Transverse Beam Dynamics

JUAS tutorial 4 (solutions)

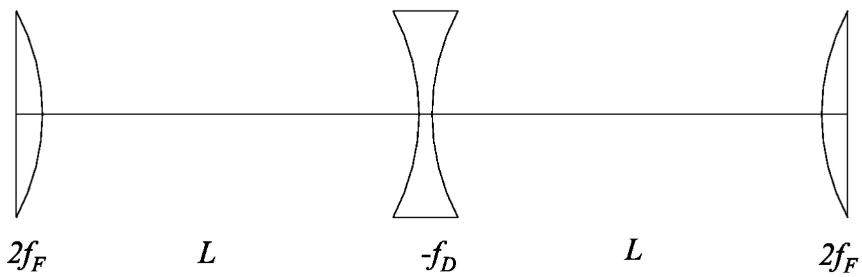
18 January 2013

1 Exercise: chromaticity in a FODO cell

Consider a ring made of N identical FODO cells with equally spaced quadrupoles. Assume that the two quadrupoles are both of length l_q, but their strengths may differ.

1.1 Calculate the maximum and the minimum betatron function in the FODO cell. (*Use the thin-lens approximations*)

Answer: First we calculate the transfer matrix for a FODO cell (see figure). We start from the center of the focusing quadrupole where the betatron function is maximum:



This exercise considers a general case where f_F is not necessarily equal to f_D. Using the thin lens approximation for the FODO cell with drifts of length L we get the following matrix:

$$
\begin{aligned}
M_{cell} &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_F} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_F} & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 - L(\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{2f_F f_D}) & 2L + \frac{L^2}{f_D} \\ \frac{1}{f_D} - \frac{1}{f_F}(1 - \frac{L}{2f_F} + \frac{L}{f_D} - \frac{L^2}{4f_F f_D}) & 1 - L(\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{2f_F f_D}) \end{pmatrix}
\end{aligned}
\tag{1}
$$

REMEMBER that in terms of betatron functions and phase advance the matrix of a FODO cell is given by:

$$
M_{cell} = \begin{pmatrix} cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix} \tag{2}
$$

Since β is maximum at the center of the focusing quadrupole: $\alpha = -\beta'/2 = 0$, and we can also write:

$$
M_{cell} = \begin{pmatrix} cos\mu & \beta \sin\mu \\ -\frac{\sin\mu}{\beta} & \cos\mu \end{pmatrix}
$$

Then, doing Eq. (1) equal to Eq. (2) we obtain:

$$\cos\mu = \frac{1}{2}\mathrm{tr}(M_{cell}) = 1 + \frac{L}{f_D} - \frac{L}{f_F} - \frac{L^2}{2f_D f_F} = 1 - 2\sin^2\frac{\mu}{2}$$

or

$$2\sin^2\frac{\mu}{2} = \frac{L}{f_F} - \frac{L}{f_D} + \frac{L^2}{2f_D f_F} \tag{3}$$

Here we have applied the following trigonometric relation: $\cos\mu = \cos(\frac{\mu}{2}+\frac{\mu}{2}) = \cos^2\frac{\mu}{2} - \sin^2\frac{\mu}{2} = 1 - 2\sin^2\frac{\mu}{2}$.

The maximum for the betatron function β_{max} will occur at the focusing quadrupole. Since Eq. (1) is for a periodic cell starting at the center of the focusing quadrupole, the m_{12} component of the matrix gives us

$$\beta_{max}\sin\mu = 2L + \frac{L^2}{f_D}$$

Rearranging things:

$$\beta_{max} = \frac{2L + \frac{L^2}{f_D}}{\sin\mu}$$

On the other hand, the minimum for the betatron function will occur at the defocusing quadrupole position. Therefore, interchanging f_F with $-f_D$ for a FODO cell gives:

$$\beta_{min} = \frac{2L - \frac{L^2}{f_F}}{\sin\mu}$$

1.2 Calculate the natural chromaticities for this machine.

Answer:

Let us remember the definition of natural chromaticity. The so-called "natural" chromaticity is the chromaticity that derives from the energy dependence of the quadrupole focusing, i.e. the chromaticity arising only from quadrupoles. The chromaticity is defined in the following way:

$$\xi = \frac{\Delta Q}{\Delta p/p_0} \tag{4}$$

where ΔQ is the tune shift due to the chromaticity effects and $\Delta p/p_0$ is the momentum offset of the beam or the particle with respect to the nominal momentum p_0.

The natural chromaticity is defined as (remember from the lecture):

$$\xi_N = -\frac{1}{4\pi}\oint \beta(s)k(s)ds \tag{5}$$

Sometimes, especially for small accelerators, the chromaticity is normalized to the machine tune Q and defined also as:

$$\xi = \frac{\Delta Q/Q}{\Delta p/p_0} \tag{6}$$

$$\xi_N = -\frac{1}{4\pi Q}\oint \beta(s)k(s)ds \tag{7}$$

For this exercise, either you decide to use Eq. (5) or Eq. (7) it is fine! From now on let me use Eq. (7):

$$
\begin{aligned}
\xi_N &= -\frac{1}{4\pi Q}\oint \beta(s)k(s)ds \\
&= -\frac{1}{2N_{cell}\mu} \times N_{cell}\int_{cell} \beta(s)k(s)ds \\
&= -\frac{1}{2\mu}\sum_{i\in\{quads\}} \beta_i(kl_q)_i
\end{aligned}
$$

Here we have used $Q = (N_{cell}\mu)/(2\pi)$ and the following approximation valid for thin lens:

$$
\int_{cell} \beta(s)k(s)ds \simeq \sum_{i\in\{quads\}} \beta_i(kl_q)_i
$$

where we sum over each quadrupole i in the cell. In the case of the FODO cell we have two half focusing quadrupoles and one defocusing quadrupole. Taking into account that $(kl_q)_i = 1/f_i$, we have:

$$
\begin{aligned}
\xi_N &\simeq -\frac{1}{2\mu}\sum_{i\in\{quads\}} \beta_i(kl_q)_i \\
&= -\frac{1}{2\mu}\left[\beta_{max}\left(\frac{1}{2f_F}\right) + \beta_{min}\left(-\frac{1}{f_D}\right) + \beta_{max}\left(\frac{1}{2f_F}\right)\right] \\
&= -\frac{1}{2\mu}\left[\beta_{max}\left(\frac{1}{f_F}\right) + \beta_{min}\left(-\frac{1}{f_D}\right)\right] \\
&= -\frac{1}{2\mu\sin\mu}\left[\left(2L + \frac{L^2}{f_D}\right)\frac{1}{f_F} - \left(2L - \frac{L^2}{f_F}\right)\frac{1}{f_D}\right] \\
&= -\frac{L}{\mu\sin\mu}\left[\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{f_F f_D}\right]
\end{aligned}
$$

1.3 Show that for short quadrupoles, if $f_F \simeq f_D$,

$$
\xi_N \simeq -\frac{2\tan\frac{\mu}{2}}{\mu}
$$

Answer: If $f_F \simeq f_D$, we have

$$
\begin{aligned}
\xi_N &\simeq -\frac{1}{\mu\sin\mu}\frac{L^2}{f_F f_D} \\
&= -\frac{1}{2\mu\sin\frac{\mu}{2}\cos\frac{\mu}{2}} 4\sin^2\frac{\mu}{2}
\end{aligned}
$$

where we do $\sin\mu = \sin(\frac{\mu}{2} + \frac{\mu}{2}) = 2\sin\frac{\mu}{2}\cos\frac{\mu}{2}$
and considering Eq. (3): $4\sin^2\frac{\mu}{2} = \frac{L^2}{f_F f_D}$
we finally obtain

$$
\xi_N \simeq -\frac{2\tan\frac{\mu}{2}}{\mu}
$$

2 Exercise: measurement of Twiss parameters

One of the possible ways to determine experimentally the Twiss parameters at a given point makes use of a so-called quadrupole scan. One can measure the transverse size of the beam in a profile monitor, called Wire Beam Scanner (WBS), located at a distance L downstream a focusing quadrupole, as a function of the normalized gradient in this quadrupole. This allows to compute the emittance of the beam, as well as the β- and the α- functions at the entrance of the quadrupole.

Let's consider a quadrupole Q with a length of $l = 20$ cm. This quadrupole is installed in an electron transport line where the particle momentum is 300 MeV$/c$. At a distance $L = 10$ m from the quadrupole the transverse beam size is measured with a WBS, for various values of the current I_Q. The maximum value of the quadrupole gradient G is obtained for a current of 100 A, and is $G = 1$ T/m. G is proportional to the current.

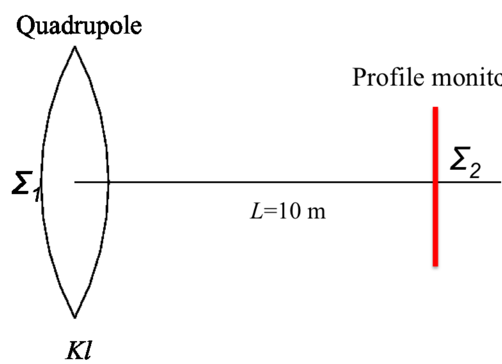
Advice: use thin-lens approximation.

2.1 How does the normalized focusing strength K vary with I_Q?

Answer:

If G proportional to I_Q: $G = C \cdot I_Q$ where C is the proportionality coefficient. We know that $G = 1$ T/m when $I_Q = 100$ A, therefore $C = 0.01$ T/(A·m).

2.2 Let Σ_1 and Σ_2 be the 2×2 matrices with the twiss parameters, $\Sigma = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$, at the quadrupole entrance and at the wire scanner, respectively.



It is worth explaining that the matrix Σ multiplied by the emittance ϵ is the covariance matrix of the beam distribution:

$$\Sigma\epsilon = \begin{pmatrix} \beta\epsilon & -\alpha\epsilon \\ -\alpha\epsilon & \gamma\epsilon \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

The transverse beam size of the beam is given by $\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta_x \epsilon_x}$ (horizontal beam size), and $\sigma_y = \sqrt{\langle y^2 \rangle} = \sqrt{\beta_y \epsilon_y}$ (vertical beam size). Here we will simply use the following notation: $\sigma_1 = \sqrt{\beta_1 \epsilon}$ for the beam size (horizontal or vertical) at position 1, and $\sigma_2 = \sqrt{\beta_2 \epsilon}$ for the beam size (horizontal or vertical) at position 2.

- Give the expression Σ_2 as function of α_1, β_1, and γ_1

Answer:

The matrix Σ propagates from position 1 to position 2 as follows:

$$\Sigma_2 = M \Sigma_1 M^T$$

where M is the transfer matrix of the system and M^T its transpose. We have:

$$\begin{aligned} \Sigma_2 = \begin{pmatrix} \beta_2 & -\alpha_2 \\ -\alpha_2 & \gamma_2 \end{pmatrix} &= \begin{pmatrix} 1 - KlL & L \\ -Kl & 1 \end{pmatrix} \begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} 1 - KlL & -Kl \\ L & 1 \end{pmatrix} \\ &= \begin{pmatrix} \beta_1 L^2 (Kl)^2 + 2L(\alpha_1 L - \beta_1) Kl + \beta_1 - 2\alpha_1 L + \gamma_1 L^2 & \beta_1 L (Kl)^2 + (2\alpha_1 L - \beta_1) Kl + \gamma_1 L - \alpha_1 \\ \beta_1 L (Kl)^2 + (2\alpha_1 L - \beta_1) Kl + \gamma_1 L - \alpha_1 & \beta_1 (Kl)^2 + 2\alpha_1 Kl + \gamma_1 \end{pmatrix} \end{aligned} \tag{8}$$

- Show that β_2 can be written in the form: $\beta_2 = A_2 (Kl)^2 + A_1 (Kl) + A_0$

- Express A_0, A_1, and A_2 as a function of L, α_1, β_1, and γ_1

Answer:

We can see from Eq. (8) that:

$$\beta_2 = \beta_1 L^2 (Kl)^2 + 2L(\alpha_1 L - \beta_1)Kl + \beta_1 - 2\alpha_1 L + \gamma_1 L^2$$

and therefore:

$$\begin{aligned} A_2 &= \beta_1 L^2 \\ A_1 &= 2L(\alpha_1 L - \beta_1) \\ A_0 &= \beta_1 - 2\alpha_1 L + \gamma_1 L^2 \end{aligned}$$

Hint for the next questions: show that if you express β_2 as

$$\beta_2 = B_0 + B_1 (Kl - B_2)^2$$

you have:

$B_0 = A_0 - A_1^2/4A_2^2 = L^2/\beta_1$

$B_1 = A_2 = L^2\beta_1$

$B_2 = -A_1/A_2 = 1/L - \alpha_1/\beta_1$

2.3 The transverse beam r.m.s. beam size is $\sigma = \sqrt{\epsilon\beta}$, where ϵ is the transverse emittance. Express σ_2 as a function of Kl and find its minimum, $(Kl)_{\mathbf{min}}$. Give the expression for $\frac{\mathrm{d}\sigma_2}{\mathrm{d}(Kl)}$.

As we have seen in the previous questions β_2 depends quadratically on Kl: $\beta_2 = B_0 + B_1 (Kl - B_2)^2$. Since ϵ is constant, if we want to minimize σ_2, we have to minimize β_2:

$$\frac{\mathrm{d}\beta_2}{\mathrm{d}(Kl)} = 0 \longrightarrow 2B_1(Kl - B_2) = 0 \longrightarrow (Kl)_{min} = B_2 = \frac{1}{L} - \frac{\alpha_1}{\beta_1} \tag{9}$$

We can write:

$$\sigma_2^2 = \beta_2\epsilon = \frac{L^2}{\beta_1}\left(1 + \beta_1^2(Kl - (Kl)_{min})^2\right)\epsilon$$

Why is this useful? By means of a quadrupole scan (changing the strength of the quadrupole) we look for the strength Kl which minimizes the value σ_2^2. We fit a parabola to the measurements σ_2^2 vs. Kl, and select then $\sigma_2^2((Kl)_{min})$. The minimum beam size is given by:

$$\mathrm{Min}(\sigma_2) = L\sqrt{\frac{\epsilon}{\beta_1}} = \sqrt{B_0\epsilon} \tag{10}$$

The derivative of σ_2 is: $\frac{\mathrm{d}\sigma_2}{\mathrm{d(Kl)}} = \frac{L^2\beta_1}{\sigma_2}(Kl - (kl)_{min})\epsilon$

2.4 How does σ_2 vary with Kl when $|Kl - (Kl)_{\mathbf{min}}| \gg 1/\beta_1$?

Under this condition:

$$\sigma_2^2 = \frac{L^2}{\beta_1}\left(1 + \beta_1^2(Kl - (Kl)_{min})^2\right)\epsilon \longrightarrow \sigma_2 \simeq L\sqrt{\beta_1\epsilon}(Kl - (Kl)_{min})$$

For $|Kl - (Kl)_{\mathrm{min}}| \gg 1/\beta_1$, σ_2 depends linearly on Kl, with slope $\frac{\mathrm{d}\sigma_2}{\mathrm{d(Kl)}} = L\sqrt{\beta_1\epsilon} = L\sigma_1$.

2.5 Deduce the values of α_1, β_1, and γ_1 from the measurement σ_2, as a function of the quadrupole current I_Q.

We know that

$$Kl = \frac{G \cdot l}{p/e} = \frac{C \cdot l \cdot I_Q}{p/e} = \frac{0.01[\mathrm{T}/(\mathrm{Am})] \cdot 0.2[\mathrm{m}]}{(0.3[\mathrm{GeV}]/0.3)[\mathrm{Tm}]} \cdot I_Q = 2 \times 10^{-3} \cdot I_Q$$

If we measure σ_2 as a function of the quadrupole current I_Q, from the minimum value we can get β_1 (Eq. (10)), and since from the measurement we obtain $(Kl)_{min} = 2 \times 10^{-3}(I_Q)_{min}$, using Eq. (9) we can calculate α_1. Once we know β_1 and α_1, it is then straightforward to calculate $\gamma_1 = (1 + \alpha_1^2)/\beta_1$.