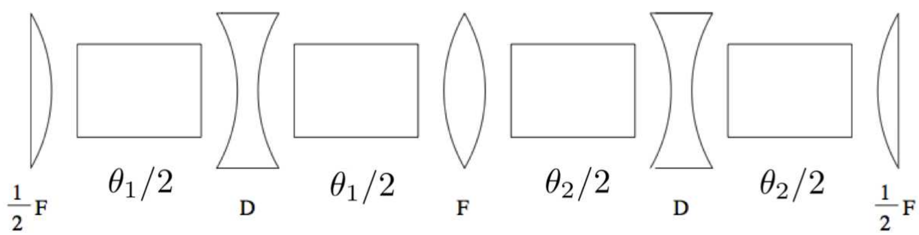
Transverse Beam Dynamics

JUAS tutorial 5 (solutions)

21 January 2013

1 Exercise: Dispersion suppressor

In several straight sections of the accelerator, like the ones hosting RF cavities, extraction systems or other devices such as detectors, it is preferable to have no dispersion $\eta(s) = \eta'(s) = 0$. For example, in big colliders, such as the LHC, where small spot sizes are required at the interaction points, the dispersion is reduced to zero at the detector positions. The most common dispersion suppressors consists of two FODO cells of equal length L and equal quadrupole strengths. Bending magnets are placed in the space between the quadrupoles with a different bending field in each FODO. Figure below shows a typical dispersion suppressor.



1.1 Considering two FODO cells with different total bend angles, $\theta_1 \neq \theta_2$, calculate the relation between the angles θ_1 and θ_2 which must be satisfied to cancel the dispersion at the end of the lattice.

Hint:

For each FODO cell, $M_{\text{FODO}} = M_{1/2\text{F}} \cdot M_{\text{dipole}} \cdot M_{\text{D}} \cdot M_{\text{dipole}} \cdot M_{1/2\text{F}}$, in thin-lens approximation we have the following 3×3 matrix:

$$M_{\text{FODO } j} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_j \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_j \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\mu & \beta\sin\mu & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_j \\ -\frac{\sin\mu}{\beta} & \cos\mu & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_j \\ 0 & 0 & 1 \end{pmatrix}$$

where $j = 1, 2$ (1=first cell, 2=second cell).
The following condition must be satisfied:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = M_{\text{suppressor}} \begin{pmatrix} \eta_0 \\ 0 \\ 1 \end{pmatrix} \tag{1}$$

where η_0 is the initial dispersion (at the middle of the first focusing quadrupole). It can be demonstrated that for a FODO lattice the dispersion has its maximum at the middle of the focusing quadrupole:

$$\eta_0 = \frac{4f^2}{L}\left(1 + \frac{L}{8f}\right)\theta \tag{2}$$

with $\theta = \theta_1 + \theta_2$ the total bend in the suppressor.

Answer:
Performing the corresponding matrix multiplication yields

$$M_{\text{suppressor}} = \begin{pmatrix} \cos 2\mu & \beta \sin 2\mu & D_x \\ -\frac{\sin 2\mu}{\beta} & \cos 2\mu & D'_x \\ 0 & 0 & 1 \end{pmatrix}$$

where:

$$\begin{aligned}
\cos 2\mu &= 1 - \frac{L^2}{2f^2} + \frac{L^4}{34f^4} \\
\beta \sin 2\mu &= 2L\left(1 - \frac{L^2}{8f^2}\right)\left(1 + \frac{L}{4f}\right) \\
\frac{\sin 2\mu}{\beta} &= \frac{L}{2f^2}\left(1 - \frac{L^2}{8f^2}\right)\left(1 - \frac{L}{4f}\right) \\
D_x &= \cos\mu \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_1 + \beta \sin\mu \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_1 + \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_2 \\
D'_x &= -\frac{\sin\mu}{\beta}\frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_1 + \cos\mu\left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_1 + \left(1 - \frac{L}{8f^2} - \frac{L^2}{32f^2}\right)\theta_2
\end{aligned} \tag{3}$$

Taking into account:
$\cos\mu = 1 - \frac{L^2}{8f^2}$; $\beta \sin\mu = L + \frac{L^2}{4f}$ and $\frac{\sin\mu}{\beta} = \frac{1}{4f^2}\left(1 - \frac{L}{4f}\right)$
the elements D_x and D'_xmay also be written as

$$\begin{aligned}
D_x &= \frac{L}{2}\left(1 + \frac{L}{8f}\right)\left[\left(3 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right] \\
D'_x &= \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\left[\left(1 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]
\end{aligned} \tag{4}$$

From the condition Eq. (1) we have

$$\begin{aligned}
\eta_0 \cos 2\mu + D_x &= 0 \\
-\eta_0 \frac{\sin 2\mu}{\beta} + D'_x &= 0
\end{aligned} \tag{5}$$

Substituting Eq. (2) in Eq. (5) one obtains:

$$\begin{aligned}
\left(3 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2 &= \left(4 - \frac{L^2}{4f^2} - \frac{8f^2}{L^2}\right)\theta \\
\left(1 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2 &= \left(2 - \frac{L^2}{4f^2}\right)\theta
\end{aligned}$$

In terms of phase advance μ this can be written as:

$$\begin{aligned}
\theta_1 &= \left(1 - \frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta \\
\theta_2 &= \frac{1}{4\sin^2\frac{\mu}{2}}\theta
\end{aligned} \tag{6}$$

where $\theta_1 + \theta_2 = \theta$.

1.2 Obtain the relation between the angles for the cases of phase-advance per cell $\mu = \pi/3$ and $\pi/2$

Answer:

- For $\mu = \pi/3 \longrightarrow 4\sin^2\frac{\mu}{2} = 1$ and therefore (using Eq. (6)) $\theta_1 = 0$ and $\theta_2 = \theta$. This corresponds to a dispersion suppressor with missing magnets.

- For $\mu = \pi/2 \longrightarrow 4\sin^2\frac{\mu}{2} = 2$ and therefore $\theta_1 = \theta_2 = \theta/2$.