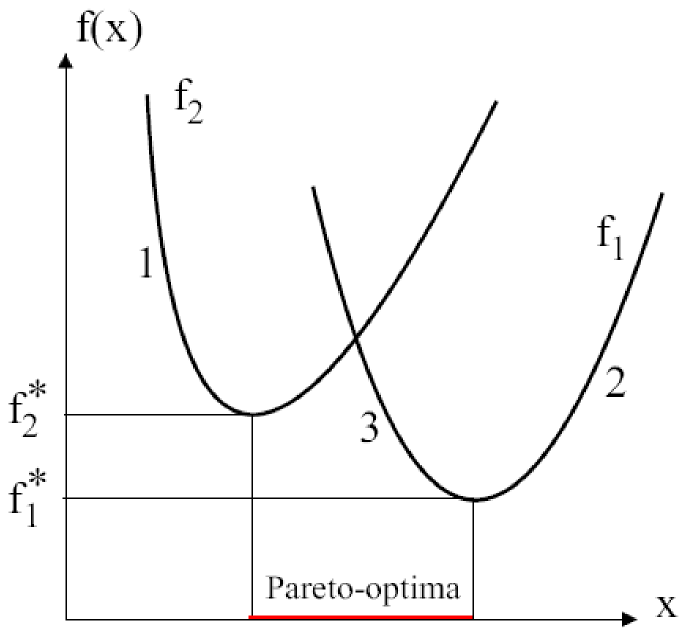
Pareto Optimality

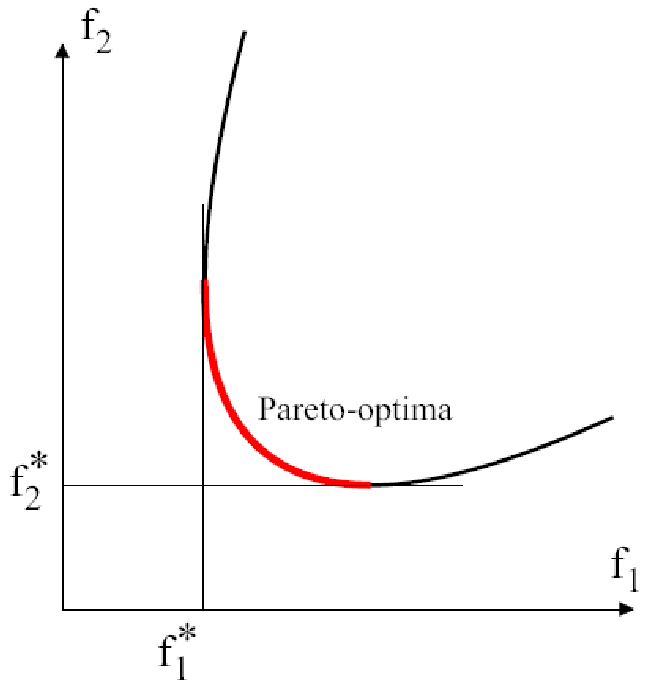
$$\text{MIN}\,\{\mathbf{f}(\mathbf{x})\} = \text{MIN}\,\{f_1(\mathbf{x}), f_2(\mathbf{x}), .., f_K(\mathbf{x})\,\}$$

A Pareto optimal solution X* is given if there exists no solution with

$$f_k(\mathbf{x}) \le f_k(\mathbf{x}^*) \qquad \forall\, k \in [1, K],$$

$$f_k(\mathbf{x}) < f_k(\mathbf{x}^*) \qquad \text{for at least one } k \in [1, K]$$





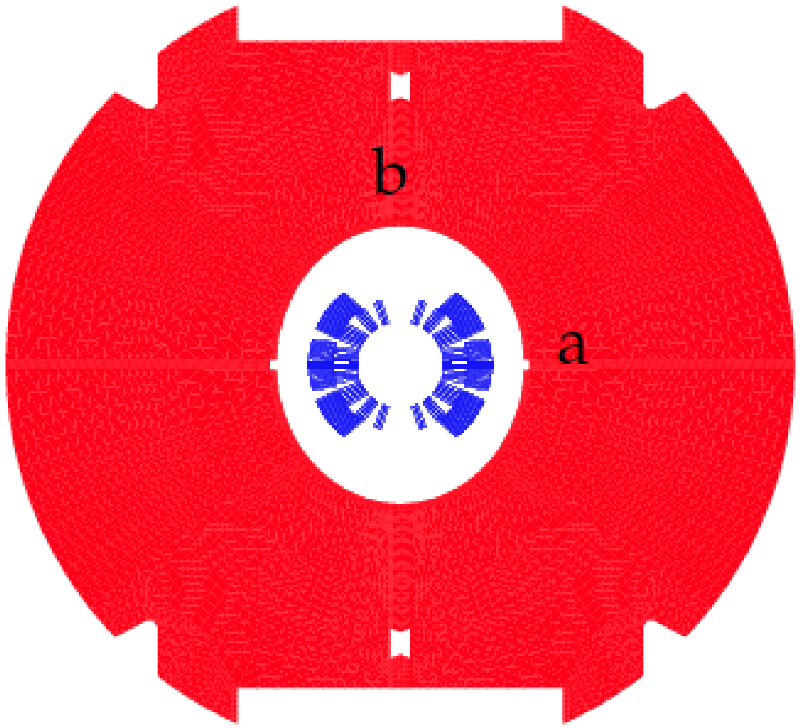
Real World Optimization Problems

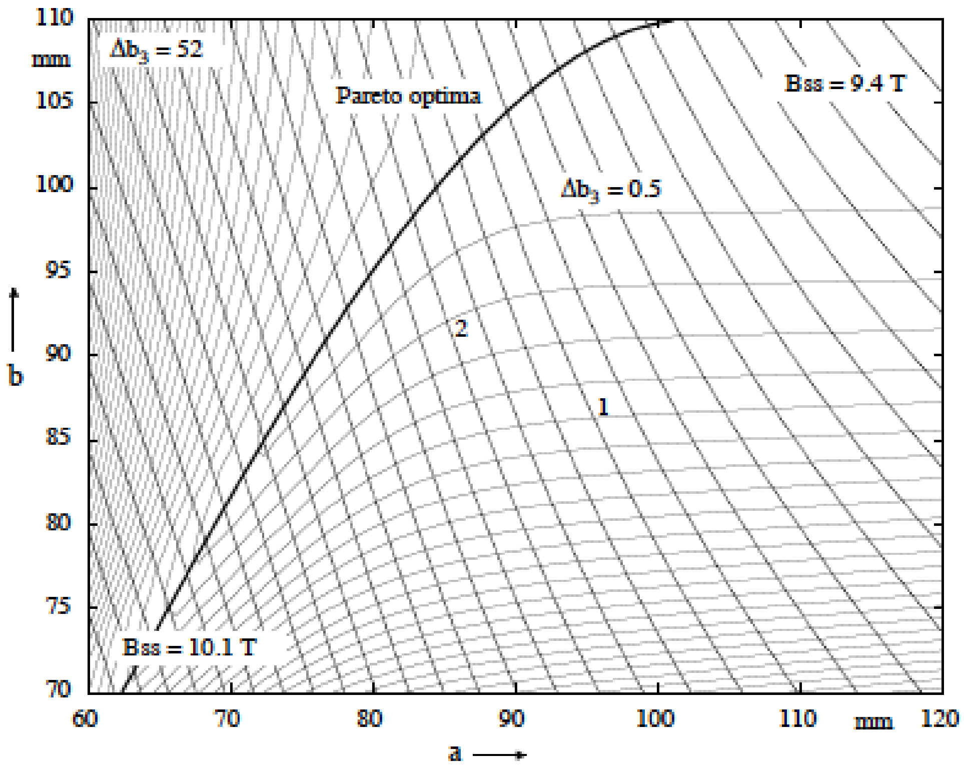
➔ There are only Pareto-optimal solutions
  – Decision making
  – Treatment of nonlinear constraints
  – Optimization algorithms

➔ The objective conflict is the characteristic of real world optimization problems

➔ Fuzzy objectives in the concept phase

Objective Conflict Superconducting Magnets



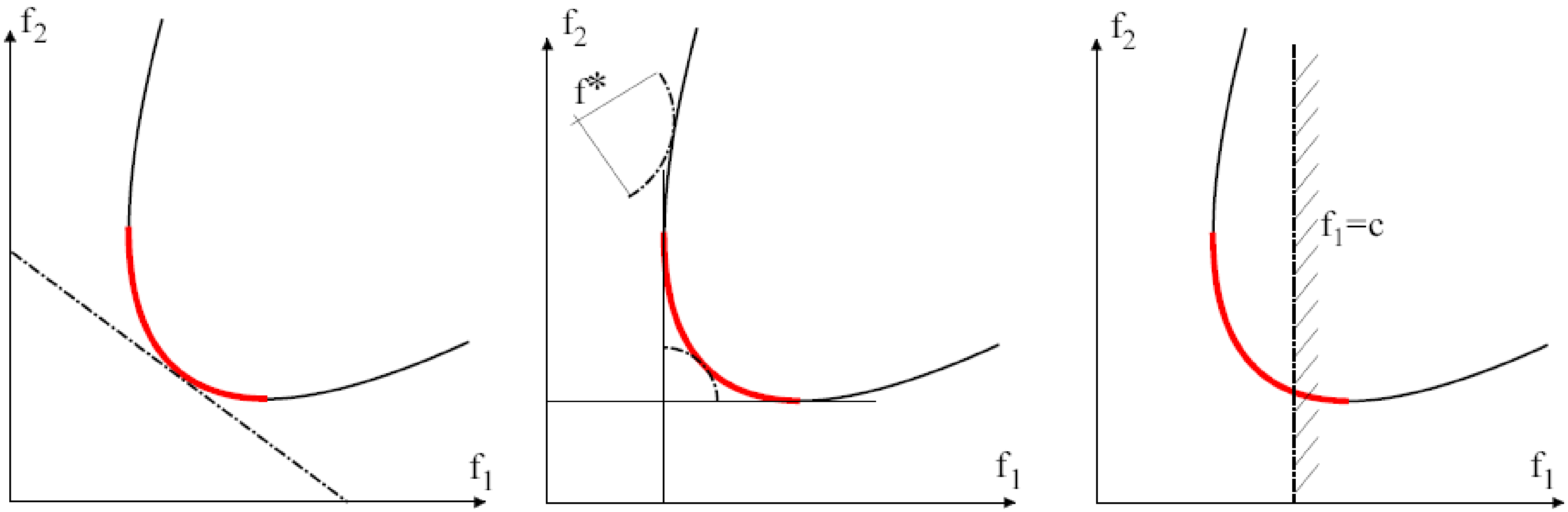


Objective Weighting, Distance Func., Constraint Form.

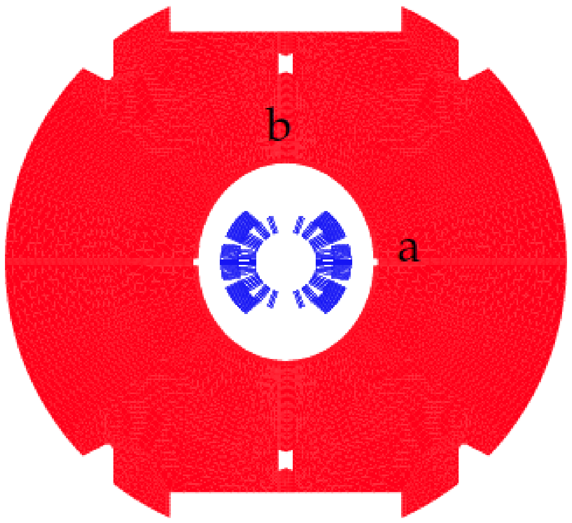
$$\min\left\{u(\mathbf{f}(\mathbf{x})) := \sum_{k=1}^{K} t_k\, f_k(\mathbf{x}) \mid \mathbf{x} \in M\right\}$$

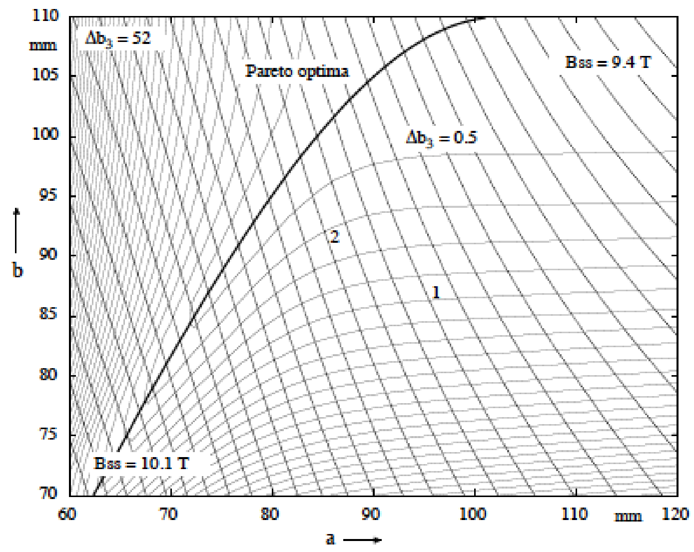
$$\min\left\{\| \mathbf{z}(\mathbf{x}) \|_2^2 := \sum_{k=1}^{K} (t_k(f_k^*(\mathbf{x}) - f_k(\mathbf{x})))^2 \mid \mathbf{x} \in M\right\}$$

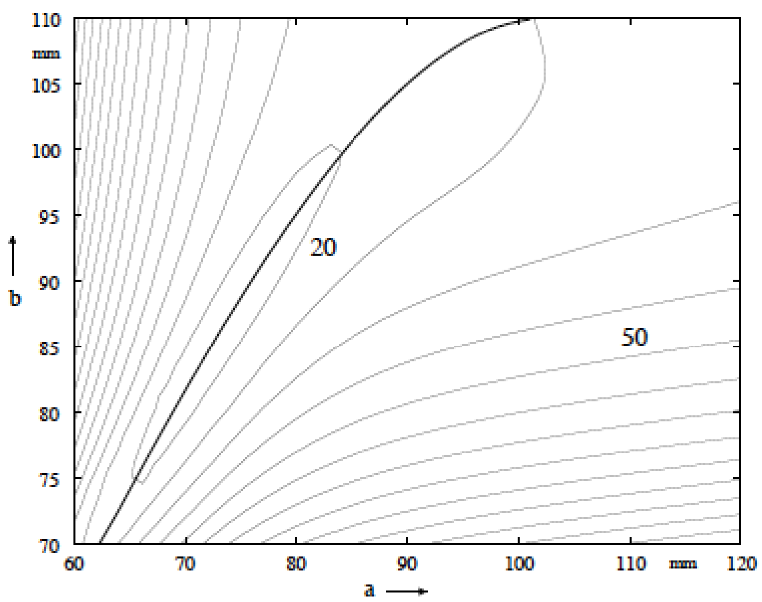
$$\min\{f_i(\mathbf{x})\} \quad \text{s.t.} \quad f_k(\mathbf{x}) - r_k \leq 0$$

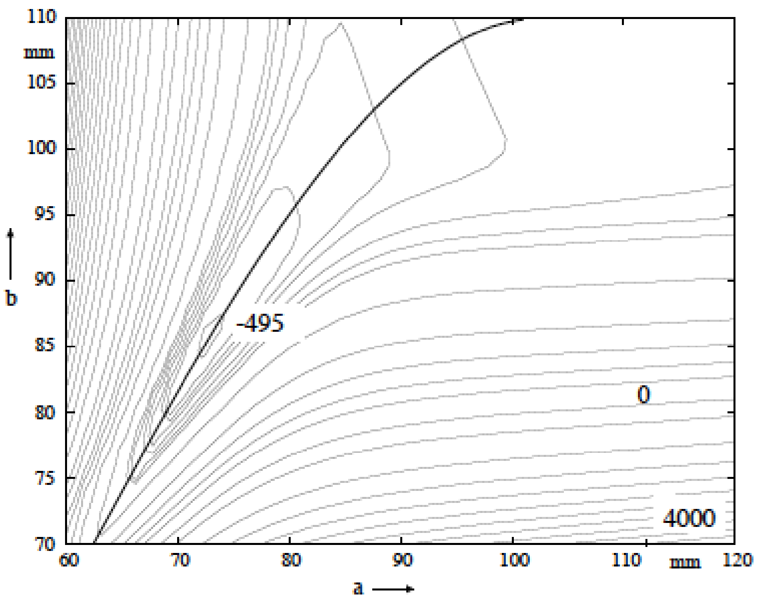


Objective Weighting and L2 Distance Function



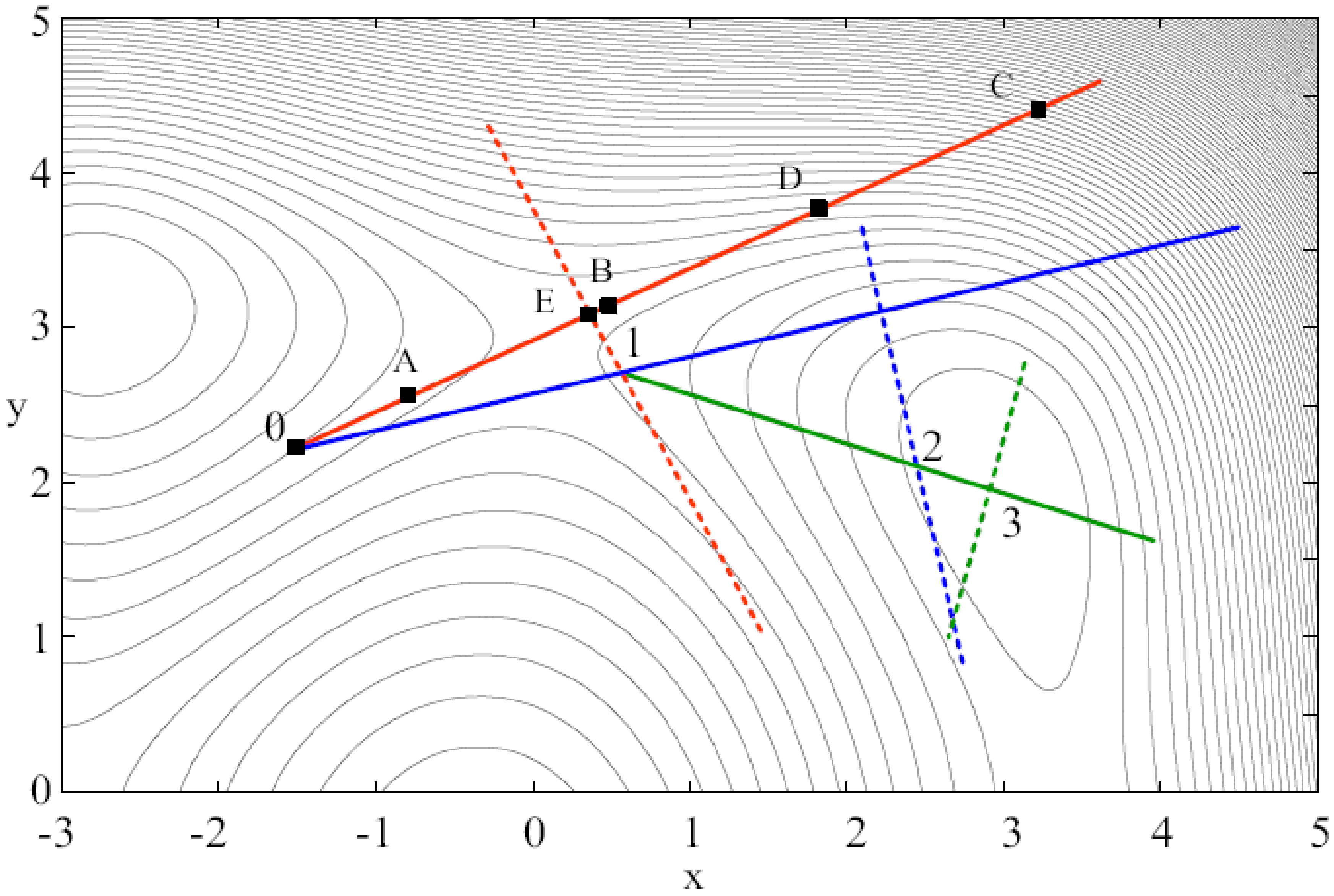






Optimization Algorithms

Search methods		
Direct search	Gauss-Seidel	
EXTREM	Jacob	1982
Rosenbrock	Rosenbrock	1960
Powell	Powell	1965
Flexible Polyhedron search	Nelder-Mead	1964
Hooke-Jeeves	Hooke-Jeeves	1962
Gradient methods		
Steepest descend	Cauchy	1847
Newton's method	Newton	1700
Levenberg-Marquard	Levenberg Marquard	1963
Conjugate gradient (CG)	Fletcher-Reeves	1964
Quasi-Newton	Davidon-Fletcher-Powell	1959
Stochastic and neural computing		
Evolutionary	Rechenberg	1964
Genetic algorithms	Fogel-Holland	1987
Neural computing (ANN)	Aarts-Korst	1989


5
4
3
2
1
0
y
-3
-2
-1
0
1
2
3
4
5
x
0
A
E
B
D
C
1
2
3

Genetics and Evolution Theory

➔ Darwin (1860)
  - Survival of the fittest
  - Variations between individuals of species
  - Reproductive populations
  - Evolutionary computation (Rechenberg, Schwefel 1964)

➔ Mendel (1850)
  - Genetic basis of variation
  - Coding (Discrete units)
  - Genetic algorithms (Holland 1970)
  - Niching decreases selective pressure
  - Niching genetic algorithms (Mahfoud 1995)

Genetic Algorithms

Phenotype	$x_1^{(1)}$	$x_2^{(1)}$	$x_3^{(1)}$	$\ldots x_n^{(1)}$	Population
	$x_1^{(j)}$	$x_2^{(j)}$	$x_3^{(j)}$	$\ldots x_n^{(j)}$	(Elder)
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
Gene pool	(0111)	(1001)	(1101)	(1011)	
(Genotype)	(0011)	(0001)	(1110)	(1101)	
	$\downarrow$	Gray coding		$\downarrow$	
Chromosomes	(0100110110111110)				
	(0010000110011011)				
	$\downarrow$				Recombination,
Crossover, Mutation,	Genetic Operators				Selection, Niching,
Genetic drift	$\downarrow$				Migration
Chromosomes	(0100110110111110)				
	(0110110010110010)				
	$\downarrow$	Gray decoding		$\downarrow$	
Genotype	(0110)	(1011)	(0101)	(0110)	
	(0100)	(1000)	(1101)	(0011)	
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
Phenotype	$x_1^{(1)^*}$	$x_2^{(1)^*}$	$x_3^{(1)^*}$	$\ldots x_n^{(1)^*}$	Offspring
	$x_1^{(j)^*}$	$x_2^{(j)^*}$	$x_3^{(j)^*}$	$\ldots x_n^{(j)^*}$	

Encoding

$$g_i = \begin{cases} b_i & \text{if } i = 1 \\ b_{i-1} \oplus b_i & \text{if } i \geq 2 \end{cases}$$

$$\mathbf{b}(13) = (1101)$$

$$(1101) \oplus (0110) = (1011)$$

Decimal	Binary		Gray
0	(0000)	↔	(0000)
1	(0001)		(0001)
2	(0010)		(0011)
3	(0011)		(0010)
4	(0100)		(0110)
5	(0101)		(0111)
6	(0110)	↔	(0101)

Decimal	Binary		Gray
7	(0111)	↔	(0100)
8	(1000)		(1100)
9	(1001)		(1101)
10	(1010)		(1111)
11	(1011)		(1110)
12	(1100)		(1010)
13	(1101)	↔	(1011)

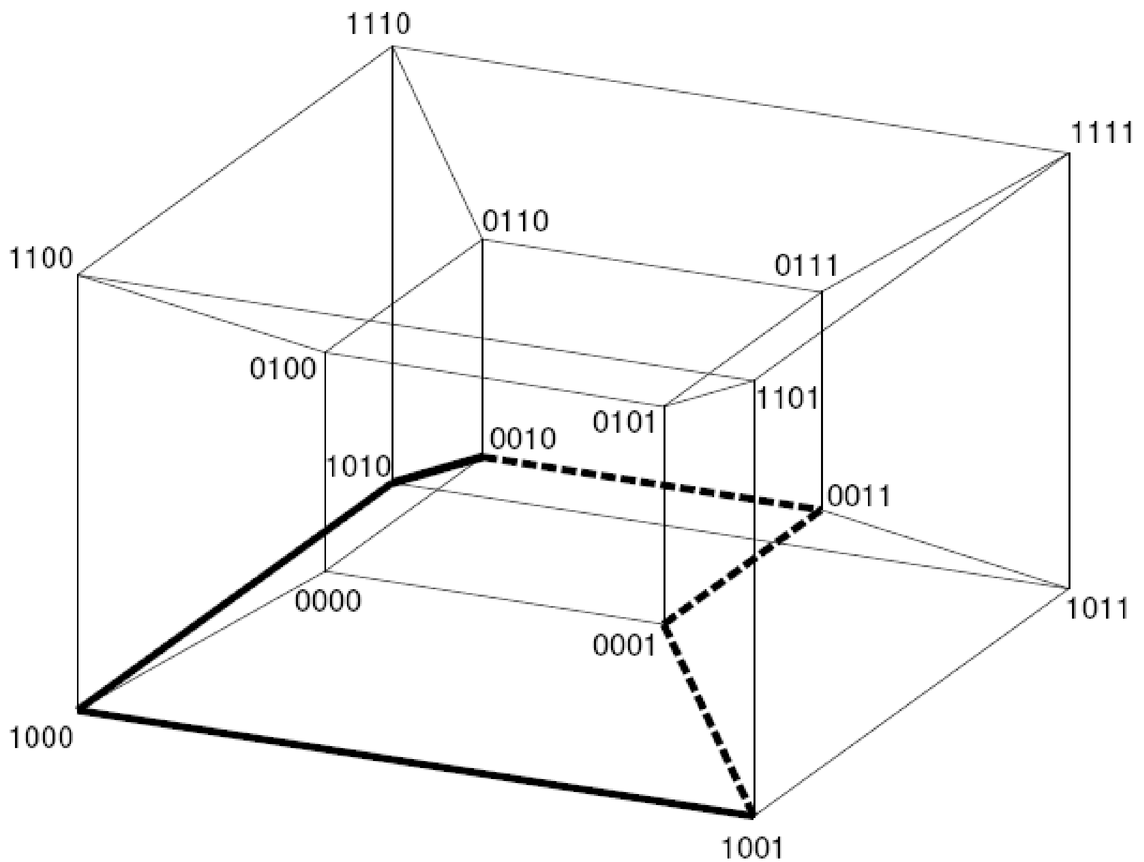
$$b_i = \bigoplus_{j=1}^{i} g_j$$

$$\mathbf{g} = (1011)$$

$$(1011) \oplus (0101) \oplus (0010) \oplus (0001) = (1101)$$

Genetic Operators: Crossover

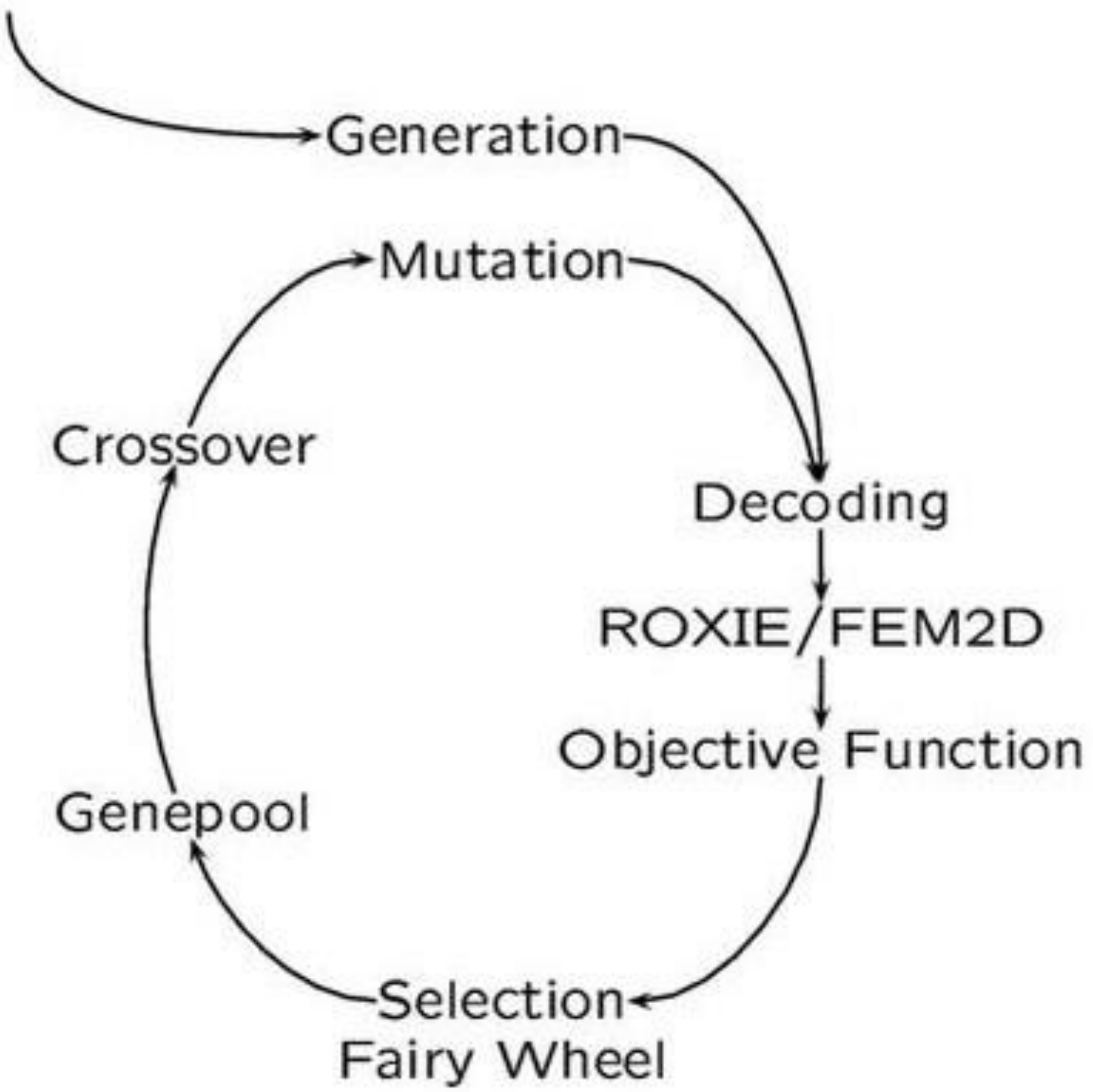
Chromosome A:	(0101001|101)		(0101001)		(011)		(0101001|011)
		:		×		:	
Chromosome B:	(1011010|011)		(1011010)		(101)		(1011010|101)



Genetic Operators: Mutation

	10	7	2	1	1	0	Phenotype, k
	1111	0100	0011	0001	0001	0000	Genotype, $\mathrm{g}(k)$
a)	10	8	2	1	1	0	Phenotype, k
	1111	**1**100	0011	0001	0001	0000	Genotype, $\mathrm{g}(k)$
b)	13	7	2	1	1	0	Phenotype, k
	1**0**11	0100	0011	0001	0001	0000	Genotype, $\mathrm{g}(k)$

Royal Road Algorithm



Fairy Wheel Selection

Index	Parent Population	Objective func. val.	Fitness value		Index of selected Chromosome	Child Population
1	(1000111010)	0.3	0.30		3	(1001101101)
2	(1110101101)	0.4	0.22		1	(1000111010)
3	(1001101101)	0.5	0.18		4	(1011010010)
4	(1011010010)	0.8	0.11	:	1	(1000111010)
5	(0111001100)	0.9	0.10		2	(1110101101)
6	(0111011010)	1.7	0.05		3	(1001101101)
7	(0011000101)	2.6	0.04		1	(1000111010)

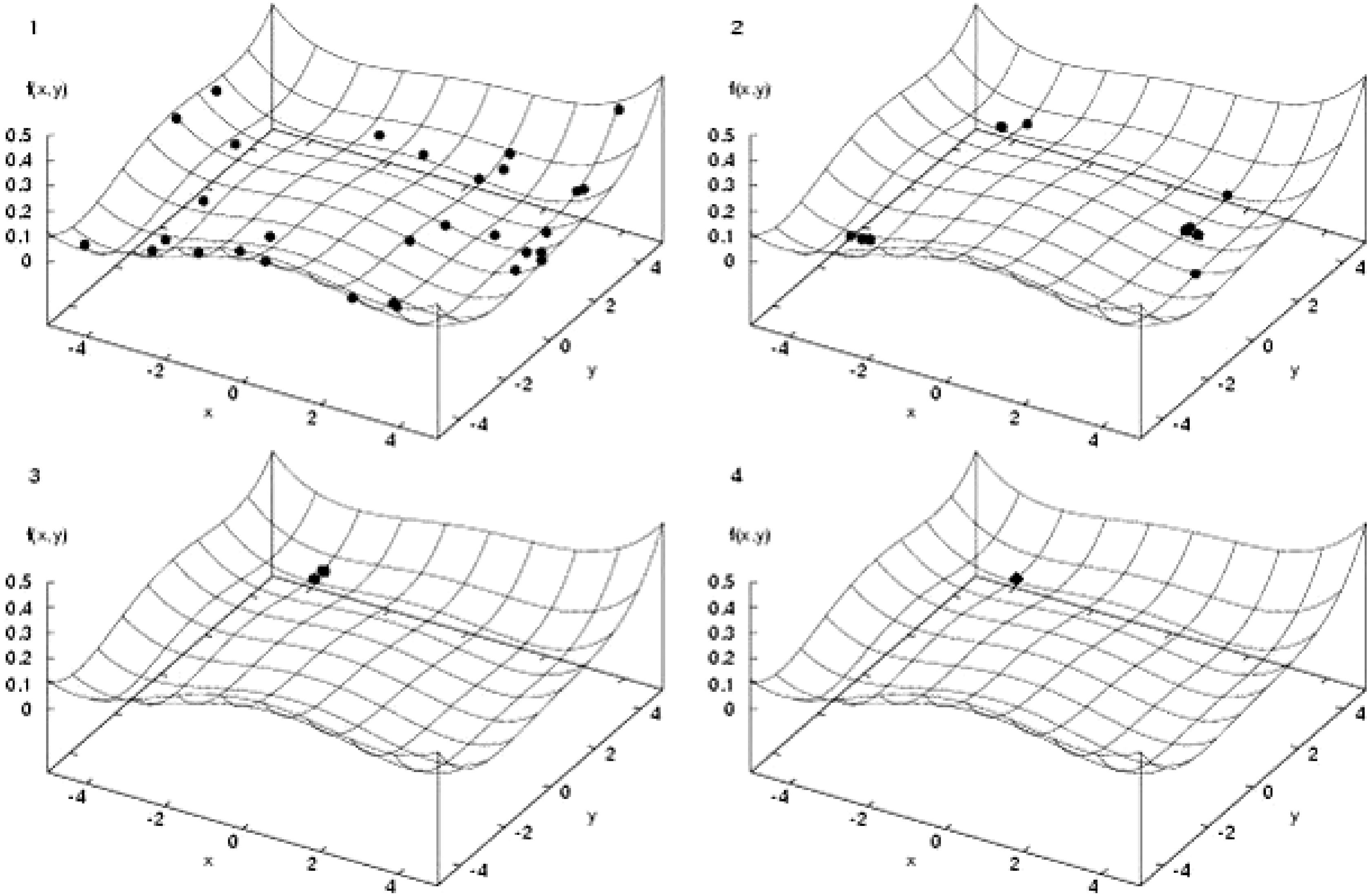
Fitness

$$e(\mathbf{b}) = f(\mathbf{x}) + c.$$

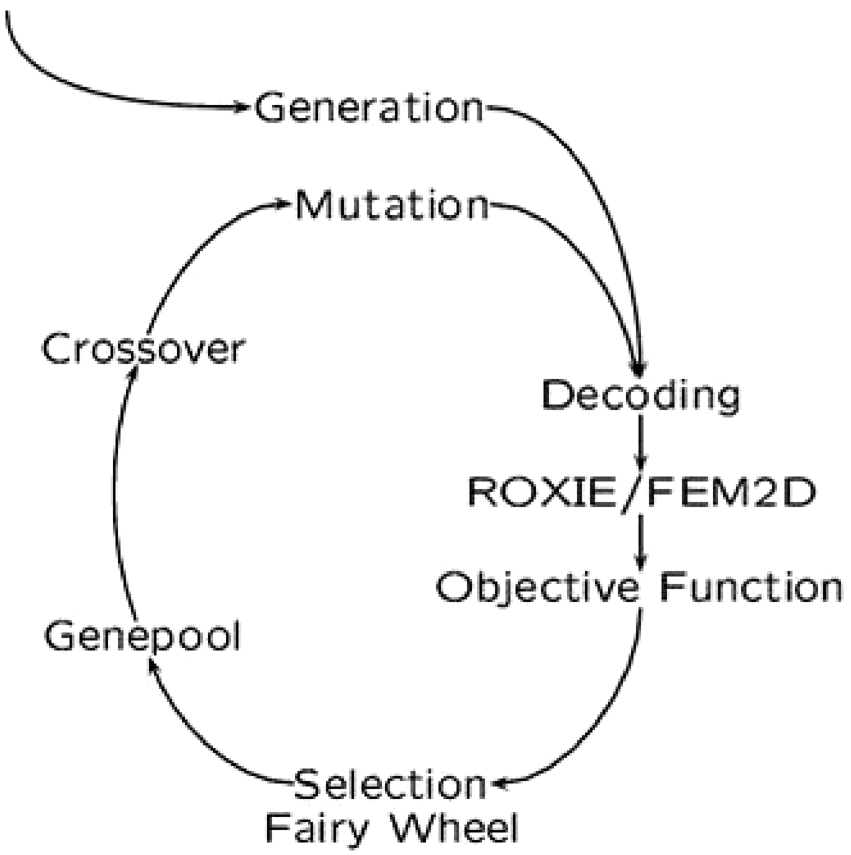
Likelihood of reproduction

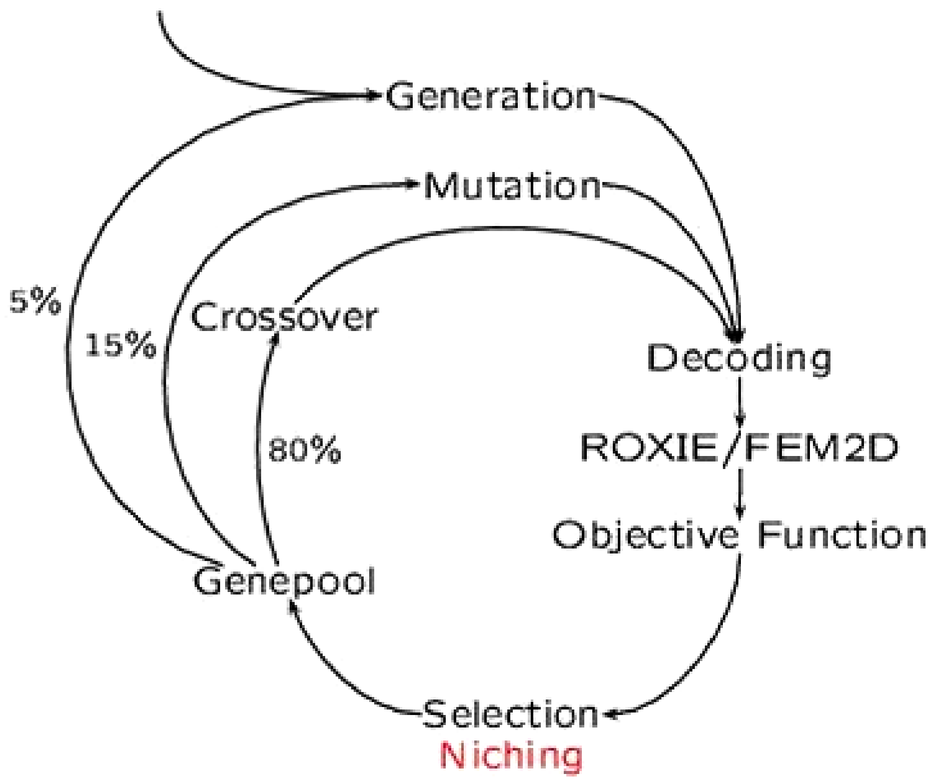
$$p_i := \frac{e(\mathbf{b}_i)}{\sum_{j=1}^{n} e(\mathbf{b}_j)}$$

Local Properties



Niching Genetic Algorithm





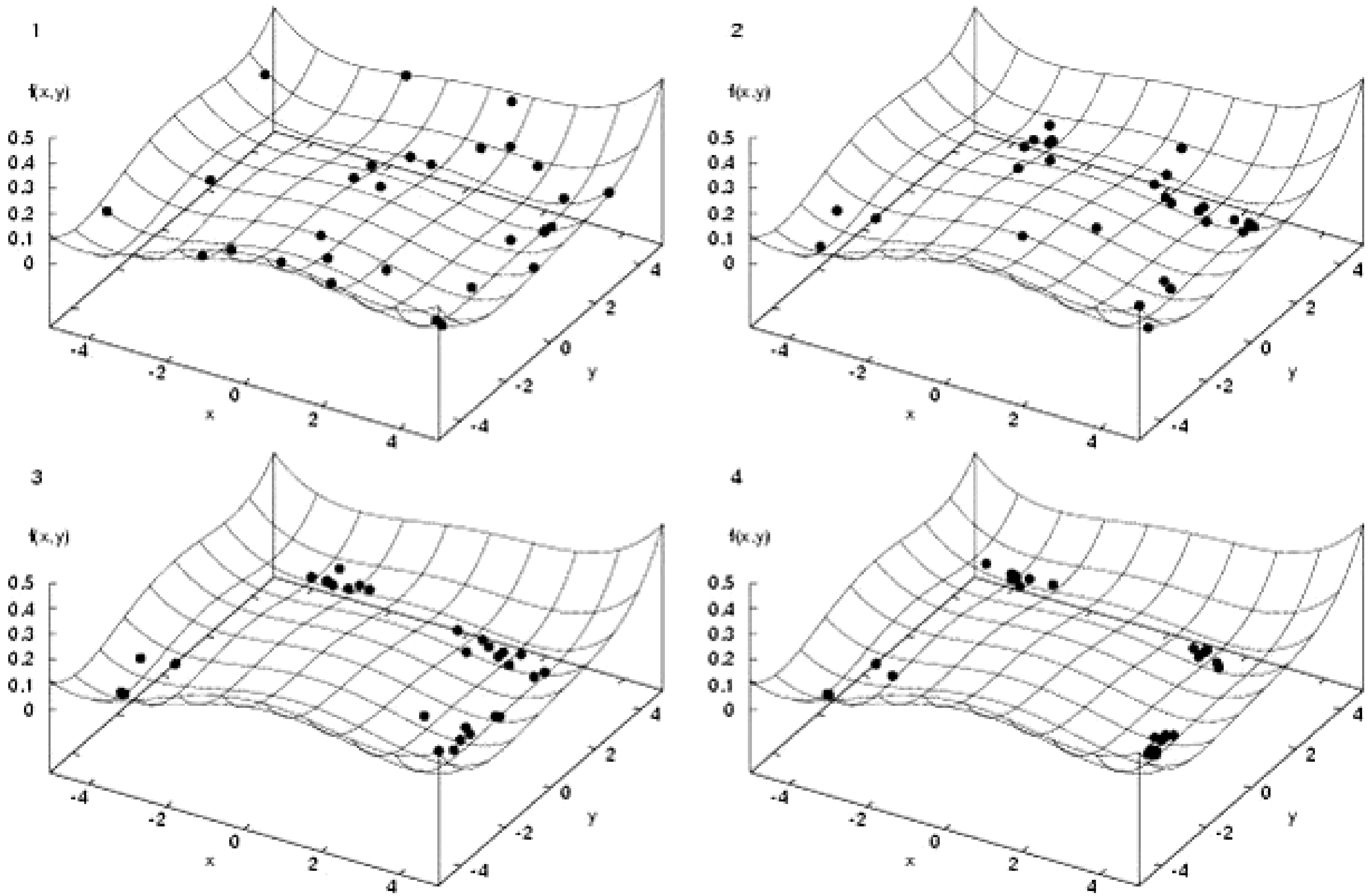
Niching Selection

Index	Parent Population	Objective func. val.	Hamming distance		Index of selected Chromosome	Child Population
1	(1000111010)	0.3	5		1	(1001101101)
2	(1110101101)	0.4	7		2	(1000111010)
3	(1001101101)	0.5	6		3	(1011010010)
4	(1011010010)	0.8	3	:	4	(1000111010)
5	(0111001100)	0.9	3		5	(0111011010)
6	(0111011010)	1.7	2		New	(0011001010)
7	(0011000101)	2.6	4		7	(1000111010)
New	(0011001010)	1.1	0			

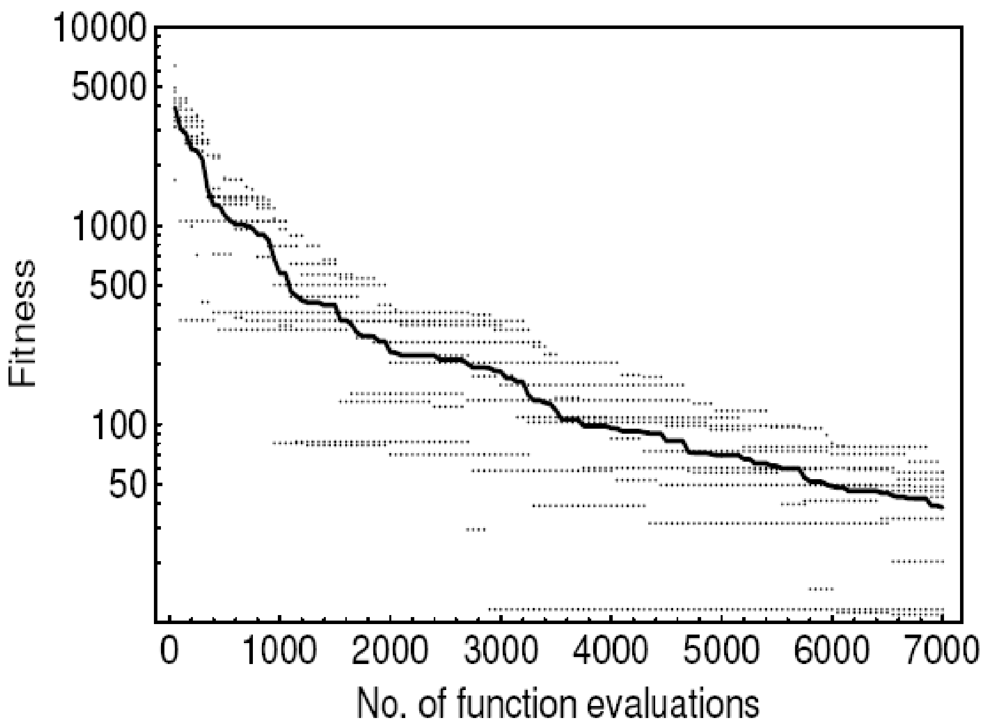
Fitness value

$$F_s(\mathbf{x}_i) := \frac{f(\mathbf{x}_i)}{\sum_{j=1}^{n} s(d(\mathbf{x}_i, \mathbf{x}_j))}$$

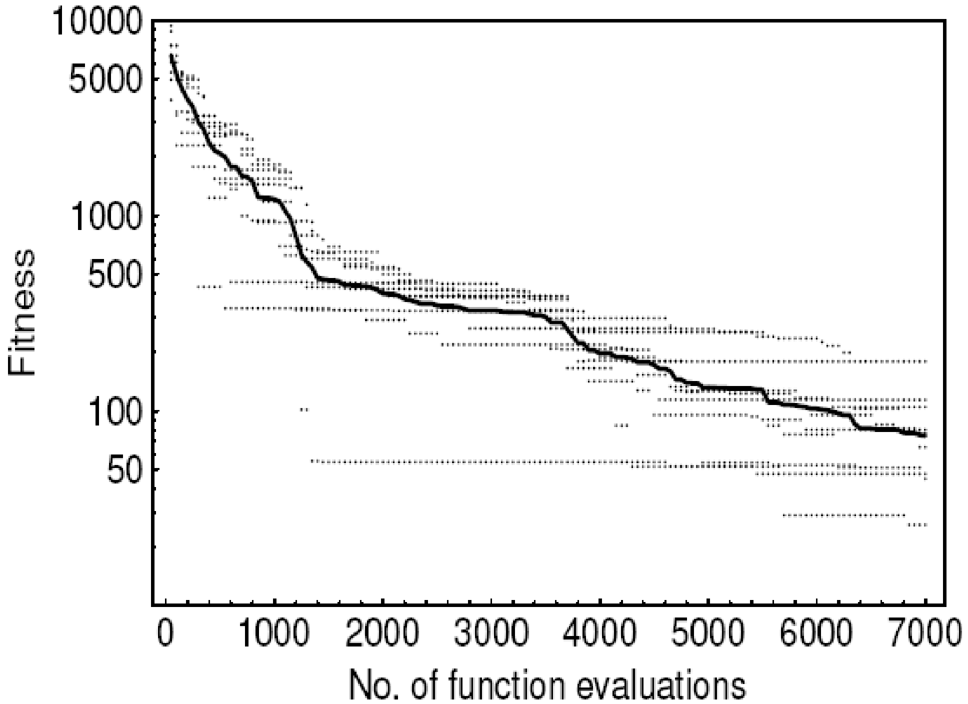
Global Properties



Convergence



Crossover rate 0.8
Mutation rate 0.15
Generation rate 0.05



Crossover rate 0.6
Mutation rate 0.35
Generation rate 0.05

Different Solutions found by Niching Genetic Algorithms