Introduction to Transverse Beam Dynamics

Lecture 5: Chromaticity correction / Insertions / Stability

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Recap: Dispersion function and orbit

$$\begin{cases} x(s) = x_\beta(s) + x_D(s) \\ x(s) = C(s)\, x_0 + S(s)\, x_0' + D(s)\, \dfrac{\Delta p}{p} \end{cases}$$

In matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}_0$$

We can rewrite the solution in matrix form:

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$

Inside a magnet, the dispersion trajectory is solution of $D''(s) + K(s)\, D(s) = \frac{1}{\rho}$:

$$D(s) = S(s) \int_0^s \frac{1}{\rho(t)} C(t)\, \mathrm{d}t - C(s) \int_0^s \frac{1}{\rho(t)} S(t)\, \mathrm{d}t$$

Exercise: show that $D(s)$ is a solution for the equation of motion, with the initial conditions $D_0 = D_0' = 0$.

Recap: Dispersion propagation through the machine

- The equation:

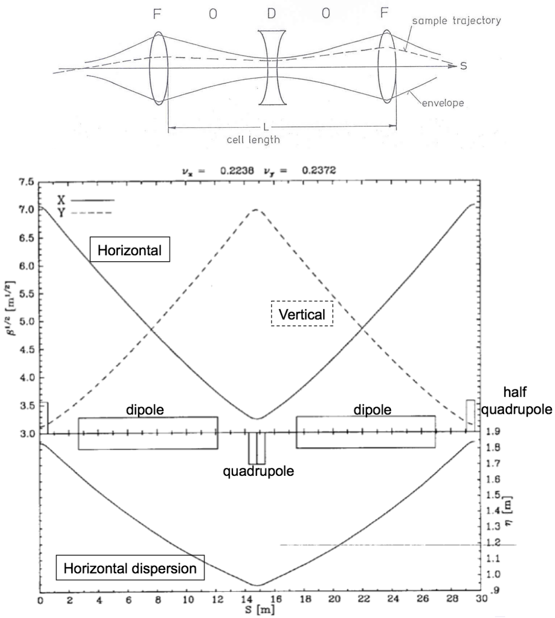
$$D\left(s\right)=S\left(s\right)\int_0^s\frac{1}{\rho\left(t\right)}C\left(t\right)\mathrm{d}t-C\left(s\right)\int_0^s\frac{1}{\rho\left(t\right)}S\left(t\right)\mathrm{d}t$$

  shows that the dispersion inside a magnet does not depend on the dispersion that might have been generated by the upstreams magnets.

- At the exit of a magnet of length L_m the dispersion reaches the value $D\left(L_\mathrm{m}\right)$, then it propagates from there on through the rest of the machine, just like any other particle:

$$\begin{pmatrix} D \\ D' \end{pmatrix}_s=\begin{pmatrix} C & S \\ C' & S' \end{pmatrix}\begin{pmatrix} D \\ D' \end{pmatrix}_0$$

Recap: FODO cell and its optical functions



Tune shift correction

Errors in the quadrupole fields induce tune shift:

$$\Delta Q = \oint_{\text{quads}} \frac{\Delta k\,(s)\,\beta\,(s)\,\mathrm{d}s}{4\pi}$$

Cure: we compensate the quad errors using other (correcting) quadrupoles

- If you use only one correcting quadrupole, with $1/f = \Delta k_1 L$
  - it changes both Q_x and Q_y:

$$\Delta Q_x = \frac{\beta_{1x}}{4\pi f_1} \quad \text{and} \quad \Delta Q_y = -\frac{\beta_{1y}}{4\pi f_1}$$

- We need to use two independent correcting quadrupoles:

$$\Delta Q_x = \frac{\beta_{1x}}{4\pi f_1} + \frac{\beta_{2x}}{4\pi f_2}$$
$$\Delta Q_y = -\frac{\beta_{1y}}{4\pi f_1} - \frac{\beta_{2y}}{4\pi f_2}$$

$$\begin{pmatrix} \Delta Q_x \\ \Delta Q_y \end{pmatrix} = \frac{1}{4\pi} \begin{pmatrix} \beta_{1x} & \beta_{2x} \\ \beta_{1y} & \beta_{2y} \end{pmatrix} \begin{pmatrix} 1/f_1 \\ 1/f_2 \end{pmatrix}$$

- Solve by inversion:

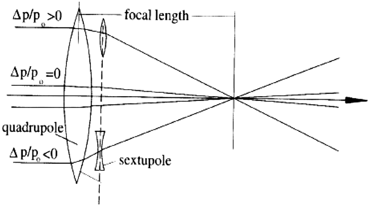
$$\begin{pmatrix} 1/f_1 \\ 1/f_2 \end{pmatrix} = \frac{4\pi}{\beta_{1x}\beta_{2y} - \beta_{2x}\beta_{1y}} \begin{pmatrix} \beta_{2y} & -\beta_{2x} \\ -\beta_{1y} & \beta_{1x} \end{pmatrix} \begin{pmatrix} \Delta Q_x \\ \Delta Q_y \end{pmatrix}$$

Chromaticity correction

Remember what is chromaticity: the quadrupole focusing experienced by particles changes with energy

- it induces tune shift, which can cause beam lifetime reduction due to resonances

Cure: we need additional energy dependent focusing. This is given by sextupoles



- The sextupole magnetic field rises quadratically:

$$B_x = \tilde{g}xy \qquad\qquad \Rightarrow \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \tilde{g}x \quad \text{a "gradient"}$$
$$B_y = \frac{1}{2}\tilde{g}\left(x^2 - y^2\right)$$

it provides a linearly increasing quadrupole gradient

Chromaticity correction (cont.)

Now remember:

- Normalised quadrupole strength is

$$k = \frac{g}{p/e} \; [\mathrm{m}^{-2}]$$

- Sextupoles are characterised by a normalised sextupole strength k_2, which carries a focusing quadrupolar component k_1:

$$k_2 = \frac{\tilde{g}}{p/e} \; [\mathrm{m}^{-3}]; \qquad \tilde{k}_1 = \frac{\tilde{g}x}{p/e} \; [\mathrm{m}^{-2}]$$

Cure for chromaticity: we need sextupole magnets installed in the storage ring in order to increase the focusing strength for particles with larger energy

- A sextupole at a location with dispersion does the trick: $x \to x + D \cdot \frac{\Delta p}{p}$

$$\tilde{k}_1 = \frac{\tilde{g}\left(x + D\frac{\Delta p}{p}\right)}{p/e} \; [\mathrm{m}^{-2}]$$

- for $x = 0$ it corresponds to an energy-dependent focal length

$$\frac{1}{f_{\mathrm{sext}}} = \tilde{k}_1 L_{\mathrm{sext}} = \overbrace{\underbrace{\frac{\tilde{g}}{p/e}}_{k_2} D\frac{\Delta p}{p}}^{\tilde{k}_1} \cdot L_{\mathrm{sext}} = k_2 D \cdot \frac{\Delta p}{p} \cdot L_{\mathrm{sext}}$$

Now the formula for the chromaticity rewrites:

$$\xi = \underbrace{-\frac{1}{4\pi}\oint k(s)\,\beta(s)\,\mathrm{d}s}_{\text{chromaticity due to quadrupoles}} + \underbrace{\frac{1}{4\pi}\oint k_2(s)\,D\beta(s)\,\mathrm{d}s}_{\text{chromaticity due to sextupoles}}$$

Design rules for sextupole scheme

- Chromatic aberrations must be corrected in both planes $\Rightarrow$ you need at least two sextupoles

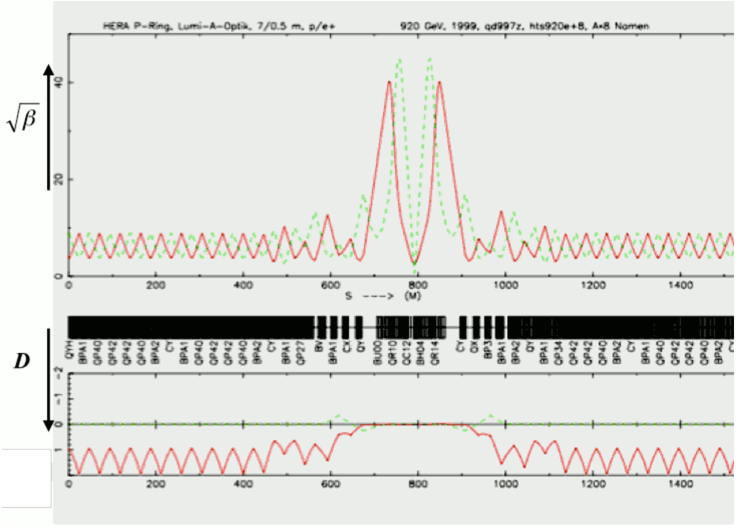
- In each plane the sextupole fields contribute with different signs to the chromaticity ξ_x and ξ_y:

$$\xi_x = -\frac{1}{4\pi} \oint \left[k - S_F D_x + S_D D_x\right] \beta_x (s) \, \mathrm{d}s$$
$$\xi_y = \frac{1}{4\pi} \oint \left[k - S_F D_x + S_D D_x\right] \beta_y (s) \, \mathrm{d}s$$

- To minimise chromatic sextupoles strengths, sextupoles should be located near quadrupoles where $\beta_x D_x$ and $\beta_y D_x$ are maximum.

- Important remark: for offset orbits, the sextupoles introduce *geometric aberrations*. They can be reduced by adopting a $-I$ transformation scheme:
  - put sextupoles in $(2n+1)\,\pi$ phase advance apart in a periodic lattice to compensate the effect

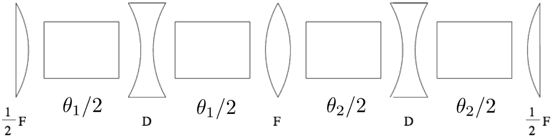
Insertions


HERA P-Ring, Lumi-A-Optik, 7/0.5 m, p/e+
920 GeV, 1999, qd997z, hts920e+8, A*8 Nomen
$\sqrt{\beta}$
D
S ---> (M)

Dispersion suppressor

In an arc, the FODO dispersion is non-zero everywhere. However, in straight sections, we often want to have $\eta = \eta' = 0$. $\Rightarrow$ for instance to keep small the beam size at the interaction point.

We can "match" between these two conditions with a "dispersion suppressor": a non-periodic set of magnets that transforms FODO η, η' to zero



Consider two FODO cells with length L and different total bend angles: θ_1, θ_2: we want to have

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix}_{\text{entrance}} = \begin{pmatrix} \eta_0 \\ 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} \eta \\ \eta' \end{pmatrix}_{\text{exit}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Note:

- the two cells have the same quadrupole strengths, so that they have also the same β, and μ (phase advance per cell)
- remember that $\alpha = 0$ at both ends, and that, if the incoming beam comes from a FODO cell with the same length L, phase advance μ, and with a total bending angle θ, then the initial dispersion is

$$\eta_0 = \frac{4f^2}{L}\left(1 + \frac{L}{8f}\right)\theta$$

Dispersion suppressor (cont.)

Transport for the dispersion:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix}_{suppressor} \begin{pmatrix} \eta_0 \\ 0 \\ 1 \end{pmatrix}$$

In 2×2 form reads

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \eta_0 \\ 0 \end{pmatrix} + \begin{pmatrix} D \\ D' \end{pmatrix}$$

which has solution

$$\begin{pmatrix} D \\ D' \end{pmatrix} = - \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \eta_0 \\ 0 \end{pmatrix}$$

The transfer matrix for the suppressor is

$$M_{\text{suppressor}} = M_{\text{FODO 2}} \cdot M_{\text{FODO 1}}$$

For each FODO cell, $M_{\text{FODO}} = M_{1/2\text{F}} \cdot M_{\text{dipole}} \cdot M_{\text{D}} \cdot M_{\text{dipole}} \cdot M_{1/2\text{F}}$, in thin-lens approximation:

$$M_{FODO\ j} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{l}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_j \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_j \\ 0 & 0 & 1 \end{pmatrix}$$

where $j = 1, 2$ (1=first cell, 2=second cell)

Dispersion suppressor (cont.)

If we do the math, we find

$$\begin{cases} D(s) = \frac{L}{2}\left(1 + \frac{L}{8f}\right)\left[\left(3 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right] \\ D'(s) = \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\left[\left(1 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right] \end{cases}$$

From lecture 3, we remember that the phase advance μ for a FODO cell, in terms of the length L and the focal length f, is

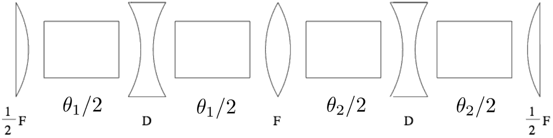
$$\left|\sin\frac{\mu}{2}\right| = \frac{L}{4f}$$

Thus, one can write the solution as a function of the phase advance μ, and of $\theta = \theta_1 + \theta_2$:

$$\begin{cases} \theta_1 = \left(1 - \frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta \\ \theta_2 = \frac{1}{4\sin^2\frac{\mu}{2}}\theta \end{cases}$$

Dispersion suppressor (summary)

Dispersion suppressor, a non-periodic set of magnets that transforms FODO η, η' to zero:



One possibility: two FODO cells with length *L*, phase advance μ, and different total bend angles: θ_1, θ_2:

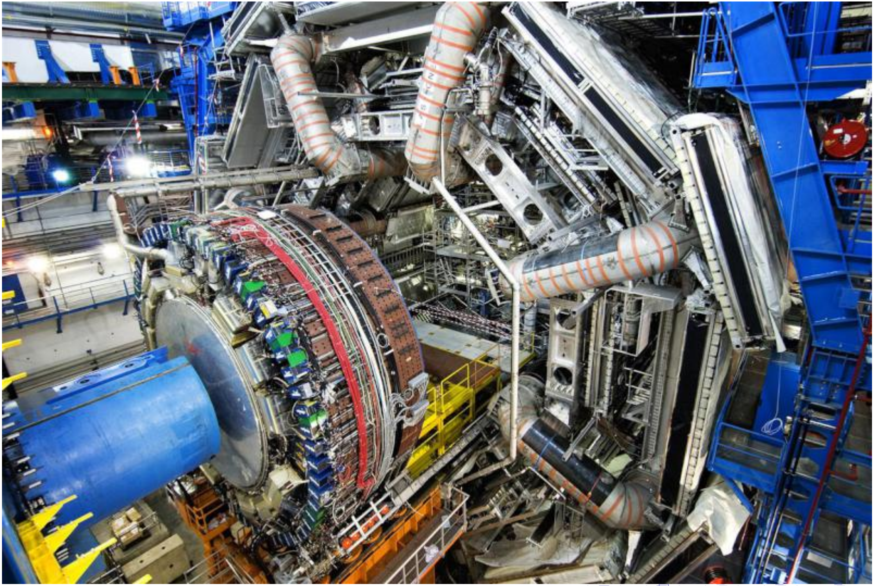
$$\begin{cases} \theta_1 = \left(1 - \dfrac{1}{4\sin^2\frac{\mu}{2}}\right)\theta \\ \theta_2 = \dfrac{1}{4\sin^2\frac{\mu}{2}}\theta \end{cases}$$

An interesting solution is for $\mu = 60^\circ$: in this case

- then $\theta_1 = 0$, and $\theta_2 = \theta \Rightarrow$ we just leave out two dipole magnets in the first FODO cell insertion
- this is called the "missing-magnet" scheme

Intermezzo

Often the insertions are larger than few meters...



The drift space

The most problematic insertion: the drift space !

Let's see what happens to the Twiss parameters α, β, and γ if we stop focusing for a while

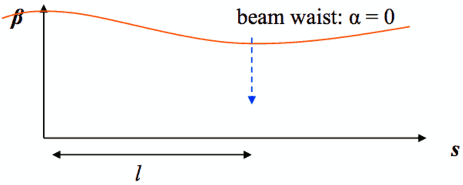
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

for a drift:

$$M_{\text{drift}} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \begin{cases} \beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2 \\ \alpha(s) = \alpha_0 - \gamma_0 s \\ \gamma(s) = \gamma_0 \end{cases}$$

Let's study the location of the waist: $\alpha = 0$

- the location of the point of smallest beam size, $\beta^\star$



Beam waist:

$$\alpha(s) = \alpha_0 - \gamma_0 s = 0 \qquad \rightarrow \qquad s = \frac{\alpha_0}{\gamma_0} = l_{\text{waist}}$$

Beam size at that point

$$\left.\begin{array}{l} \gamma(l) = \gamma_0 \\ \alpha(l) = 0 \end{array}\right\} \qquad \rightarrow \gamma(l) = \frac{1+\alpha^2(l)}{\beta(l)} = \frac{1}{\beta(l)} \qquad \rightarrow \beta_{\text{min}} = \frac{1}{\gamma_0}$$

This beta, at $l = l_{\text{waist}}$, is also called "beta star":

$$\Rightarrow \beta^\star = \beta_{\text{min}}$$

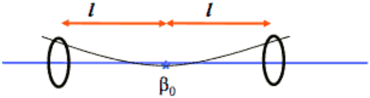
It's here that the interaction point (IP) is located.

Drift space with $L = l_{\text{waist}}$: The low β-insertion

We can assume we have a symmetry point at a distance l_{waist}:

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2, \text{ at} \alpha(s) = 0 \quad \rightarrow \beta^\star = \frac{1}{\gamma_0}$$

On each side of the symmetry point

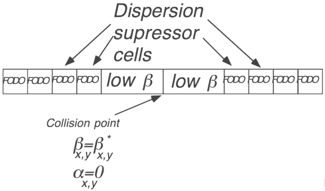


we have

$$\beta(s) = \beta^\star + \frac{s^2}{\beta^\star}$$

$\Rightarrow \beta$ grows quadratically with s.

A drift space at the interaction point, with length $L = l_{\text{waist}}$, is called "low-β insertion":



Phase advance in a low-β insertion

We have:

$$\beta(s) = \beta^\star + \frac{s^2}{\beta^\star}$$

The phase advance across the straight section is:

$$\Delta\mu = \int_{-L_{\text{waist}}}^{L_{\text{waist}}} \frac{\mathrm{d}s}{\beta^\star + \frac{s^2}{\beta^\star}} = 2\arctan\frac{L_{\text{waist}}}{\beta^\star}$$

which is close to $\Delta\mu = \pi$ for $L_{\text{waist}} \gg \beta^\star$.

In other words: the tune will increase by half an integer!

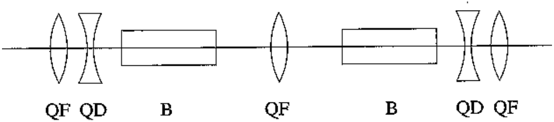
Achromatic insertions

There exist insertions (arcs) that don't introduce dispersion: they are called *achromatic arcs*

- In principle, dispersion can be suppressed by one focusing quadrupole and one bending magnet
- With one focusing quad in between two dipoles, one can get achromat condition: In between two bends, we call it arc section. Outside the arc section, we can match dispersion to zero. This is called "Double Bend Achromat" (DBA) structure
- We need quads outside the arc section to match the betatron functions, tunes, etc.
- Similarly, one can design "Triple Bend Achromat" (TBA), "Quadruple Bend Achromat" (QBA), and "Multi Bend Achromat" (MBA or nBA) structure
- For FODO cells structure, dispersion suppression section at both ends of the standard cells (see previous slides)

The Double Bend Achromat lattice (DBA)

Consider a simple DBA cell with a single quadrupole in the middle (plus external quadrupoles for matching).



$$M_{\text{DBA}} = M_{\text{B}} \cdot M_{\text{drift}} \cdot \underbrace{M_{1/2\text{F}} \cdot M_{1/2\text{F}}}_{M_{\text{F}}} \cdot M_{\text{drift}} \cdot M_{\text{B}}$$

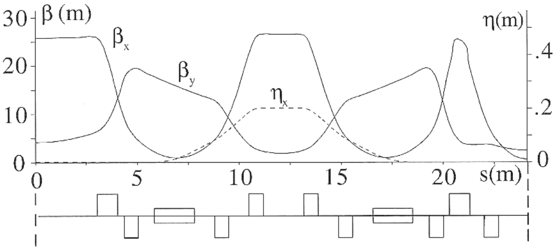
In thin-lens approximation, the dispersion matching condition:

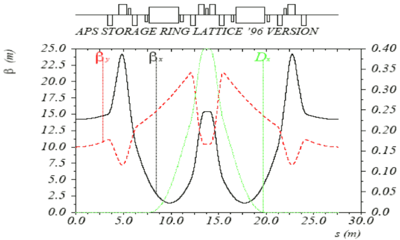
$$\begin{pmatrix} D_{\text{center}} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & L\theta/2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

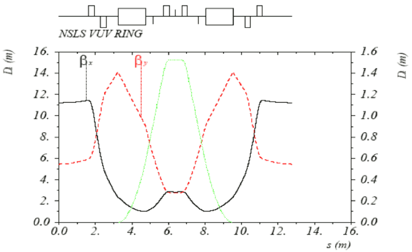
where f is the focal length of the quad, θ and L are the bend angle and the length of the dipole, and L_1 is the distance between the dipole and the centre of the quad.

$$f = \frac{1}{2}\left(L_1 + \frac{1}{2}L\right); \qquad D_{\text{center}} = \left(L_1 + \frac{1}{2}L\right)\theta$$

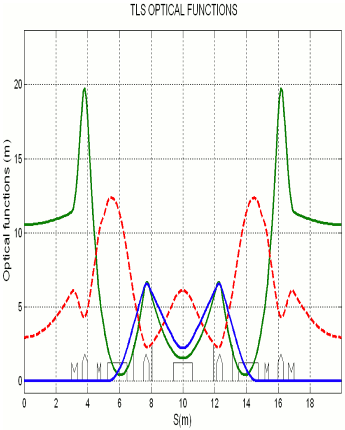
DBA optical functions

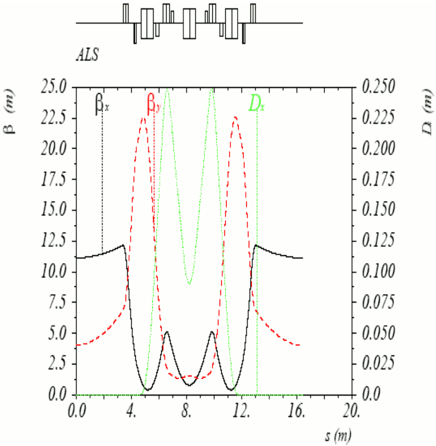






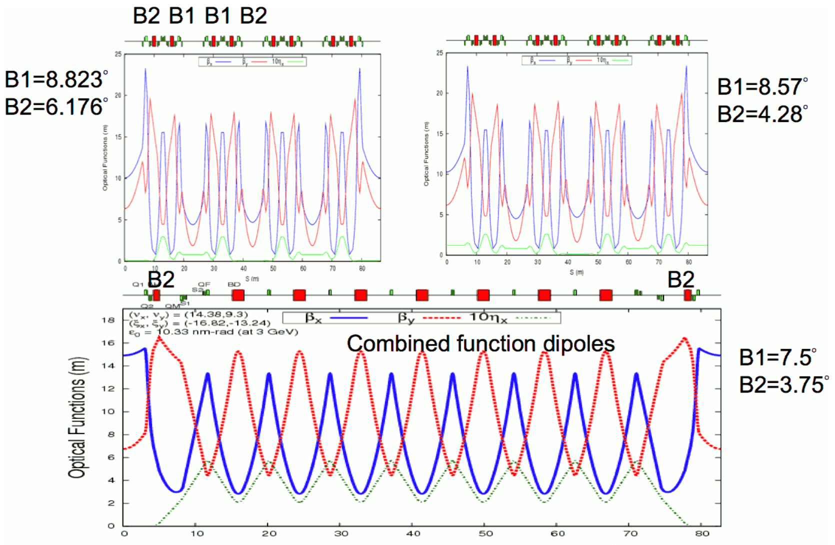
Triple Bend Achromat (TBA)





Combined function dipoles

QBA, OBA, and nBA



Last steps: 6-D phase space

In the real life the state vector is six-dimensional:

$$\begin{pmatrix} x & x' & y & y' & z & \Delta p/p \end{pmatrix}^T$$

and the transfer matrix is typically

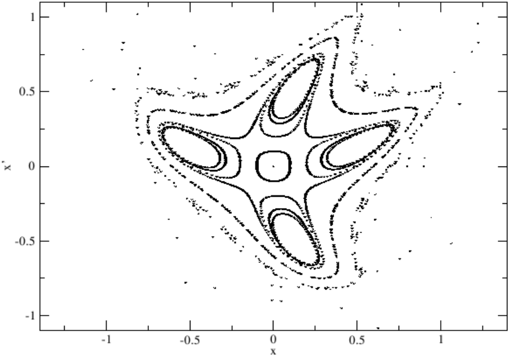
$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \frac{\Delta p}{p} \end{pmatrix}_s = \begin{pmatrix} R_{11} & R_{12} & \mathbf{0} & \mathbf{0} & 0 & R_{16} \\ R_{21} & R_{22} & \mathbf{0} & \mathbf{0} & 0 & R_{26} \\ \mathbf{0} & \mathbf{0} & R_{33} & R_{34} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \frac{\Delta p}{p} \end{pmatrix}_0$$

in bold the elements that would couple the $x - y$ motion.

Nota bene: this matrix can still represent **only** linear elements.

- if we want to consider high-order elements: e.g. sextupoles, octupoles, etc. $\Rightarrow$ we need computer simulations ! "particle tracking"
- because such elements introduce non-linear motion, which is too difficult to treat analytically

Non-linear dynamics



$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos(2\pi Q) & \sin(2\pi Q) \\ -\sin(2\pi Q) & \cos(2\pi Q) \end{pmatrix} \begin{pmatrix} x_n \\ x'_n + x_n^2 \end{pmatrix}$$

- Q=0.2516
- linear motion near center (circles)
- More and more square
- Non-linear tuneshift
- Islands
- Limit of stability
- Dynamic Aperture
- Crucial if strong quads and chromaticity correction in s.r. light sources
- many non-linearities in LHC due to s.c. magnet and finite manufacturing tolerances

Particle tracking with dynamic aperture

Dynamic aperture: *is a method used to calculate the amplitude threshold of stable motion of particles. Numerical simulations of particle tracking aims at determining the "dynamic aperture".*

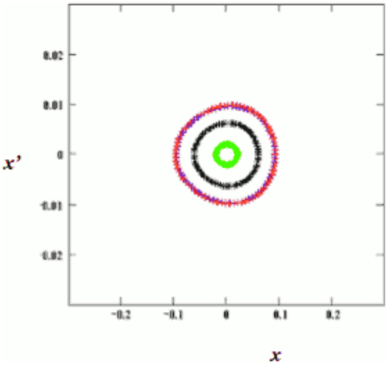
Dynamic aperture for hadrons

- in the case of protons or heavy ion accelerators, (or synchrotrons, or storage rings), there is minimal radiation, and hence the dynamics is symplectic
- for long term stability, a tiny dynamical diffusion can lead an initially stable orbit slowly into an unstable region
- this makes the dynamic aperture problem particularly challenging: One may need to consider the stability over billions of turns

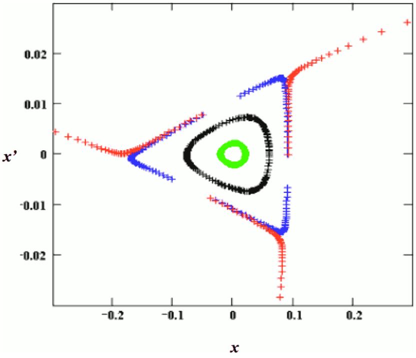
For the case of electrons

- in bending magnetic fields, the electrons radiate which causes a damping effect.
- this means that one typically only cares about stability over thousands of turns

Dynamic Aperture and tracking simulations



a beam of four particles in a storage ring composed by only linear elements



a beam of four particles in a storage ring where there is a strong sextupole: it's a catastrophe!!!

The end !

I'd like to thank a lot:

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