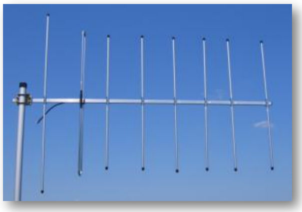
Preliminary remarks

An important consequence of classical electrodynamics is the generation of electomagnetic waves by accelerated charges particles.

Example: The antenna



The RF-voltage produces an electric field

$$E(t) = E_0 \sin \omega t$$

It causes in the antenna rod onto the electrons the force

$$F(t) = e E_0 \sin \omega t$$

and consequently the accelleration

$$a(t) = \frac{e}{m} E_0 \sin \omega t$$

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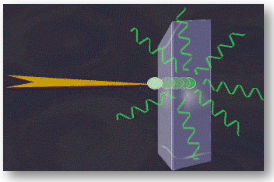
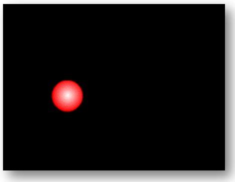
As soon as a fast moving electron hits a solid state body it is decelerated. Actually it is transversly bend by the coulomb field of the atoms. Bending a charged particle is a transverse acceleration. According to classical electrodynamics theese particles emit electromagnetic radiation.

$\Rightarrow$ X-ray radiation or „Bremsstrahlung“





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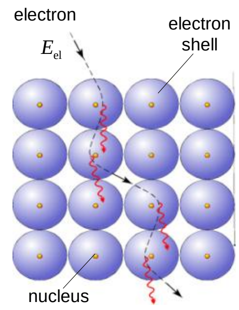

The energy of the electrons is

$$E_{el} = eU$$

Then the energy of the X-ray is in the range of

$$0 < E_{X-ray} \le E_{el}$$

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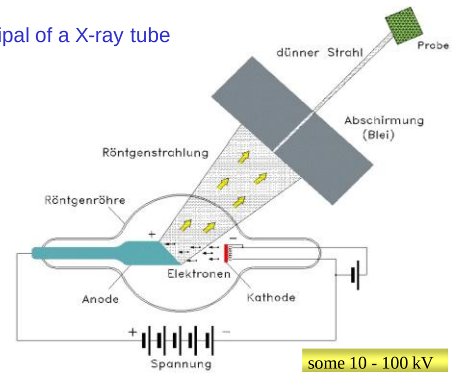
Principal of a X-ray tube



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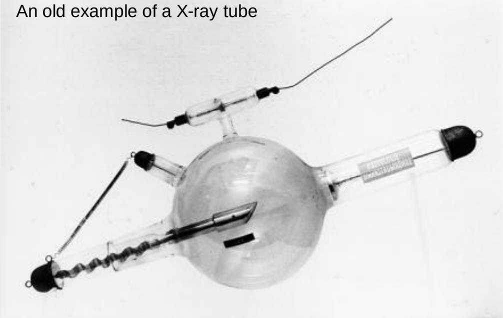
An old example of a X-ray tube



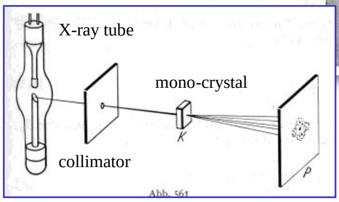
The X-ray radiation has been discovered by Wilhelm Conrad Röntgen

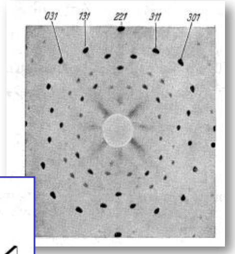
1895: Discovery of the X-ray radiation

The hand of Mrs. Röntgen

X-rays are a powerful tool to study the properties of all kinds of material.

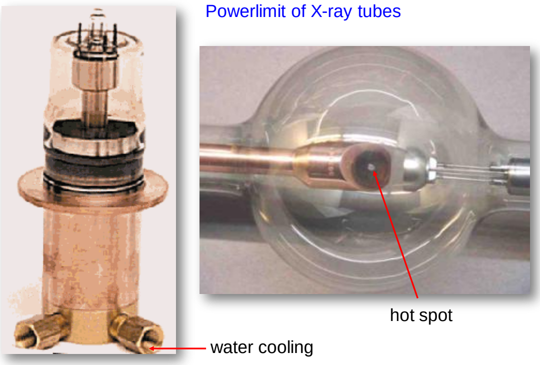
The X-ray tube provides a wide wavelength spectrum of radiation.



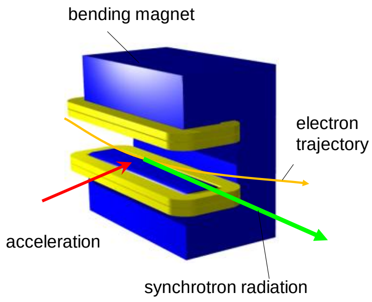


Laue-interference of a NaCl-cristal

Powerlimit of X-ray tubes



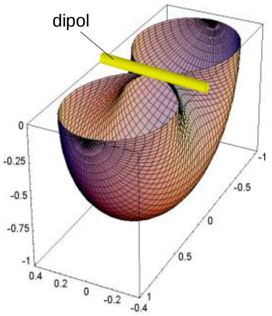
Relativistic electons passing through a vertical magnetic field



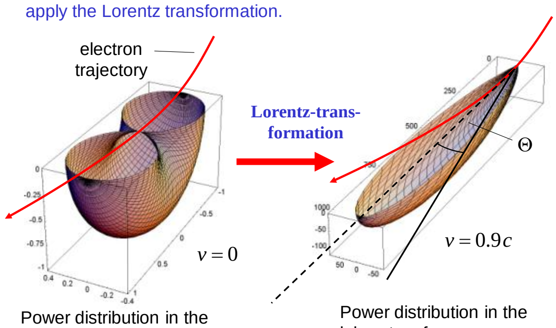
In the dipole magnet the electrons feel a horizontal acceleration.

This causes also a kind of electromagnetic radiation

In the center of mass frame of the electron the spartial power distribution of the radiation is the same as of the Hertz' dipole



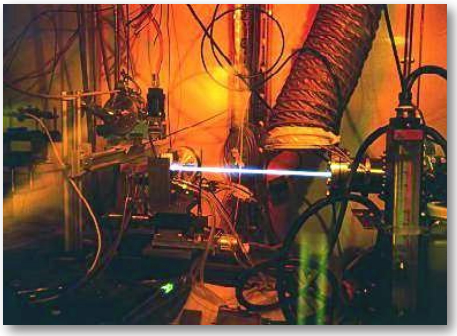
Because of the relativistic velocity of the electrons one has to apply the Lorentz transformation.



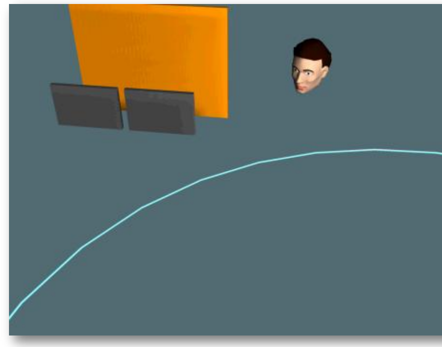
Power distribution in the center of mass frame

Power distribution in the laboratory frame

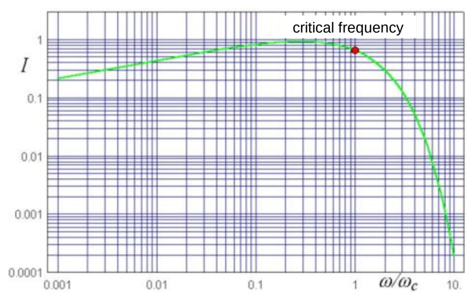
A synchrotron radiation beam



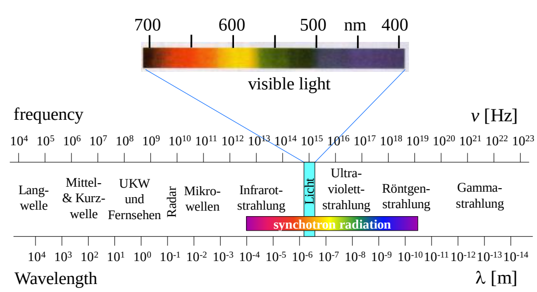
Time structure of the synchrotron radiation



Because of the short radiation flash we have a wide frequency spectrum of the radiation emitted by the relativistic elevtrons.



Spektrum of electromagnetic radiation



Alfred-Marie Liénard
1869 - 1958

In 1898 *Alfred-Marie Liénard* has calculated the radiation emitted by a moving charged particle.

Due to his results the radiated power by relativistic particles is given by the relativistic invariant expression

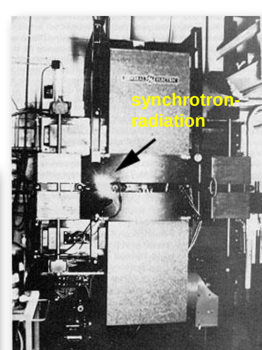
$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 \left(m_0 c^2\right)^2}\left[\left(\frac{d\vec{p}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2\right]$$

At that time the possible electron energy in a laboratory was strongly limited to some 100 keV. Therefore, it was not possible to produce this kind of radiation.

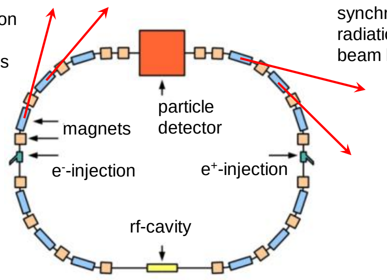
In 1947 a **70 MeV-Synchrotron** was built by General-Electric.

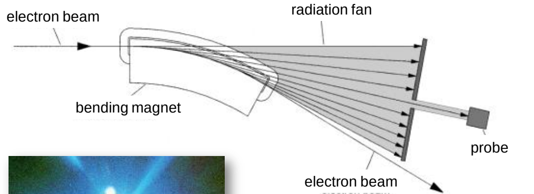
This energy was high enough to produce sufficient radiation power.

⇒ **synchrotron radiation**



Use of synchrotron radiation emitted by an electron storage ring for high energy particle physics.

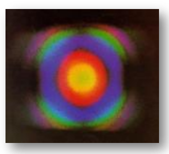


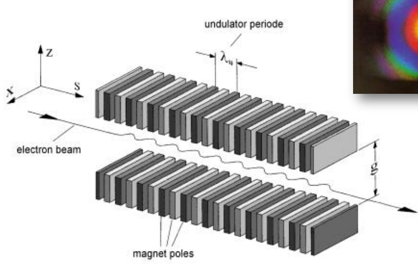


the synchrotron radiation from a bending magnet is horizontally spread out over a wide radiation fan.

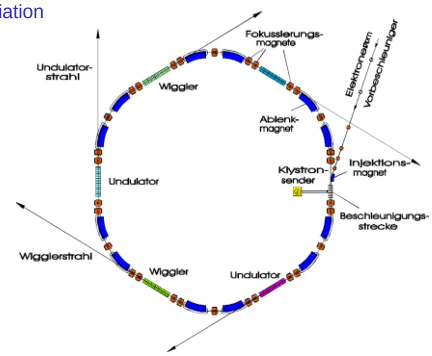
⇒ The radiation power at the probe is limited.

Much more intensity is provided by wigglers and undulators

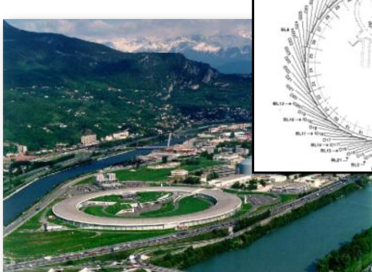


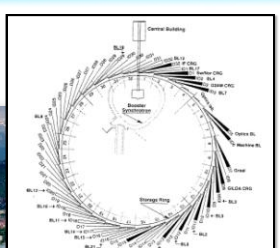


Principal of a modern, dedicated storage ring for syncrotron radiation



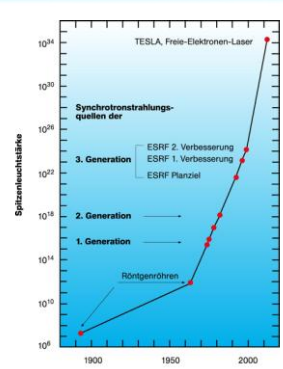
ESRF
Grenoble





$E_{max} = 6$ GeV

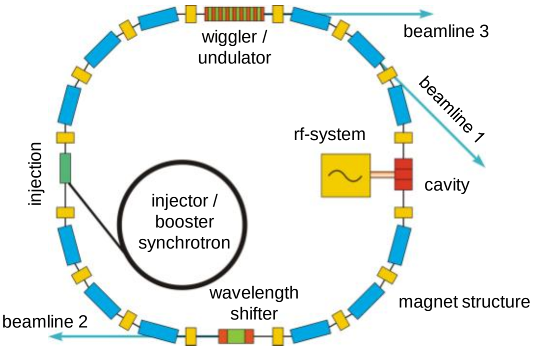
Development of radiation power



Workshop

Design of a dedicated synchrotron radiation source

The main elements of the SR-storage ring



1. Beamline requirements

Beamline 1 (bending magnet)

critical energy $E_c \ge 3.5\ \text{keV}$

photon flux @ E_c $\frac{d\dot{N}}{(d\varepsilon/\varepsilon)d\Theta} \ge 10^{12} \frac{\text{photons}}{0.1\%\text{BW}\cdot\text{mrad}\cdot\text{s}}$

important formulas

radiated power	$P_0 = \frac{e\gamma^4 I_b}{3\varepsilon_0\rho}$
critical frequency	$\omega_c = \frac{3c\gamma^3}{2\rho}$
photon flux	$\frac{d\dot{N}}{d\varepsilon/\varepsilon} = \frac{P_0}{\omega_c\hbar} S\left(\frac{\omega}{\omega_c}\right)$
spectral function	$S(\xi) = \frac{9\sqrt{3}}{8\pi}\xi\int_\xi^\infty K_{5/3}(t)\,dt$

Beamline 2 (wavelength shifter)

critical energy $E_c \ge 20\ \text{keV}$

photon flux @ E_c $\frac{d\dot{N}}{(d\varepsilon/\varepsilon)d\Theta} \ge 10^{12} \frac{\text{photons}}{0.1\%\text{BW}\cdot\text{mrad}\cdot\text{s}}$

Beamline 3 (undulator)

photon wavelength $\lambda = 2 - 20\ \text{nm}$

line width $\frac{\Delta\lambda}{\lambda} \le 1\%$

important formulas

undulator field	$\tilde{B} = \frac{B_0}{\cosh(\pi g/\lambda_u)}$
undulator parameter	$K = \frac{\lambda_u e\tilde{B}}{2\pi m_e c}$
coherence condition	$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}+\gamma^2\Theta^2\right)$

2. General beam requirements

horizontal beam emittance $\varepsilon_x \le 1\cdot 10^{-8}\ \text{m rad}$

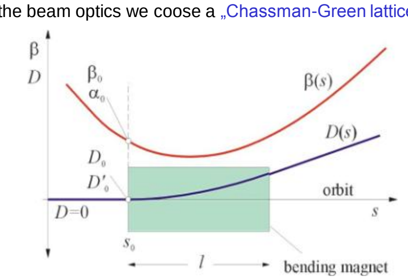
vertical beam emittance $\varepsilon_z \le 0.1\cdot\varepsilon_x = 1\cdot 10^{-9}\ \text{m rad}$

important formula

beam emittance	$\varepsilon_x = \frac{55}{32\sqrt{3}}\frac{\hbar}{m_e c}\gamma^2 \frac{\left\langle \frac{1}{\rho^3}H(s)\right\rangle}{J_x\left\langle\frac{1}{\rho^2}\right\rangle}$

including the optics calculations of the storage ring Optics

For the beam optics we coose a „Chassman-Green lattice".



$$\varepsilon_x = C_\gamma\gamma^2\Theta^3\left(\frac{\gamma_0 l}{20}-\frac{\alpha_0}{4}+\frac{\beta_0}{3l}\right) < 1\cdot 10^{-8}\ \text{m rad}$$

$$C_\gamma = 3.832\cdot 10^{-13}\ \text{m}$$

For the minimum emittance the initial conditions are

$$\beta_0 = 2\sqrt{\frac{3}{5}}\,l = 1.549\,l$$

$$\alpha_0 = \sqrt{15} = 3.873$$

$$\gamma_0 = \frac{1+\alpha_0^2}{\beta_0} = \frac{10.329}{l}$$

This extreme slope α_0 is too high, it causes problems finding stable beam optics. Therefore, it is recommended not to exceed this value beond $\alpha_0 \approx 3{,}0$.

3. The machine

type: electron storage ring

beam energy $E_b = ?$

beam current $I_0 = ?$

bending magnets: bending radius $\rho = ?$

magnet length $l = ?$

bending angle / magnet $\Delta\Theta = ?$

total number of magnets $N = ?\ (N \cdot \Delta\Theta = 2\pi)$

beam optics (recommended: Cassman-Green lattice)

insertion optics: WLS (strong magnet)

undulator (weak magnet)

rf-system: rf-frequency $f_{rf} = ?$

rf-power $P_{rf} = ?$

cavity type: pillbox, 3-cell, 5 cell, superconductive etc.

injection

injection energy: $E_{inj} = E_b$

$E_{inj} < E_b$ (+ SR-ramping)

injection rate (maximum rate limited by radiaton damping)

damping constant $a_x = \frac{W_0}{2E_bT_0}(1-D)$

generally: keep the design simple and cheep !

1. Introduction to Electromagnetic Radiation

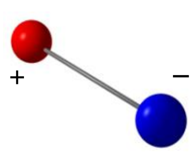
1.1. Units and Dimensions

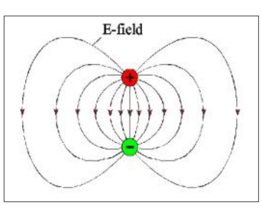
In the following only **MKSA** units will be used.

physical quantity	symbol	dimension
length	l	meter [m]
mass	m	kilogram [kg]
time	t	second [s]
current	I	Ampere [A]
velocity of light	c	$2.997925 \cdot 10^8$ m/s
charge	q	1 C = 1 A s
charge of an electron	e	$1.60203 \cdot 10^{-19}$ C
dielectric constant	ε_0	$8.85419 \cdot 10^{-12}$ As/Vm
permeability	μ_0	$4\pi \cdot 10^{-7}$ Vs/Am
voltage	V	1 volt [V]
electric field	E	V / m
magnetic field	B	1 tesla [T]

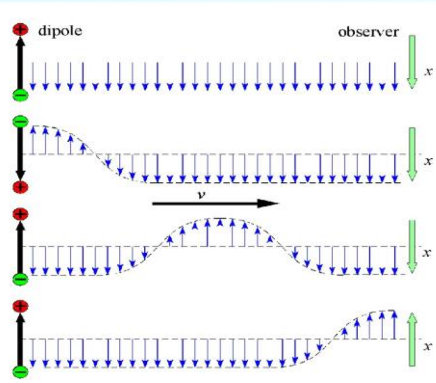
1.2. Rotating electric dipole

At first we will look at a static electrical dipole



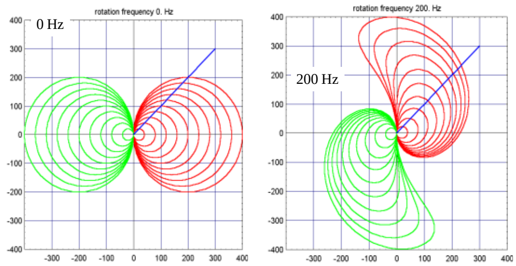


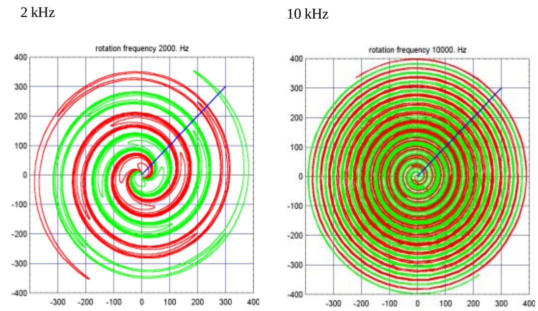
One can see that the delay (or "*retardation*") of the electric field spreading immediately leads to a wave of the electric field.



1.3. Rotating magnetic dipole

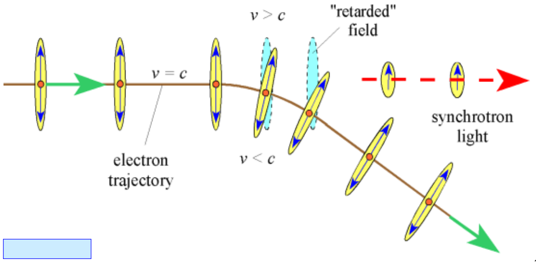
The figures show three pattern with different rotation frequencies between 200 Hz and 10 kHz. One can directly see the generation of spherical waves traveling from the center to the outside.





1.4. Relativistic charged particle traveling through a bending magnet

The last example is the radiation emitted by a charged particle moving with a velocity close to the velocity of light.



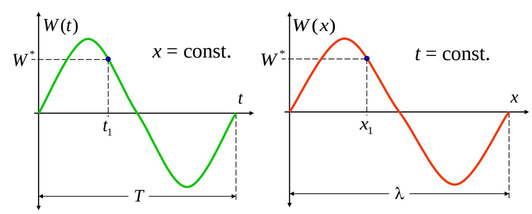
2. Electromagnetic Waves

2.1. The wave equation

Oscillations are periodic changes with time $\quad S(t) = S_0 \exp i\omega t$

It is the solution of the differential equation $\quad \ddot{S}(t) + \omega^2 S(t) = 0$

A wave describes a periodic change with time and space.



The corresponding equations are

$$\ddot{W}(t) + \omega^2 W(t) = 0 \qquad \omega = \frac{2\pi}{T} \quad \text{(frequency)} \qquad (2.1)$$

$$\frac{\partial^2 W(x)}{\partial x^2} + k^2 W(x) = 0 \qquad k = \frac{2\pi}{\lambda} \quad \text{(wave number)} \qquad (2.2)$$

and for all 3 dimensions

$$\Delta W(\vec{r}) + \vec{k}^2 W(\vec{r}) = 0 \qquad \vec{k} = (k_x, k_y, k_z)$$

At the time t_1 the wave has at the point x_1 the value W^*. At the time t_2 the wave point has moved to the point x_2

$$W^*(x,t) = W_0 \exp i(\omega t_1 - k x_1) = W_0 \exp i(\omega t_2 - k x_2)$$

$$\Rightarrow \quad \omega t_1 - k x_1 = \omega t_2 - k x_2$$

$$\Rightarrow \quad \omega(t_1 - t_2) = k(x_1 - x_2)$$

The wave velocity (phase velocity) becomes

$$v = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\omega}{k} \qquad (2.3)$$

From (2.1) we get

$$\ddot{W}(x,t) + \omega^2 W(x,t) = 0 \quad \Rightarrow \quad W(x,t) = -\frac{1}{\omega^2}\ddot{W}(x,t)$$

Inserting this result into (2.2) we get

$$\frac{\partial^2 W(x,t)}{\partial x^2} + k^2 W(x,t) = 0$$

$$\Rightarrow \quad \frac{\partial^2 W(x,t)}{\partial x^2} - \frac{k^2}{\omega^2}\ddot{W}(x,t) = 0$$

With the phase velocity (2.3) we find the one dimensional wave equation

$$\frac{\partial^2 W(x,t)}{\partial x^2} - \frac{1}{v^2}\ddot{W}(x,t) = 0$$

The general tree dimensional wave equation has then the form

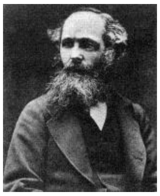
$$\Delta W(\vec{r},t) - \frac{1}{v^2}\ddot{W}(\vec{r},t) = 0 \qquad (2.4)$$

with the *Laplace operator*

$$\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = \nabla^2$$

2.2 Maxwell's equations

The electromagnetic radiation is based on Maxwell's equations. In MKSA units these equations have the form



$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{Coulomb's law} \qquad (2.5)$$

$$\nabla \cdot \vec{B} = 0 \qquad (2.6)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (2.7)$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \text{Ampere's law} \qquad (2.8)$$

One can easily show that time dependent electric or magnetic fields generates an electromagnetic wave. In the vacuum there is no current and therefore $\vec{j} = 0$.

From (2.7) and (2.8) we get

$$\nabla \times \vec{E} = -\dot{\vec{B}} \qquad \Big|\frac{\partial}{\partial t} \qquad \Rightarrow \qquad \nabla \times \dot{\vec{E}} = -\ddot{\vec{B}}$$

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \dot{\vec{E}} \qquad \Big|\nabla \times \qquad\qquad \nabla \times \left(\nabla \times \vec{B}\right) = \mu_0 \varepsilon_0 \nabla \times \dot{\vec{E}}$$

Inserting the first equation into the second one we get

$$\nabla \times \left(\nabla \times \vec{B}\right) = -\mu_0 \varepsilon_0 \ddot{\vec{B}}$$

Using the vector relation

$$\nabla \times \left(\nabla \times \vec{a}\right) = \nabla\left(\nabla \cdot \vec{a}\right) - \nabla^2 \vec{a}$$

and equation (2.6) we finally find

$$\nabla^2 \vec{B} - \mu_0 \varepsilon_0 \ddot{\vec{B}} = 0$$

This is a wave equation of the form of (2.4). The phase velocity is

$$c = 1/\sqrt{\mu_0 \varepsilon_0} = 2.997925 \cdot 10^8 \ \frac{\text{m}}{\text{s}}$$

2.3 Wave equation of the vector and scalar potential

With the Maxwell equation $\nabla \vec{B} = 0$ and the relation $\nabla\left(\nabla \times \vec{a}\right) = 0$ we can derive the magnetic field from a vector potential $\vec{A}$ as

$$\vec{B} = \nabla \times \vec{A} \qquad (2.9)$$

We insert this definition into Maxwell 's equation (2.7) and get

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\nabla \times \left(\frac{\partial \vec{A}}{\partial t}\right) \quad \Rightarrow \quad \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

The expression $\left(\vec{E} + \partial \vec{A}/\partial t\right)$ can be written as a gradient of a scalar potential $\phi(\vec{r},t)$ in the form

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi \qquad (2.10)$$

The electric field becomes

$$\vec{E} = -\left(\nabla \phi + \frac{\partial \vec{A}}{\partial t}\right) \qquad (2.11)$$

With Coulomb's law (2.5) we find

$$\nabla \vec{E} = -\nabla\left(\nabla \phi + \frac{\partial \vec{A}}{\partial t}\right) = \frac{\rho}{\varepsilon_0}$$

or

$$\nabla^2 \phi + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{A}\right) = -\frac{\rho}{\varepsilon_0} \qquad (2.12)$$

We take now the formula of Ampere's law (2.8) and insert the relations for the magnetic and electric field (2.9) and (2.11) and get

$$\underbrace{\nabla \times \left(\nabla \times \vec{A}\right)}_{\nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}} = \mu_0 \vec{j} - \mu_0 \varepsilon_0 \left(\frac{\partial}{\partial t}\nabla \phi + \frac{\partial^2 \vec{A}}{\partial t^2}\right) \qquad (2.13)$$

$$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \left(\nabla \frac{\partial \phi}{\partial t} + \frac{\partial^2 \vec{A}}{\partial t^2}\right) - \nabla \cdot \left(\nabla \cdot \vec{A}\right) = -\mu_0 \vec{j}$$

The relation becomes

$$\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla \cdot \left(\nabla \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial \phi}{\partial t}\right) = -\mu_0 \vec{j} \qquad (2.14)$$

Equations (2.12) and (2.14) create a coupled system for the potentials $\vec{A}$ and ϕ. We define now the following gauge transformation

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \nabla \Lambda \qquad \phi \rightarrow \phi' = \phi - \frac{\partial \Lambda}{\partial t}$$

The free choice of $\Lambda(\vec{r},t)$ provides a set of potentials satisfying the *Lorentz condition*

$$\nabla \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

With the gauge transformation we get

$$\nabla\left(\vec{A}+\nabla\Lambda\right)+\frac{1}{c^2}\frac{\partial}{\partial t}\left(\phi-\frac{\partial\Lambda}{\partial t}\right)=\underbrace{\nabla\vec{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}}_{=0}+\nabla(\nabla\Lambda)-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}=0$$

If the function $\Lambda(\vec{r},t)$ is a solution of the wave equation

$$\nabla^2\Lambda-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}=0$$

the Lorentz condition is fulfilled. In (2.12) we replace $\nabla\vec{A}$ by $-\dot{\phi}/c^2$ (*Lorentz condition*) and get

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=-\frac{\rho}{\varepsilon_0} \qquad (2.15)$$

With $c^2=1/\mu_0\varepsilon_0$ the expression (2.14) becomes

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}-\nabla\cdot\underbrace{\left(\nabla\cdot\vec{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}\right)}_{=0}=-\mu_0\vec{j}$$

The result is then

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{j} \qquad (2.16)$$

The two expressions (2.15) and (2.16) are the decoupled equations for the potentials $\vec{A}(\vec{r},t)$ and $\phi(\vec{r},t)$. These inhomogeneous wave equations are the basis of all kind of electromagnetic radiation.

2.4.The solution of the inhomogeneous wave equations

We have now to find the solution of the inhomogeneous wave equations (2.15) and (2.16). We start assuming a point charge in the origin of the coordinate system of the form

$$dq=\rho(\vec{r},t)\,\delta^3(\vec{r})\,dV$$

Outside the origin, i.e. $|\vec{r}|\neq 0$ the charge density ρ vanishes. The wave equations of the potential becomes

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=0$$

The potential has now a spherical symmetry as

$$\phi(\vec{r},t)=\phi(|\vec{r}|,t)=\phi(r,t)$$

We have now to evaluate the expression $\nabla^2\phi(r)$ for a point charge. A straight forward calculation yields

$$\nabla^2\phi(r)=\nabla\cdot\nabla\phi(r)=\nabla\left(\frac{\vec{r}}{r}\frac{\partial\phi}{\partial r}\right)=\left(\nabla\frac{\vec{r}}{r}\right)\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial r^2}=\frac{2}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial r^2}$$

On the other hand we find the relation

$$\frac{\partial^2}{\partial r^2}(r\phi)=\frac{\partial}{\partial r}\left(\phi+r\frac{\partial\phi}{\partial r}\right)=2\frac{\partial\phi}{\partial r}+r\frac{\partial^2\phi}{\partial r^2}=r\nabla^2\phi$$

Combining these two expressions we get the wave equation in the form

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=\frac{1}{r}\left(\frac{\partial^2}{\partial r^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)(r\phi)=0$$

with the general solution

$$\phi(r,t)=\frac{1}{r}f_1(r-ct)+\frac{1}{r}f_2(r+ct)$$

The second term on the right hand side represents a reflected wave, which doesn't exist in this case. Therefore, the solution is reduced to

$$\phi(r,t)=\frac{1}{r}f(r-ct)$$

In order to evaluate the function $f(r-ct)$ one has to calculate the potential $\phi(r,t)$ in the origin of the coordinate system. The problem is that

$$r\to 0 \quad\Rightarrow\quad \phi(r,t)=\frac{f(r-ct)}{r}\to\infty$$

A better way is to compare the first and second derivatives of the potential. For $r\to 0$ we get

$$\frac{\partial\phi}{\partial r}\propto\frac{f(-ct)}{\underline{r^2}}\gg\frac{\partial\phi}{\partial t}\propto\frac{1}{\underline{r}}\frac{\partial f(-ct)}{\partial t}$$

The ratio of the second spatial derivative to the second time derivative is even much larger

$$\frac{\partial^2\phi}{\partial r^2}\gg\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\quad\text{for}\quad r\to 0$$

and we can simplify the wave equation (2.15) to

$$\nabla^2\phi(r,t)=-\frac{\rho}{\varepsilon_0}\qquad(r\to 0)$$

This is the well known Poisson equation for a static point charge. For $r\to 0$ the potential $\phi(r,t)$ approaches the Coulomb potential. Therefore, we can write

$$\phi(r,t)=\frac{1}{r}f(r-ct)\xrightarrow{r\to 0}\frac{1}{r}f(-ct)=\frac{1}{4\pi\varepsilon_0}\frac{\rho(0,t)}{r}\Delta V$$

Because of the limited velocity c of the electromagnetic fields, at a point r outside the origin the time dependent potential is delayed by

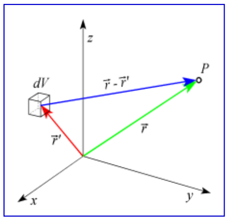
$$\Delta t = \frac{r}{c} \qquad \Rightarrow \qquad t \rightarrow t - \frac{r}{c}$$

At this point we have the *"retarded"* potential

$$d\phi(r,t) = \frac{1}{4\pi\varepsilon_0}\frac{\rho\left(0,\, t-\frac{r}{c}\right)}{r}dV$$

If the charge is not in the origin but at any point $\vec{r}\,'$ in a Volume dV we get

$$d\phi(r,t) = \frac{1}{4\pi\varepsilon_0}\frac{\rho\left(\vec{r}\,',t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}dV$$



retarded by $\Delta t = \frac{|\vec{r}-\vec{r}\,'|}{c}$

Since under real conditions one do not has a point charge the potential must be integrated over a finite volume containing the charge distribution. The result is then

$$\phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho\left(\vec{r}\,',t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}dV \qquad (2.17)$$

The vector potential $\vec{A}(\vec{r},t)$ can according to (2.15) and (2.16) easily evaluated by replacing the expression ρ/ε_0 by $\mu_0\vec{j}$. In this way we find

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{j}\left(\vec{r}\,',t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}dV \qquad (2.18)$$

These solutions of the two wave equations are called ***Liénard-Wiechert potentials.***

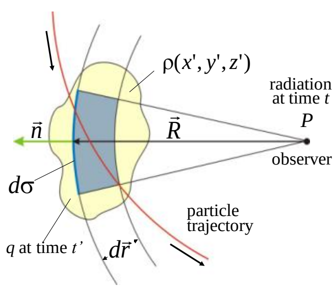
2.5. Liénard-Wiechert potentials of a moving charge

We now replace the distance between the charge and the observer by

$$\vec{R} = \vec{r}\,' - \vec{r}$$

Radiation observed at the point P comes from all charges within a spherical shell with the center P, the radius $|\vec{R}|$ and the thickness $|d\vec{r}|$. If $d\sigma$ is the surface element of the shell the volume element is

$$dV = d\sigma dr$$



The retarded time for radiation from the outer surface of the shell is

$$t' = t - \frac{|\vec{R}|}{c}$$

and from the inner surface

$$t'' = t' - \frac{|d\vec{r}|}{c}$$

The electromagnetic field at P at time t is generated by the charge within the volume element dV. The charge in this volume element is with $dr = |d\vec{r}|$

$$dq_1 = \rho\, d\sigma\, dr$$

For charges moving with the velocity $\vec{v}$ one has to add all charge that penetrate the inner shell surface during the time $dt = dr/c$, i.e.

$$dq_2 = \rho\, \vec{v}\, \vec{n}\, dt\, d\sigma$$

with the vector $\vec{n}$ normal to the outer surface defined by $\vec{n} = \vec{R}/|\vec{R}|$

The total effective charge element is then

$$dq = dq_1 + dq_2 = \rho\, d\sigma\left(dr + \vec{v}\vec{n}dt\right) = \rho\, d\sigma\left(dr + \vec{v}\vec{n}\frac{dr}{c}\right)$$
$$= \rho\left(1 + \vec{n}\vec{\beta}\right)dr d\sigma$$

With this relation we can write

$$\rho\, dr\, d\sigma = \rho\, dV = \frac{dq}{1+\vec{n}\vec{\beta}} \qquad (2.19)$$

Insertion into equation (2.17) gives

$$\phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{R\left(1+\vec{n}\vec{\beta}\right)} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}\frac{1}{\left(1+\vec{n}\vec{\beta}\right)} \qquad (2.20)$$

The current density an be written as $\vec{j} = \rho\vec{v}$. With this relation the vector potential (2.18) becomes

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{v}}{R}\rho\, dV$$

With (2.19) we get finally

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{v}dq}{R\left(1+\vec{n}\vec{\beta}\right)} = \frac{c\,\mu_0}{4\pi}\frac{q}{R}\frac{\vec{\beta}}{\left(1+\vec{n}\vec{\beta}\right)}\Bigg|_{t'} \qquad (2.21)$$

It is important to notice that the parameter in the expression on the right hand side must be taken at the retarded time t'. The equations (2.20) and (2.21) are the Liénard-Wiechert potentials for a moving point charge.

2.6 The electric field of a moving charged particle

Using the formula (2.10) we can derive the electric field at the point P by inserting the potentials as

$$\vec{E} = -\left(\nabla'\phi + \frac{\partial \vec{A}}{\partial t}\right) = -\frac{q}{4\pi\varepsilon_0}\nabla'\frac{1}{R(1+\vec{n}\vec{\beta})} - \frac{c\mu_0 q}{4\pi}\frac{\partial}{\partial t}\frac{\vec{\beta}}{R(1+\vec{n}\vec{\beta})}$$

After longer calculations (see script) the electrical field finally becomes

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left\{-\frac{1-\vec{\beta}^2}{a^3}\left(\vec{R}+\vec{\beta}R\right) + \frac{1}{ca^3}\vec{R}\times\left[\left(\vec{R}+\vec{\beta}R\right)\times\dot{\vec{\beta}}\right]\right\} \qquad (2.28)$$

with $a := R(1+\vec{n}\vec{\beta})$

Since the expression $1/R^2$ drops down with the distance R the first term vanishes at longer distances. The second term, however, reduces only inversely proportional to the distance R. It determines the radiation far away from the source charge. Therefore, we can we can neglect the first term in (2.28) and get

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{ca^3}\left\{\vec{R}\times\left[\left(\vec{R}+\vec{\beta}R\right)\times\dot{\vec{\beta}}\right]\right\} \qquad (2.29)$$

2.8 The magnetic field of a moving charged particle

With the relations (2.9) and (2.21) we can calculate the magnetic field of a moving charged particle and we find

$$\vec{B} = \nabla'\times\vec{A} = \frac{c\mu_0 q}{4\pi}\nabla'\times\left(\frac{\vec{\beta}}{a}\right) = \frac{c\mu_0 q}{4\pi}\left(\frac{1}{a}\nabla'\times\vec{\beta} - \frac{1}{a^2}(\nabla' a)\times\vec{\beta}\right) \qquad (2.30)$$

Again after longer calculations (see script) the magnetic field becomes

$$\vec{B} = \frac{c\mu_0 q}{4\pi}\left\{-\frac{\left[\vec{\beta}\times\vec{n}\right]}{a^2} - \frac{R}{ca^2}\left[\dot{\vec{\beta}}\times\vec{n}\right] + \frac{R}{a^3}\left(\vec{n}\vec{\beta}+\vec{\beta}^2+\frac{\vec{R}}{c}\dot{\vec{\beta}}\right)\left[\vec{\beta}\times\vec{n}\right]\right\} \qquad (2.33)$$

For the long distance field only termes proportional to $1/R$ are important. We get

$$\vec{B} = \frac{c\mu_0 q}{4\pi}\left(-\frac{\left[\dot{\vec{\beta}}\times\vec{n}\right]}{cR\left(1+\vec{n}\vec{\beta}\right)^2} + \frac{\left(\dot{\vec{\beta}}\vec{n}\right)\left[\vec{\beta}\times\vec{n}\right]}{cR\left(1+\vec{n}\vec{\beta}\right)^3}\right)$$

we modify the formula (2.26) in the following way

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{a^2}\left[-\vec{n}-\vec{\beta}+b\vec{R}\right] - \frac{R}{ca^2}\dot{\vec{\beta}} + \frac{R\vec{\beta}}{a^2}b\right\}$$

The vector multiplication of this equation with the unit vector $\vec{n}$ gives

$$\left[\vec{E}\times\vec{n}\right] = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{a^2}\left[-\vec{n}-\vec{\beta}+b\vec{R}\right] - \frac{R}{ca^2}\dot{\vec{\beta}} + \frac{R\vec{\beta}}{a^2}b\right\}\times\vec{n}$$

$$\left[\vec{E}\times\vec{n}\right] = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{a^2}\left(-\underbrace{\left[\vec{n}\times\vec{n}\right]}_{=0} - \left[\vec{\beta}\times n\right] + b\underbrace{\left[\vec{R}\times\vec{n}\right]}_{=0}\right) - \frac{R}{ca^2}\left[\dot{\vec{\beta}}\times\vec{n}\right] + \frac{Rb}{a^2}\left[\vec{\beta}\times\vec{n}\right]\right\}$$

$$= \frac{q}{4\pi\varepsilon_0}\left\{-\frac{\left[\vec{\beta}\times\vec{n}\right]}{a^2} - \frac{R}{ca^2}\left[\dot{\vec{\beta}}\times\vec{n}\right] + \frac{R}{a^3}\left(\vec{n}\vec{\beta}+\vec{\beta}^2+\frac{\vec{R}}{c}\dot{\vec{\beta}}\right)\left[\vec{\beta}\times\vec{n}\right]\right\}$$

Comparison with the equation (2.33) leads directly to the following simple relation between the magnetic and electric field

$$\vec{B} = \frac{1}{c}\left[\vec{E}\times\vec{n}\right]$$

We can now state the Poynting vector of the radiation in the form

$$\vec{S} = \frac{1}{\mu_0}\left[\vec{E}\times\vec{B}\right] = \frac{1}{c\mu_0}\left[\vec{E}\times\left(\vec{E}\times\vec{n}\right)\right]$$

We apply again the vector relation $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \vec{b}\left(\vec{a}\vec{c}\right) - \vec{c}\left(\vec{a}\vec{b}\right)$ and get

$$\vec{E}\times\left(\vec{E}\times\vec{n}\right) = \vec{E}\left(\vec{E}\vec{n}\right) - \vec{n}\vec{E}^2 = -\vec{n}\vec{E}^2$$

The Poynting vector finally becomes

$$\vec{S} = -\frac{1}{c\mu_0}\vec{E}^2\vec{n}$$

This is the power density of the radiation parallel to $\vec{n}$ observed at the point P per unit cross section. We now evaluate the Poynting vector at the retarded time t'. With (2.23) we find

$$\vec{S}' = \vec{S}\frac{dt}{dt'} = -\frac{1}{c\mu_0}\vec{E}^2\,\vec{n}\frac{dt}{dt'} = -\frac{1}{c\mu_0}\vec{E}^2\frac{a}{R}\vec{n}$$

and finally

$$\vec{S}' = -\frac{1}{c\mu_0}\vec{E}^2\left(1+\vec{n}\vec{\beta}\right)\vec{n}$$

3 Synchrotron Radiation

3.1 Radiation power and energy loss

We choose a coordinate system K^* which moves with the particle of the charge $q = e$. In this reference frame the particle velocity vanishes and the charge oscillates about a fixed point. We get

$$\vec{v}^* = 0 \quad \to \quad \vec{\beta}^* = 0 \quad \to \quad a = R$$

It is important to notice that $\dot{\vec{\beta}}^* \neq 0$! The expression (2.29) is then modified to

$$\vec{E}^* = \frac{e}{4\pi\varepsilon_0}\frac{1}{cR^3}\left(\vec{R}\times\left[\vec{R}\times\dot{\vec{\beta}}^*\right]\right) = \frac{e}{4\pi\varepsilon_0}\frac{1}{cR}\left(\vec{n}\times\left[\vec{n}\times\dot{\vec{\beta}}^*\right]\right)$$

The radiated power per unit solid angle at the distance R from the generating charge is

$$\frac{dP}{d\Omega} = -\vec{n}\vec{S}R^2 = \frac{1}{c\mu_0}\frac{e^2}{(4\pi\varepsilon_0)^2}\frac{1}{c^2}\left(\vec{n}\times\left[\vec{n}\times\dot{\vec{\beta}}^*\right]\right)^2$$
$$= \frac{e^2}{(4\pi)^2 c\varepsilon_0}\left(\vec{n}\times\left[\vec{n}\times\dot{\vec{\beta}}^*\right]\right)^2 \qquad (3.1)$$

With the vector relation $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b})$ and $\vec{n}\vec{n} = \vec{n}^2 = 1$ we find

$$\left(\vec{n}\times\left[\vec{n}\times\dot{\vec{\beta}}^*\right]\right)^2 = \left(\vec{n}\left(\vec{n}\dot{\vec{\beta}}^*\right) - \dot{\vec{\beta}}^*(\vec{n}\vec{n})\right)^2$$
$$= \vec{n}^2\left(\vec{n}\dot{\vec{\beta}}^*\right)^2 - 2\vec{n}\left(\vec{n}\dot{\vec{\beta}}^*\right)\dot{\vec{\beta}}^* + \dot{\vec{\beta}}^{*2} = \dot{\vec{\beta}}^{*2} - \left(\vec{n}\dot{\vec{\beta}}^*\right)^2 \qquad (3.2)$$

Since

$$\vec{n}\dot{\vec{\beta}}^* = |\vec{n}|\left|\dot{\vec{\beta}}^*\right|\cos\Theta = \left|\dot{\vec{\beta}}^*\right|\cos\Theta$$

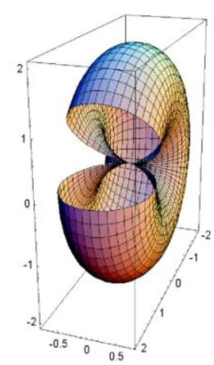
Θ is the angle between the direction of the particle acceleration and the direction of observation the relation (3.2) becomes

$$\left(\vec{n}\times\left[\vec{n}\times\dot{\vec{\beta}}^*\right]\right)^2 = \dot{\vec{\beta}}^{*2} - \dot{\vec{\beta}}^{*2}\cos^2\Theta = \dot{\vec{\beta}}^{*2}\left(1-\cos^2\Theta\right) = \dot{\vec{\beta}}^{*2}\sin^2\Theta$$

The power per unit solid angle is then

$$\frac{dP}{d\Omega} = \frac{e^2}{(4\pi)^2 c\varepsilon_0}\dot{\vec{\beta}}^{*2}\sin^2\Theta \qquad (3.3)$$

The spatial power distribution corresponds to the power distribution of a Hertz' dipole.



The total power radiated by the charged particle can be achieved by integrating (3.3) over all solid angle. With

$$d\Omega = \sin\Theta' d\Theta' d\phi$$

we can write

$$P = \frac{e^2}{(4\pi)^2 c\varepsilon_0}\dot{\vec{\beta}}^{*2}\int_0^{2\pi}\int_0^{\pi}\sin^3\Theta d\Theta d\phi$$

where ϕ is the azimuth angle with respect to the direction of the acceleration. the total power becomes

$$P = \frac{e^2}{6\pi\varepsilon_0 c}\dot{\vec{\beta}}^{*2}$$

This result was first found by Lamor . One can directly see that radiation only occurs while the charged particle is accelerated. With the modification

$$\dot{\vec{\beta}}^* = \frac{\dot{\vec{v}}^*}{c} = \frac{m\dot{\vec{v}}^*}{mc} = \frac{\dot{\vec{p}}}{mc}$$

we get

$$P = \frac{e^2}{6\pi\varepsilon_0 m^2 c^3}\left(\frac{d\vec{p}}{dt}\right)^2$$

This is the radiation of a non-relativistic particle. To get an expression for extreme relativistic particles we have to replace the time t by the Lorentz-invariant time $d\tau = dt/\gamma$ and the momentum $\vec{p}$ by the 4-momentum P_μ.

$$dt \quad \to \quad d\tau = \frac{1}{\gamma}dt \quad \text{with} \quad \gamma = \frac{E}{m_0c^2} = \frac{1}{\sqrt{1-\beta^2}}$$
$$\vec{p} \quad \to \quad P_\mu \quad \text{(4-momentum)}$$

or

$$\left(\frac{d\vec{p}}{dt}\right)^2 \to \left(\frac{dP_\mu}{d\tau}\right)^2 = \left(\frac{d\vec{p}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2$$

With this modification we get the radiated power in the relativistic invariant form

$$P_s = \frac{e^2c}{6\pi\varepsilon_0\left(m_0c^2\right)^2}\left[\left(\frac{d\vec{p}}{d\tau}\right)^2 - \frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2\right] \qquad (3.4)$$

There are two different cases:

1. linear acceleration: $\frac{d\vec{v}}{d\tau} \parallel \vec{v}$

2. circular acceleration: $\frac{d\vec{v}}{d\tau} \perp \vec{v}$

3.1.1 Linear acceleration

The particle energy is

$$E^2 = \left(m_0 c^2\right)^2 + p^2 c^2$$

After differentiating we get

$$E\frac{dE}{d\tau} = c^2 p \frac{dp}{d\tau}$$

Using $E = \gamma m_0 c^2$ and $p = \gamma m_0 v$ we have

$$\frac{dE}{d\tau} = v\frac{dp}{d\tau}$$

Insertion into (3.4) gives

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 \left(m_0 c^2\right)^2}\left[\left(\frac{dp}{d\tau}\right)^2 - \left(\frac{v}{c}\right)^2\left(\frac{dp}{d\tau}\right)^2\right] = \frac{e^2 c}{6\pi\varepsilon_0 \left(m_0 c^2\right)^2}\left(1-\beta^2\right)\left(\frac{dp}{d\tau}\right)^2$$

With $1-\beta^2 = 1/\gamma^2$ we can write

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 \left(m_0 c^2\right)^2}\left(\frac{dp}{\gamma d\tau}\right)^2 = \frac{e^2 c}{6\pi\varepsilon_0 \left(m_0 c^2\right)^2}\left(\frac{dp}{dt}\right)^2$$

For linear acceleration holds

$$\frac{dp}{dt} = \frac{c\,dp}{c\,dt} = \frac{dE}{dx}$$

and we get

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 \left(m_0 c^2\right)^2}\left(\frac{dE}{dx}\right)^2$$

In modern electron linacs one can achieve

$$\frac{dE}{dx} \approx 15\,\frac{\text{MeV}}{\text{m}} \quad \Rightarrow \quad P_s = 4\cdot 10^{-17}\ \text{Watt (!)}$$

3.1.2 Circular acceleration

Completely different is the situation when the acceleration is perpendicular to the direction of particle motion. In this case the particle energy stays constant. Equation (3.4) reduces to

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 \left(m_0 c^2\right)^2}\left(\frac{dp}{d\tau}\right)^2 = \frac{e^2 c\gamma^2}{6\pi\varepsilon_0 \left(m_0 c^2\right)^2}\left(\frac{dp}{dt}\right)^2 \qquad (3.5)$$

On a circular trajectory with the radius ρ a change of the orbit angle $d\alpha$ causes momentum variation

$$dp = p\,d\alpha$$

With $v = c$ and $E = pc$ follows

$$\frac{dp}{dt} = p\,\omega = \frac{p\,v}{\rho} = \frac{E}{\rho}$$

We insert this result in (3.5) and get with $\gamma = E/m_0 c^2$

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 \left(m_0 c^2\right)^4}\frac{E^4}{\rho^2} \qquad (3.6)$$

Comparison of radiation from an electron and a proton with the same energy gives

$$m_e c^2 = 0.511\text{MeV}$$

$$m_p c^2 = 938.19\text{MeV}$$

$$\frac{P_{s,e}}{P_{s,p}} = \left(\frac{m_p c^2}{m_e c^2}\right)^4 = 1.13\cdot 10^{13}(!)$$

This radiation is therefore observed in most of the cases from electrons.

In a circular accelerator the energy loss per turn is

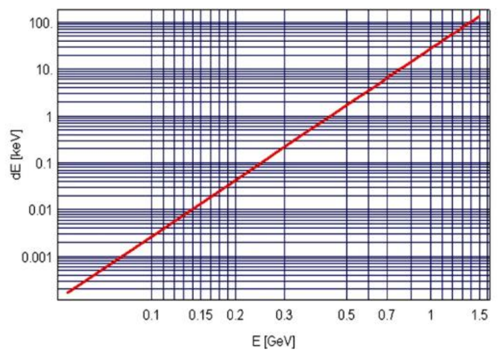
$$\Delta E = \oint P_s dt = P_s t_b = P_s \frac{2\pi\rho}{c} \qquad (3.7)$$

We insert (3.6) into (3.7) and get

$$\Delta E = \frac{e^2}{3\varepsilon_0 \left(m_0 c^2\right)^4}\frac{E^4}{\rho}$$

For electrons one can reduce this formula to a very simple expression

$$\Delta E\,[\text{keV}] = 88.5\,\frac{E^4\,[\text{GeV}^4]}{\rho\,[\text{m}]}$$



The synchrotron radiation was investigated the first time by Liénard at the end of the 20th century. It was observed almost 50 years later at the 70 GeV-synchrotron of General Electric in the USA.

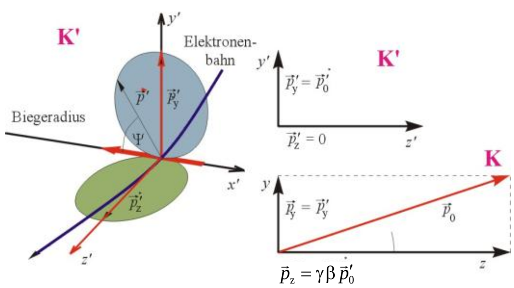
The energy loss per revolution is

$$\Delta E \propto \frac{E^4}{\rho}$$

	L [m]	E [GeV]	ρ [m]	B [T]	ΔE [keV]
BESSY I	62.4	0.80	1.78	1.500	20.3
DELTA	115	1.50	3.34	1.500	134.1
DORIS	288	5.00	12.2	1.370	$4.53\cdot 10^3$
ESRF	844	6.00	23.4	0.855	$4.90\cdot 10^3$
PETRA	2304	23.50	195.0	0.400	$1.38\cdot 10^5$
LEP	$27\cdot 10^3$	70.00	3000	0.078	$7.08\cdot 10^5$

3.2. Opening angle of synchrotron radiation

In the center of mass system **K'** the spartial intensity distribution is the same as at the Hertz' dipole.



A photon emitted parallel to the *y'*-axis has the momentum

$$\vec{p}'_y = \vec{p}'_0 = \frac{E'_s}{c}\vec{n}$$

E'_s is the photon energy. The 4-momentum becomes

$$P'_\mu = (p_t, p_x, p_y, p_z) = (E'_s/c,\, 0,\, p'_0,\, 0)$$

Using the Lorentztransformation we get the 4-momentum in **K**

$$P_\mu = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} E'_s/c \\ 0 \\ p'_0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma E'_s/c \\ 0 \\ p'_0 \\ \gamma\beta E'_s/c \end{pmatrix}$$

With $p'_0 = E'_s/c$ we get the opening angle

$$\tan\Theta = \frac{p_y}{p_z} = \frac{p'_0}{\gamma\beta p'_0} \approx \frac{1}{\gamma}$$

3.3 Spatial distribution of the radiation of a relativistic particle

The power per unit solid angle was given in (3.3) as

$$\frac{dP}{d\Omega} = \frac{e^2}{(4\pi)^2 c\varepsilon_0}\dot{\beta}^{*2}\sin^2\Theta$$

for the radiation of a charged particle in the reference frame K^*. The angular distribution corresponds to that of the Hertz' dipole. The radiation of relativistic particles is focused with the opening angle of .

The radiation power per unit solid angle is given in (3.1)

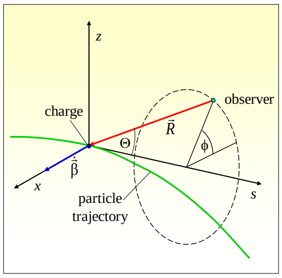
$$\frac{dP}{d\Omega} = -\vec{n}\,\vec{S}' R^2$$

With the relation for the Poynting vector at the radiated time we get

$$\frac{dP}{d\Omega} = \frac{1}{c\mu_0}\vec{E}^2(1+\vec{n}\,\vec{\beta})R^2$$

Inserting the electrical field (2.29) and with the charge of an electron $q = e$ we find

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{1}{c\mu_0}\frac{e^2}{(4\pi\varepsilon_0)^2}\frac{1}{c^2a^6}\cdot\left\{\vec{R}\times\left[\left(\vec{R}+\vec{\beta}R\right)\times\dot{\vec{\beta}}\right]\right\}^2\left(1+\vec{n}\vec{\beta}\right)R^2 \\ &= \frac{1}{c\mu_0}\frac{e^2}{(4\pi\varepsilon_0)^2}\frac{R^5}{c^2a^5}\left\{\vec{n}\times\left[\left(\vec{n}+\vec{\beta}\right)\times\dot{\vec{\beta}}\right]\right\}^2 \end{aligned} \qquad (3.8)$$



The vector $\vec{R}$ pointing from the observer to the moving particle is

$$\vec{R} = -R\begin{pmatrix} \sin\Theta\cos\phi \\ \sin\Theta\sin\phi \\ \cos\Theta \end{pmatrix}$$

and the correlated unit vector

$$\vec{n} = \begin{pmatrix} -\sin\Theta\cos\phi \\ -\sin\Theta\sin\phi \\ -\cos\Theta \end{pmatrix} \qquad (3.9)$$

The Lorentz force of an electron traveling through a magnet is

$$\vec{F} = -e\vec{v}\times\vec{B} = -e\begin{pmatrix} -vB_z \\ 0 \\ 0 \end{pmatrix} = \gamma m_0\dot{\vec{v}}$$

with

$$\vec{v} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}, \quad \dot{\vec{v}} = \begin{pmatrix} \dot{v}_x \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{B} = \begin{pmatrix} 0 \\ B_z \\ 0 \end{pmatrix} \qquad (3.10)$$

A straight forward calculation yields

$$\gamma m_0 \dot{v}_x = evB_z = ec\beta B_z$$

On the other hand the bending radius ρ of a trajectory in a magnet can be evaluated according to

$$\frac{1}{\rho} = \frac{e}{p}B_z = \frac{eB_z}{\gamma m_0 v} \quad \Rightarrow \quad B_z = \frac{\gamma m_0 v}{e\rho}$$

The transverse acceleration of the particle can now be written in the form

$$\dot{v}_x = \frac{c^2\beta^2}{\rho} \qquad (3.11)$$

With (3.10) and (3.11) we get

$$\vec{\beta} = \frac{\vec{v}}{c} = \begin{pmatrix} 0 \\ 0 \\ v/c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix} \qquad (3.12)$$

and

$$\dot{\vec{\beta}} = \begin{pmatrix} \dot{v}_x/c \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (c\beta^2)/\rho \\ 0 \\ 0 \end{pmatrix} \qquad (3.13)$$

Using again the vector relation

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b})$$

The double product in (3.8) becomes

$$\left\{\vec{n}\times\left(\left[\vec{n}+\vec{\beta}\right]\times\dot{\vec{\beta}}\right)\right\} = \left(\vec{n}+\vec{\beta}\right)\left(\vec{n}\dot{\vec{\beta}}\right) - \dot{\vec{\beta}}\left(1+\vec{n}\vec{\beta}\right)$$

Inserting (3.9), (3.12) and (3.13) we get

$$\left(\vec{n}+\vec{\beta}\right)\left(\vec{n}\dot{\vec{\beta}}\right) - \dot{\vec{\beta}}\left(1+\vec{n}\vec{\beta}\right) =$$

$$= \begin{pmatrix} -\sin\Theta\cos\phi \\ -\sin\Theta\sin\phi \\ \beta-\cos\Theta \end{pmatrix}\left(-\sin\Theta\cos\phi\frac{c\beta^2}{\rho}\right) - \begin{pmatrix} (c\beta^2)/\rho \\ 0 \\ 0 \end{pmatrix}(1-\beta\cos\Theta)$$

$$= \frac{c\beta^2}{\rho}\left\{\begin{pmatrix} \sin^2\Theta\cos^2\phi \\ \sin^2\Theta\sin\phi\cos\phi \\ -(\beta-\cos\Theta)\sin\Theta\cos\phi \end{pmatrix} - \begin{pmatrix} 1-\beta\cos\Theta \\ 0 \\ 0 \end{pmatrix}\right\}$$

From the definition of a we derive with (3.9) and (3.12)

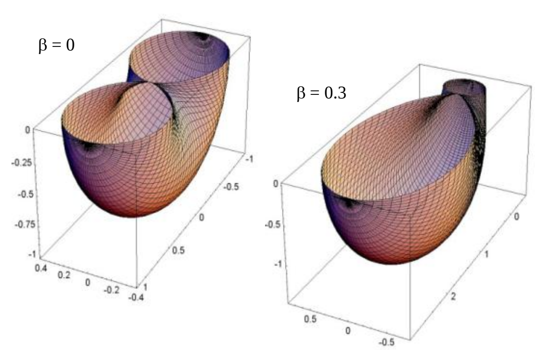
$$a = R\left(1+\vec{n}\vec{\beta}\right) = R\left(1-\beta\cos\Theta\right) \qquad (3.14)$$

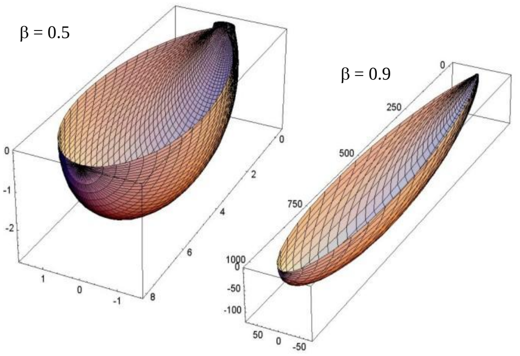
Some further calculations finally privide

$$\frac{dP}{d\Omega} = \frac{1}{c^3\mu_0}\frac{e^4}{(4\pi\varepsilon_0)^2}\frac{\beta^4}{\rho^2}\frac{(\beta^2-1)\sin^2\Theta\cos^2\phi + (1-\beta\cos\Theta)^2}{(1-\beta\cos\Theta)^5} \qquad (3.15)$$

Acceleration

$$a = \frac{dv}{dt} = \frac{v^2}{\rho} = c^2\frac{\beta^2}{\rho}$$





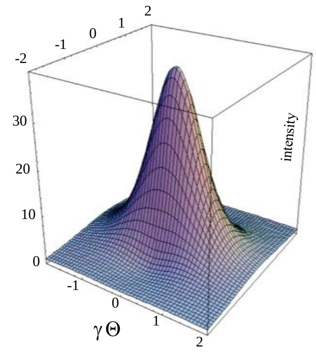
With the dimensionless particle energy

$$\gamma = \frac{E}{m_0 c^2} = \frac{1}{\sqrt{1+\beta^2}}$$

we vary the angle Θ between the direction of particle motion and the direction of photon emission according to

$$\Theta = \frac{u}{\gamma} \quad (u = \text{dimensionless number})$$

and calculate the photon intensity using equation (3.15).



It is directly to see that the radiation is mainly concentrated within a cone of an opening angle of. In equation (3.15) we set $\phi = \pi/2$ and the fraction on the right hand side reduces to

$$w(\Theta) = \frac{1}{(1-\beta\cos\Theta)^3} \qquad (3.16)$$

With the conditions $\gamma \gg 1$ and $\Theta \ll 1$ we find the approximations

$$\beta = \sqrt{1-\frac{1}{\gamma^2}} \approx 1-\frac{1}{2\gamma^2} \qquad \text{and} \qquad \cos\Theta \approx 1-\frac{\Theta^2}{2}$$

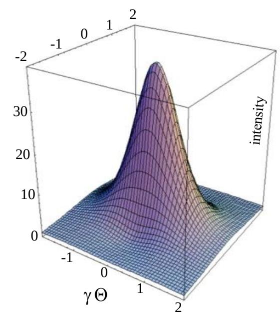
and we get from (3.16)

$$w(\Theta) \approx \left[1-\left(1-\frac{1}{2\gamma^2}\right)\left(1-\frac{\Theta^2}{2}\right)\right]^{-3} = \left[1-1+\frac{\Theta^2}{2}+\frac{1}{2\gamma^2}-\frac{\Theta^2}{4\gamma^2}\right]^{-3}$$

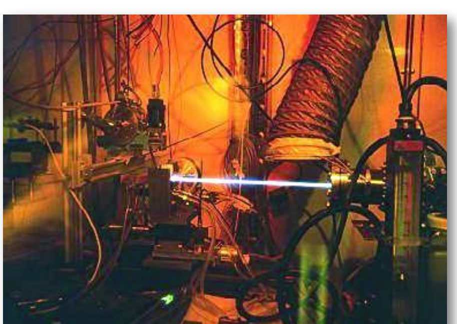
$$w(\Theta) \approx \left(\frac{\Theta^2}{2}+\frac{1}{2\gamma^2}\right)^{-3}$$

We chose now an angle of $\Theta = 1/\gamma$ and find the relation

$$\frac{w(1/\gamma)}{w(0)} = \frac{\left(\frac{1}{2\gamma^2}+\frac{1}{2\gamma^2}\right)^{-3}}{\left(\frac{1}{2\gamma^2}\right)^{-2}} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$



A sychrotron radiation beam



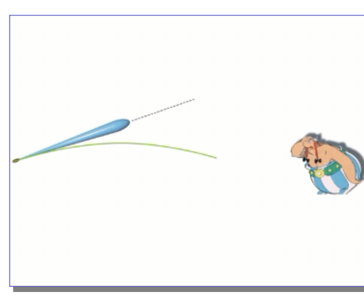
3.4. Time structure and radiation spectrum

A detailed evaluation of the spectral functions can be derived in

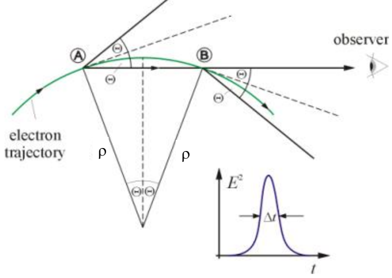
J.D. Jackson, *Classical Electrodynamics*, Sect. 14

or in

H. Wiedemann, *Particle Accelerator Physics II*, chapter 7.4



The synchrotron radiation is focused within a cone of an angle $\Theta = 1/\gamma$. An observer locking onto the particle trajectory can see the radiation the first time when the electron has reached the point **A.**



The photons from **A** fly directly to the observer with the velocity of light. The electron takes the circular trajectory and its velocity is less than the velocity of light. **B** is the last position from which radiation can be observed. The duration of the light flash is the difference of the time used by the electron and by the photon moving from the point **A** to point **B**

$$\Delta t = t_e - t_\gamma = \frac{2\rho\Theta}{c\beta} - \frac{2\rho\sin\Theta}{c}$$

or

$$\Delta t = \frac{2\rho}{c}\left(\frac{\Theta}{\beta} - \Theta + \frac{\Theta^3}{3!} - \cdots\right) = \frac{2\rho}{c}\left(\frac{1}{\gamma - 1/2\gamma} - \frac{1}{\gamma} + \frac{1}{6\gamma^3}\right)$$

With

$$\frac{1}{\gamma - 1/2\gamma} = \frac{1}{\gamma}\frac{1}{1 - 1/2\gamma^2} \approx \frac{1}{\gamma}\left(1 + \frac{1}{2\gamma^2}\right) = \frac{1}{\gamma} + \frac{1}{2\gamma^3}$$

We get

$$\Delta t \approx \frac{2\rho}{c}\left(\frac{1}{\gamma} + \frac{1}{2\gamma^3} - \frac{1}{\gamma} + \frac{1}{6\gamma^3}\right) = \frac{4\rho}{3c\gamma^3}$$

In order to calculate the pulse length we assume a bending radius of $\rho = 3.3$ m and a beam energy of $E = 1.5$ GeV, i.e. $\gamma = 2935$. With this parameters the pulse length becomes

$$\Delta t = 5.8\cdot 10^{-19}\,\mathrm{sec}$$

This extremely short pulse causes a broad frequency spectrum with the *typical frequency*

$$\omega_{typ} = \frac{2\pi}{\Delta t} = \frac{3\pi c\gamma^3}{2\rho}$$

More often the *critical frequency*

$$\omega_c = \frac{\omega_{typ}}{\pi} = \frac{3c\gamma^3}{2\rho}$$

is used. The exact calculation of the radiation spectrum has been carried out the first time by *Schwinger.* He found

$$\frac{d\dot{N}}{d\varepsilon/\varepsilon} = \frac{P_0}{\omega_c\hbar}S_s\left(\frac{\omega}{\omega_c}\right) \qquad (3.17)$$

With the radiation power given in (3.6)

$$P_s = \frac{e^2 c}{6\pi\varepsilon_0 (m_0 c^2)^4}\frac{E^4}{\rho^2}$$

the total power radiated by N electrons is

$$P_0 = \frac{e^2 c\gamma^4}{6\pi\varepsilon_0\rho^2}N = \frac{e\gamma^4}{3\varepsilon_0\rho}I_b$$

with the beam current

$$I_b = \frac{Nec}{2\pi\rho}$$

The spectral function in (3.17) has the form

$$S_s(\xi) = \frac{9\sqrt{3}}{8\pi}\xi\int_\xi^\infty K_{5/3}(\xi)d\xi$$

where $K_{5/3}(\xi)$ is the modified Bessel function and $\xi = \omega/\omega_c$.

Because of energy conservation the spectral function satisfies the normalization condition

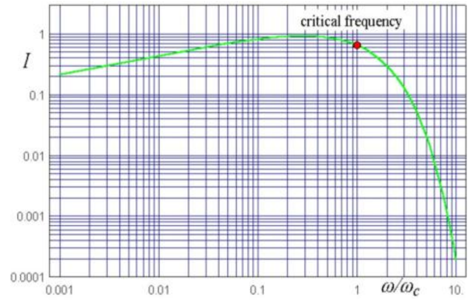
$$\int_0^\infty S_s(\xi)\,d\xi = 1$$

Integrating until the upper limit $\xi = 1$, i.e. $\omega = \omega_c$, gives

$$\int_0^1 S_s(\xi)\,d\xi = \frac{1}{2}$$

This result shows that the critical frequency ω_c divides the spectrum into two parts of identical radiation power.

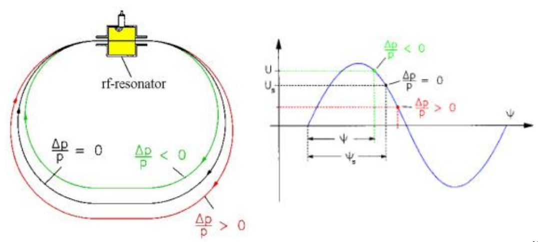
Synchrotron radiation spectrum from a bending magnet



4 Electron Dynamics with Radiation

4.1 The particles as harmonic oscillators

In cyclic machines we have synchrotron and betatron oscillations. In a good approximation we can consider the system to be a harmonic oscillator.



4.1.1 Synchrotron oscillation

In a circular accelerator we have to compensate the energy loss by a rf-cavity ("phase focusing").

For an on-momentum particle ($\Delta p/p = 0$) the energy change per revolution is

$$E_0 = eU_0 \sin\Psi_s - W_0 \tag{4.1}$$

with the reference phase Ψ_s, the peak voltage U_0 and the energy loss W_0. For any particle with a phase deviation $\Delta\Psi$ we find

$$E = eU_0 \sin(\Psi_s + \Delta\Psi) - W \tag{4.2}$$

The energy loss can be expanded as

$$W = W_0 + \frac{dW}{dE}\Delta E$$

The difference between (4.1) and (4.2) is

$$\Delta E = E - E_0 = eU_0[\sin(\Psi_s + \Delta\Psi) - \sin\Psi_s] - \frac{dW}{dE}\Delta E$$

The frequency of the phase oscillations is very low compared to the revolution frequency $f_u = 1/T_0$. It follows

$$\Delta\dot{E} = \frac{\Delta E}{T_0} = \frac{eU_0}{T_0}[\sin(\Psi_s + \Delta\Psi) - \sin\Psi_s] - \frac{dW}{dE}\frac{\Delta E}{T_0} \tag{4.3}$$

The phase difference $\Delta\Psi$ is caused by the variation of the revolution time of the particles

$$\Delta T = T_0 \frac{\Delta L}{L_0} = T_0 \alpha \frac{\Delta E}{E} \tag{4.4}$$

with the *momentum-compaction-factor* α defined as

$$\frac{\Delta L}{L_0} = \alpha \frac{\Delta p}{p}$$

With the period of the rf-voltage T_{rf} we get

$$\Delta\Psi = 2\pi \frac{\Delta T}{T_{rf}} = \omega_{rf} \Delta T \tag{4.5}$$

The ratio of the rf-frequency and the revolution frequency must be an integer number

$$q = \frac{\omega_{rf}}{\omega_u} \quad \text{with} \quad q = \text{integer}$$

With (4.4) and (4.5) we get

$$\Delta\Psi = q\,\omega_u \Delta T = 2\pi q \frac{\Delta T}{T_0} = 2\pi q \alpha \frac{\Delta E}{E}$$

and after differentation

$$\Delta\dot{\Psi} = \frac{\Delta\Psi}{T_0} = \frac{2\pi q \alpha}{T_0}\frac{\Delta E}{E} \tag{4.6}$$

Assuming small oscillations, i.e. $\Delta\Psi \ll \Psi_s$ we can write

$$\sin(\Psi_s + \Delta\Psi) - \sin\Psi_s = \sin\Psi_s \cos\Delta\Psi + \cos\Psi_s \sin\Delta\Psi - \sin\Psi_s$$
$$\approx \Delta\Psi \cos\Psi_s$$

With this approximation equation (4.3) reduces to

$$\Delta\dot{E} = \frac{eU_0}{T_0}\Delta\Psi \cos\Psi_s - \frac{dW}{dE}\frac{\Delta E}{T_0}$$

A second differentiation provides

$$\Delta\ddot{E} = \frac{eU_0}{T_0}\Delta\dot{\Psi} \cos\Psi_s - \frac{dW}{dE}\frac{\Delta\dot{E}}{T_0}$$

Insertion of (4.6) gives

$$\Delta\ddot{E} + \frac{1}{T_0}\frac{dW}{dE}\Delta\dot{E} - \frac{2\pi q e \alpha U_0 \cos\Psi_s}{T_0^2 E}\Delta E = 0$$

or

$$\Delta\ddot{E} + 2a_s \Delta\dot{E} + \Omega^2 \Delta E = 0 \tag{4.7}$$

with the damping const

$$a_s = \frac{1}{2T_0}\frac{dW}{dE} \tag{4.8}$$

and the synchrotron frequency

$$\Omega = \omega_u \sqrt{-\frac{eU_0 q \alpha \cos\Psi_s}{2\pi E}}$$

The equation (4.7) can be solved by the ansatz

$$\Delta E(t) = \Delta E_0 \exp(-a_s t)\exp(i\Omega t)$$

This damped oscillation with the frequency Ω is called the *synchrotron oscillation*.

4.1.2 Betatron oscillation

The motion of a charged particle can be expressed by the equations

$$x''(s)+\left(\frac{1}{\rho^2(s)}-k(s)\right)x(s)=\frac{1}{\rho(s)}\frac{\Delta p}{p}$$

$$z''(s)+k(s)z(s)=0$$

Where $\rho(s)$ and $k(s)$ give the bending radius and the quadrupole strength. With $K(s) = 1/\rho^2(s) - k(s)$ we find for on-momentum particles

$$x''(s)+K(s)\,x(s)=0 \qquad (4.9)$$

According to *Floquet's theorem* we find the solution

$$x(s)=\sqrt{\varepsilon}\sqrt{\beta(s)}\cos[\Psi(s)+\phi] \qquad (4.10)$$

with the constant beam emittance ε and the variable but periodic betafunction ß(s) .

The phase can be expressed as

$$\Psi(s)=\int_0^s\frac{d\sigma}{\beta(\sigma)}$$

The solution (4.10) is a transverse spatial particle oscillation with respect to the beam orbit. We have a strong correlation between the position s at the orbit and the time t

$$s(t)=s_0+ct$$

This transverse periodic particle motion is called *betatron oscillation*.

4.2 Radiation Damping

The damping needs an energy loss due to synchrotron radiation depending on the oscillation amplitude.

4.2.1 Damping of synchrotron oscillation

The radiated power of the synchrotron radiation is

$$P_s=\frac{e^2c}{6\pi\varepsilon_0}\frac{1}{(m_0c^2)^4}\frac{E^4}{\rho^2}$$

The bending radius is

$$\frac{1}{\rho}=\frac{e}{p}B=\frac{ec}{E}B \qquad \Rightarrow \qquad \frac{E^2}{\rho^2}=e^2c^2B^2$$

We can write the radiated power in the form

$$P_s=\mathrm{C}E^2B^2 \qquad \text{with} \qquad \mathrm{C}=\frac{e^4c^3}{6\pi\varepsilon_0(m_0c^2)^4} \qquad (4.11)$$

In order to evaluate the radiation damping of the synchrotron oscillation we use the equation (4.7)

$$\Delta\ddot{E}+2a_s\Delta\dot{E}+\Omega^2\Delta E=0$$

with the damping constant (4.8)

$$a_s=\frac{1}{2T_0}\frac{dW}{dE}$$

It is necessary to calculate the ration dW/dE. We estimate the energy loss along a dispersion. It is

$$ds'=\left(1+\frac{\Delta x}{\rho}\right)ds$$

Using $ds'/dt=c$ the energy loss per revolution is

$$W=\int_0^{T_0}P_s dt=\oint P_s\frac{ds'}{c}=\frac{1}{c}\oint P_s\left(1+\frac{\Delta x}{\rho}\right)ds$$

The displacement Δx is caused by an energy deviation

$$\Delta x=D\frac{\Delta E}{E}$$

The energy loss becomes

$$W=\frac{1}{c}\oint P_s\left(1+\frac{D}{\rho}\frac{\Delta E}{E}\right)ds$$

Differentiating gives

$$\frac{dW}{dE}=\frac{1}{c}\oint\left[\frac{dP_s}{dE}+\frac{D}{\rho}\left(\frac{dP_s}{dE}\frac{\Delta E}{E}+P_s\frac{1}{E}\right)\right]ds \qquad (4.12)$$

Averaging over a long time one finds

$$\left\langle\frac{\Delta E}{E}\right\rangle=0$$

Equation (4.12) becomes

$$\frac{dW}{dE}=\frac{1}{c}\oint\left[\frac{dP_s}{dE}+\frac{DP_s}{\rho E}\right]ds \qquad (4.13)$$

We use the radiation formula (4.11) and get

$$\frac{dP_s}{dE}=2\mathrm{C}EB^2+2\mathrm{C}E^2B\frac{dB}{dE}=2P_s\left(\frac{1}{E}+\frac{1}{B}\frac{dB}{dE}\right) \qquad (4.14)$$

In quadrupoles with non vanishing dispersion the field variation with the particle energy is

$$\frac{dB}{dE}=\frac{dB}{dx}\frac{dx}{dE}=\frac{dB}{dx}\frac{D}{E}$$

It is put into the expression (4.14) and we get from (4.13)

$$\frac{dW}{dE}=\frac{1}{c}\oint\left[2P_s\left(\frac{1}{E}+\frac{D}{BE}\frac{dB}{dx}\right)+P_s\frac{D}{\rho E}\right]ds$$

$$=\underbrace{\frac{2}{cE}\oint P_s\,ds}_{=2W_0/E}+\frac{1}{cE}\oint DP_s\left(\frac{2}{B}\frac{dB}{dx}+\frac{1}{\rho}\right)ds$$

With (4.8) the damping constant is then

$$a_s=\frac{1}{2T_0}\frac{dW}{dE}=\frac{W_0}{2T_0E}\left[2+\frac{1}{cW_0}\oint DP_s\left(\frac{2}{B}\frac{dB}{dx}+\frac{1}{\rho}\right)ds\right]$$

or

$$a_s=\frac{W_0}{2T_0E}(2+\mathcal{D})\quad\text{with}\quad\mathcal{D}=\frac{1}{cW_0}\oint DP_s\left(\frac{2}{B}\frac{dB}{dx}+\frac{1}{\rho}\right)ds\qquad(4.15)$$

It is more convenient to apply the bending radius ρ and the quadru-pole strength k

$$\left.\begin{array}{l}k=\dfrac{ec}{E}\dfrac{dB}{dx}\;\rightarrow\;\dfrac{dB}{dx}=\dfrac{kE}{ec}\\[2ex]\dfrac{1}{\rho}=\dfrac{ec}{E}B\;\rightarrow\;\dfrac{1}{B}=\dfrac{ec}{E}\rho\end{array}\right\}\Rightarrow\quad\frac{1}{B}\frac{dB}{dx}=k\rho$$

We write the radiation power in the form

$$P_s=\frac{C}{e^2c^2}\frac{E^4}{\rho^2}$$

Then the integral (4.15) becomes

$$\oint DP_s\left(\frac{2}{B}\frac{dB}{dx}+\frac{1}{\rho}\right)ds=\frac{CE^4}{e^2c^2}\oint\frac{D}{\rho^2}\left(2k\rho+\frac{1}{\rho}\right)ds=\frac{CE^4}{e^2c^2}\oint\frac{D}{\rho}\left(2k+\frac{1}{\rho^2}\right)ds$$

The energy radiated by an on-momentum particle is

$$W_0=\int_0^{T_0}P_s\,dt=\frac{1}{c}\oint P_s\,ds=\frac{CE^4}{e^2c^3}\oint\frac{ds}{\rho^2}$$

The damping constant for synchrotron oscillation is

$$a_s=\frac{W_0}{2T_0E}(2+\mathcal{D})\quad\text{with}\quad\mathcal{D}=\frac{\oint\frac{D}{\rho}\left(2k+\frac{1}{\rho^2}\right)ds}{\oint\frac{ds}{\rho^2}}\qquad(4.16)$$

The damping only depends on the magnet structure of the machine.

4.2.1 Damping of betatron oscillation

Following *Floquet's transformation* we can write with

$$A:=b\sqrt{\beta(s)}$$

$$\left.\begin{array}{l}z=b\sqrt{\beta(s)}\cos\phi\\ -\dfrac{b}{\sqrt{\beta(s)}}\sin\phi\end{array}\right\}\Rightarrow\left\{\begin{array}{l}z=A\cos\phi\\ z'=-\dfrac{A}{\beta(s)}\sin\phi\end{array}\right.\qquad(4.17)$$

We calculate the amplitude A using z and z'.

$$A^2=A^2\cos^2\phi+A^2\sin^2\phi=z^2+[\beta(s)z']^2\qquad(4.18)$$

A photon is emitted and the particle momentum $\vec{p}$ is reduced by $\delta\vec{p}$

$$\vec{p}^{\,*}=\vec{p}-\delta\vec{p}$$

The longitudinal component p_s of the particle momentum is restored by the rf-cavity, the transverse component, however, stays reduced. The angle z' is reduced by the amount

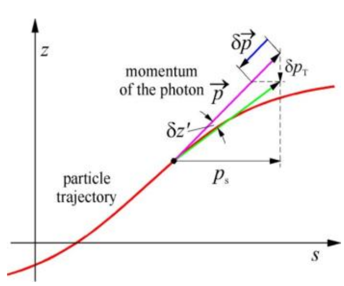
$$\delta z'=-\frac{\delta p_\perp}{|\vec{p}|}$$

The energy variation of the electron is then

$$\delta E=\frac{c^2}{v}\delta p_\perp$$

or using $v=z'c$

$$\delta E=\frac{c}{z'}\delta p_\perp$$



With the relation $E=c|\vec{p}|$ follows

$$\delta z'=-\frac{\delta E}{E}z'\qquad(4.19)$$

From (4.18) we get the variation

$$\delta\left(A^2\right)=\delta\left(z^2\right)+\delta\left(z'^2\beta^2(s)\right)=\beta^2(s)\,\delta\left(z'^2\right)$$

And we find with $\delta\left(z^2\right)=0$

$$2A\delta A=2\beta^2(s)z'\delta z'\quad\Rightarrow\quad A\delta A=\beta^2(s)z'\delta z'$$

After insertion of (4.19) we get

$$A\delta A=-\beta^2(s)z'^2\frac{\delta E}{E}\qquad(4.20)$$

Now one has to average over z'^2. Taking the formula (4.17) gives

$$\left\langle z'^2\right\rangle=\frac{A^2}{2\pi\beta^2(s)}\int_0^{2\pi}\sin^2\phi\,d\phi=\frac{A^2}{2\beta^2(s)}$$

In this way we find with the relation (4.20)

$$A\langle\delta A\rangle = -\frac{A^2}{2\beta^2(s)}\beta^2(s)\frac{\delta E}{E} = -\frac{A^2}{2}\frac{\delta E}{E}$$

After a full revolution the energy losses δE have accumulated to the total loss W_0. The average amplitude variation per revolution is

$$\Delta A = \sum\langle\delta A\rangle \qquad (4.21)$$

Then we get from (4.21)

$$\frac{\Delta A}{A} = -\frac{W_0}{2E}$$

The amplitude decreases and we have a damping of the betatron oscillation. The damping constant is

$$\frac{dA}{A} = -a_z dt$$

With the revolution time $\Delta t = T_0$ we finally find

$$a_z = -\frac{\Delta A}{A\,\Delta t} = \frac{W_0}{2ET_0} \qquad (4.22)$$

A similar calculations including the dispersion gives

$$a_x = \frac{W_0}{2ET_0}(1-\mathcal{D}) \qquad (4.23)$$

with

$$\mathcal{D} = \frac{\oint \frac{D}{\rho}\left(2k + \frac{1}{\rho^2}\right)ds}{\oint \frac{ds}{\rho^2}}$$

4.3 The Robinson theorem

With the equations (4.16), (4.22) and (4.23) we have all damping constants

$$a_s = \frac{W_0}{2T_0E}(2+\mathcal{D}) = \frac{W_0}{2T_0E}J_s \qquad a_z = \frac{W_0}{2T_0E} = \frac{W_0}{2T_0E}J_z$$

$$a_x = \frac{W_0}{2T_0E}(1-\mathcal{D}) = \frac{W_0}{2T_0E}J_x$$

with

$$J_s = 2+\mathcal{D} \qquad J_z = 1 \qquad J_x = 1-\mathcal{D}$$

From these relations we can directly derive *the Robinson criteria*

$$J_x + J_z + J_s = 4$$

The total damping is constant. The change of the damping partition is possible by varying the quantity $\mathcal{D}$. In most of the cases we have $\mathcal{D} \ll 1$ (*"natural damping partition"*).

In strong focusing machines it is possible to shift the particles onto a dispersion trajectory by variation of the particle energy. With this measure one can change the value of $\mathcal{D}$ within larger limits. The trajectory circumference L depends on the rf-frequency f as

$$L = q\lambda = q\frac{c}{f} \quad\Rightarrow\quad dL = -qc\frac{df}{f^2}$$

We get

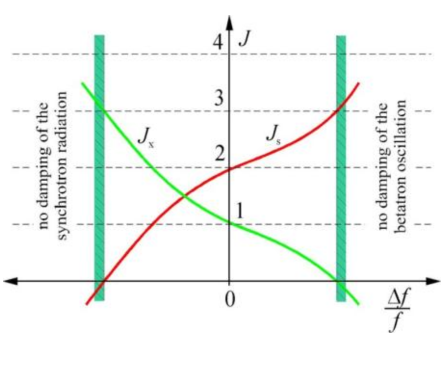
$$\frac{\Delta L}{L} = -\frac{qc}{L}\frac{\Delta f}{f^2} = -\frac{\Delta f}{f}$$

With the momentum compaction factor we get

$$\frac{\Delta L}{L} = \alpha\frac{\Delta E}{E} \quad\Rightarrow\quad \frac{\Delta E}{E} = \frac{1}{\alpha}\frac{\Delta L}{L} = -\frac{1}{\alpha}\frac{\Delta f}{f}$$

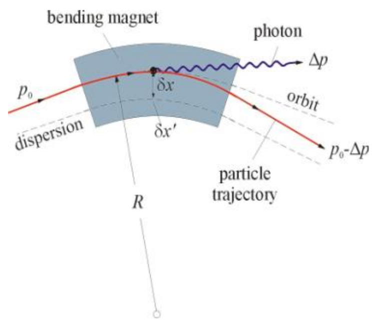
The variation of the rf-frequency f shifts the beam onto the dispersion trajectory

$$x_D(s) = -D(s)\frac{1}{\alpha}\frac{\Delta f}{f}$$



5. Particle distribution in the transversal phase space

5.1 Transversal beam emittance



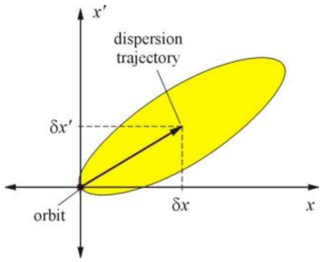
The natural beam emittance is determined by the emission of synchrotron radiation.

We start with an electron of momentum p_0 and emittance $\varepsilon_i = 0$. The particle emits a photon with the momentum Δp and continues the flight with the momentum $p_0 - \Delta p$

It now belongs to a dispersion trajectory with the displacement and angle

$$\delta x = D\frac{\Delta p}{p}$$

and $\delta x' = D'\frac{\Delta p}{p}$



The electron has therefore a finite emittance. It can be calculated using the ellipse relation.

$$\varepsilon_i = \gamma\delta x^2 + 2\alpha\delta x\delta x' + \beta\delta x'^2 = \left(\frac{dp}{p}\right)^2\left(\gamma D^2 + 2\alpha DD' + \beta D'^2\right) = \left(\frac{dp}{p}\right)^2 \mathcal{H}(s)$$

To get the beam emittance one had to integrate over all particles in the beam. For relativistic particles is

$$\frac{\Delta p}{p} = \frac{\Delta E}{E}$$

A detailled calculation gives the natural beam emittance in the form

$$\varepsilon_x = \frac{55}{32\sqrt{3}}\frac{\hbar c}{m_0 c^2}\gamma^2 \frac{\left\langle \frac{1}{R^3}\mathcal{H}(s)\right\rangle}{J_x\left\langle \frac{1}{R^2}\right\rangle}$$

The damping is represented by the amount J_x. If all bending magnets are equal, we get with $J_x \approx 1$ the simplified expression

$$\varepsilon_x = 1.47\cdot 10^{-6}\frac{E^2}{Rl}\int_0^l \mathcal{H}(s)ds$$

with E in [GeV], R in [m] and ε_x in [m rad]. Because of

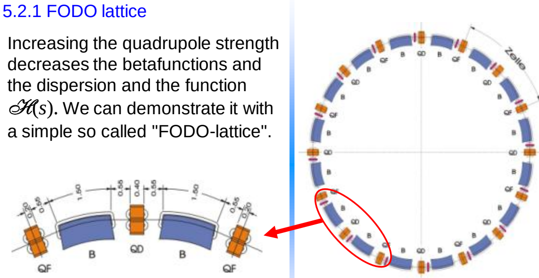
$$\mathcal{H}(s) = \left(\gamma D^2 + 2\alpha DD' + \beta D'^2\right)$$

the emittance is small whenever the betafunction and the dispersion is small inside a bending magnet.

5.2 Examples

5.2.1 FODO lattice

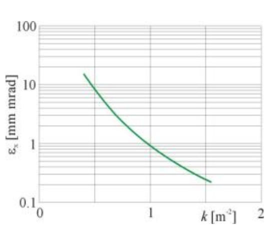
Increasing the quadrupole strength decreases the betafunctions and the dispersion and the function $\mathcal{H}(s)$. We can demonstrate it with a simple so called "FODO-lattice".



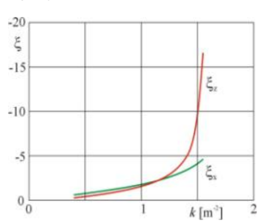
one cell of the FODO-lattice

OPTICS

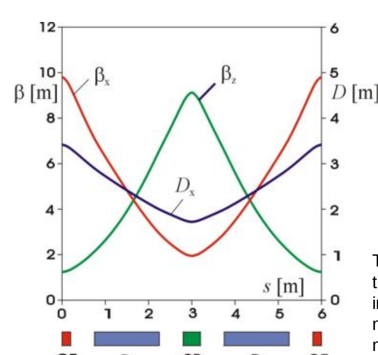
The quadrupole strengths vary from $k = 0.4$ m^{-2} to $k = 1.6$ m^{-2}. It reduces the emittance almost by two orders of magnitude !



With increasing quad strength, the chromaticity increases rapidly.

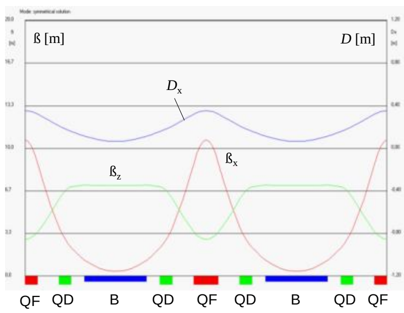


Extremely low beam emittances need very effective chromaticity compensation (dynamic aperture).



The betafunction and the dispersion have in the bending magnet not the minimum value.

5.2.2 Triplett lattice



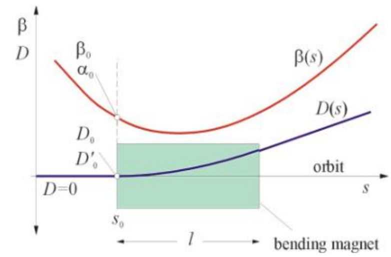
This structure has been used for the electron storage ring DELTA. The emittance at an beam energy of $E = 1.5$ GeV is $\varepsilon_x = 7{\cdot}10^{-9}$ m rad.

6. Low emittance lattice

6.1 Basic idea of low emittance lattices

What is the lowest possible beam emittance ?

In dedicated synchrotron radiation sources long straight sections for wiggler and undulator magnets are required. This straight sections have no dispersion, i.e. $D \equiv 0$.



Therefore, at the beginning of the bending magnet the dispersion has the initial value

$$\begin{pmatrix} D_0 \\ D_0' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

With this initial condition the dispersion in the bending magnet is well defined. With $s/R \ll 1$ we get

$$D(s) = R\left(1 - \cos\frac{s}{R}\right) \approx \frac{s^2}{2R} \qquad D'(s) = \sin\frac{s}{R} \approx \frac{s}{R}$$

The emittance can only be changed by varying the initial values β_0 and α_0 of the betafunction. These functions can be transformed as

$$\begin{pmatrix} \beta(s) & -\alpha(s) \\ -\alpha(s) & \gamma(s) \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$$

and after straight forward calculations

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2, \quad \alpha(s) = \alpha_0 - \gamma_0 s, \quad \gamma(s) = \gamma_0 = \text{const.}$$

We can write the function $\mathcal{H}(s)$ in the form

$$\mathcal{H}(s) = \gamma(s)D^2(s) + 2\alpha(s)D(s)D'(s) + \beta(s)D'^2(s)$$
$$= \frac{1}{R^2}\left(\frac{\gamma_0}{4}s^4 - \alpha_0 s^3 + \beta_0 s^2\right)$$

For identical bending magnets and with $J_x = 1$ we get

$$\varepsilon_x = C_\gamma \frac{\gamma^2}{Rl}\int_0^l \mathcal{H}(s)ds = C_\gamma \gamma^2 \left(\frac{l}{R}\right)^3 \left(\frac{\gamma_0 l}{20} - \frac{\alpha_0}{4} + \frac{\beta_0}{3l}\right)$$

with

$$C_\gamma = \frac{55}{32\sqrt{3}}\frac{\hbar}{m_0 c} = 3.832\cdot 10^{-13}\text{m}$$

The relation

$$\frac{l}{R} = \Theta$$

is the bending angle of the magnet. We can write

$$\varepsilon_x = C_\gamma \gamma^2 \Theta^3 \left(\frac{\gamma_0 l}{20} - \frac{\alpha_0}{4} + \frac{\beta_0}{3l}\right) \qquad (6.1)$$

Since the emittance grows with Θ^3 one should use many short bending magnets rather than a few long ones to get beams with low emittances.

In order to get the minimum possible emittance we have to vary the initial conditions β_0 and α_0 in (6.1) until the minimum is found. This is the case if

$$\frac{\partial \varepsilon_x}{\partial \alpha_0} = \mathcal{A}\frac{\partial}{\partial \alpha_0}\left(\frac{1+\alpha_0^2}{\beta_0}\frac{l}{20} - \frac{\alpha_0}{4} + \frac{\beta_0}{3l}\right) = \mathcal{A}\left(\frac{\alpha_0}{\beta_0}\frac{l}{10} - \frac{1}{4}\right) = 0$$

and

$$\frac{\partial \varepsilon_x}{\partial \beta_0} = \mathcal{A}\left(-\frac{1+\alpha_0^2}{\beta_0^2}\frac{l}{20} + \frac{1}{3}\right) = 0$$

with

$$\mathcal{A} = C_\gamma \gamma^2 \Theta^3$$

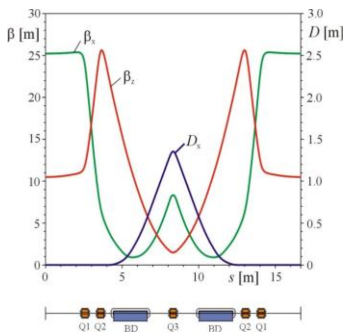
The unknown initial conditions β_0 and α_0 are

$$\beta_{0,\min} = 2\sqrt{\frac{3}{5}}\,l = 1.549\,l \qquad (6.2)$$
$$\alpha_{0,\min} = \sqrt{15} = 3.873$$

The betafunction for the minimum possible emittance is determined only by the magnet length l.

This principle is used by the Chasman Green lattice, the optical functions do not exactly fit the conditions (6.2). The reason is the extremely high chromaticity caused by the ideal initial conditions (6.2).

The simple magnet structure shown in the figure has no flexibility. Therefore, more quadrupole magnets are used in modern light sources

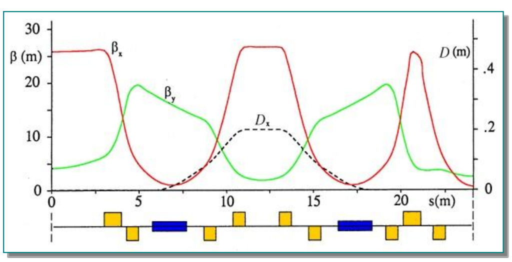


European **S**ynchrotron **R**adiation **F**acility, Grenoble

An example of a flexible low emittance storage ring of the third generation

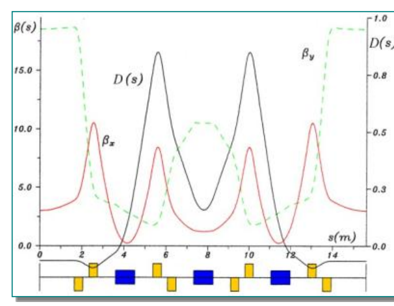
In order to get the required flexibility, a larger number of quadrupols is applied.

The lattice of one cell of the ESRF magnet structure. The ring consists on 32 cells.



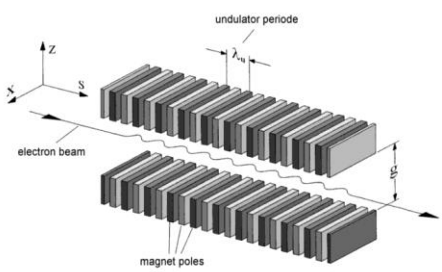
Magnet structures of this type are often called "*double bend achromat lattice*" (DBA)

Another modification of this optical principle is the "*triple bend achromat lattice*" (TBA) as applied in the storage ring BESSY II in Berlin



7. Appendix A: Undulator radiation

Synchrotron radiation is nowadays mostly generated by use of undulators (or "insertion devices").



7.1 The field of a wiggler or undulator

Along the orbit one has a periodic field with the period length λ_u. The potential is

$$\varphi(s,z) = f(z)\cos\left(2\pi\frac{s}{\lambda_u}\right) = f(z)\cos(k_u s). \qquad (7.1)$$

In x-direction the magnet is assumed to be unlimited. The function $f(z)$ gives the vertical field pattern. With the Laplace equation

$$\nabla^2\varphi(s,z) = 0$$

We get

$$\frac{d^2 f(z)}{dz^2} - f(z)k_u^2 = 0$$

and find the solution

$$f(z) = A\sinh(k_u z)$$

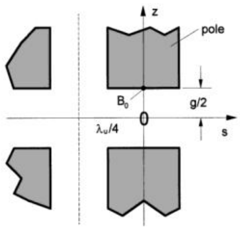
Inserting into (7.1) the potential becomes

$$\varphi(s,z) = A\sinh(k_u z)\cos(k_u s)$$

and the vertical field component

$$B_z(s,z) = \frac{\partial\varphi}{\partial z} \qquad (7.2)$$

$$= k_u A\cosh(k_u z)\cos(k_u s)$$



In order to get the integration constant A we take the pole tip field B_0 at $\{s,z\} = \{0, g/2\}$. With (7.2) we get

$$B_0 = B_z\left(0,\frac{g}{2}\right) = k_u A\cosh\left(k_u \frac{g}{2}\right)$$

$$= k_u A\cosh\left(\pi\frac{g}{\lambda_u}\right)$$

and

$$A = \frac{B_0}{k_u\cosh(\pi g/\lambda_u)}$$

Insertion into (8.2) provides

$$B_z(s,z) = \frac{B_0}{\cosh\left(\pi\frac{g}{\lambda_u}\right)}\cosh(k_u z)\cos(k_u s)$$

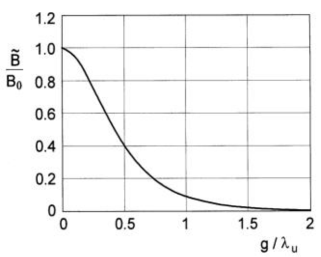
and

$$B_s(s,z) = \frac{\partial\varphi}{\partial s} = \frac{-B_0}{\cosh\left(\pi\frac{g}{\lambda_u}\right)}\sinh(k_u z)\sin(k_u s)$$

At the orbit the periodic field has the maximum value

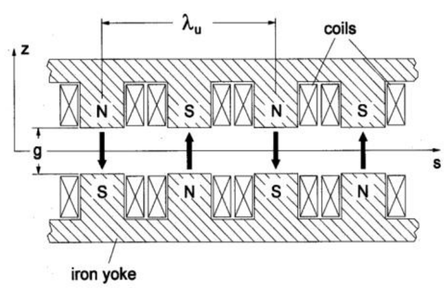
$$\tilde{B} = \frac{B_0}{\cosh(\pi g/\lambda_u)}$$

For given period length the λ_u the field decreases with increasing gap height g. Short periods require therefore small pole distances.

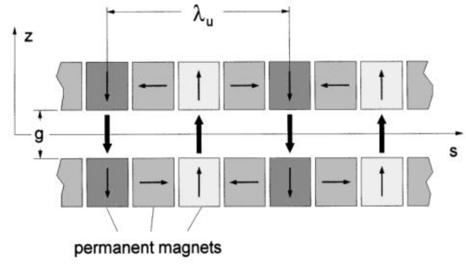


At the beam the periodic field is $B_z(s,z) = \tilde{B}\sin(k_u s)$

The most simple design is an electromagnet



Shorter period length down to a few cm are possible by use of permanent magnets. The field variation is made by changing the gap height.



Undulator of the SASE-FEL (DESY)

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Undulator U55 an DELTA

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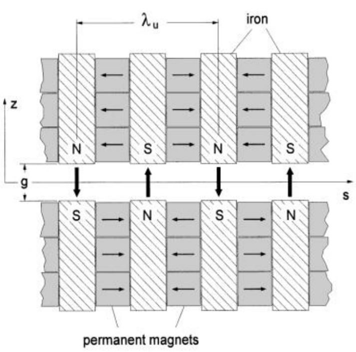
A hybrid magnet consists of permanent magnets and iron poles.



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Superconductive wiggler magnet

W/U-magnets have maximum fields at the beam about 1 T. The minimum wave length is limited because of

$$\lambda_c = \frac{4\pi}{3}\frac{R}{\gamma^3} = \frac{4\pi c (m_0 c^2)^3}{3eE^2}\frac{1}{\tilde{B}}$$

Shorter wave lengths are possible with superconductive wigglers with fields of $B > 5$T.

Superconductive asymmetric wiggler at the storage ring DELTA

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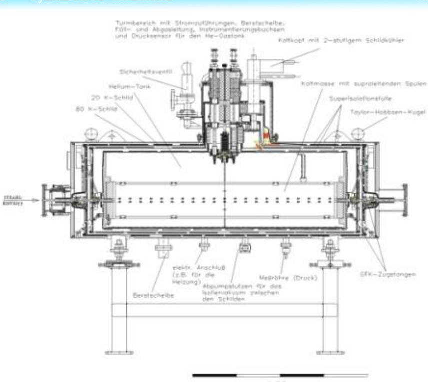

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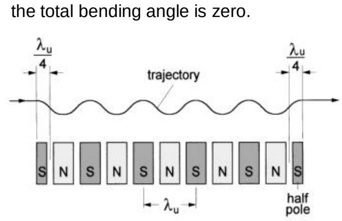
The W/U-field has to be matched that the total bending angle is zero.



We have then

$$\int_{W/U} B_z(s)ds = \tilde{B}\int_{s_1}^{s_2} \cos(k_u s)ds = 0$$

The condition is fulfilled if

$$s_1 = 0$$

and $s_2 = n\lambda_u + \frac{\lambda_u}{2}$

with $n = 1, 2, \ldots$. It is possible to utilize at both ends short magnet pieces of half pole length. In addition one has to shim the single poles to compensate the unavoidable tolerances.

7.2 Equation of motion in a W/U-magnet

In a W/U-magnet we have the Lorentz force

$$\vec{F} = \dot{\vec{p}} = m_0\gamma\dot{\vec{v}} = e\vec{v}\times\vec{B}$$

With the approximation

$$\vec{B} = \begin{pmatrix} 0 \\ B_z \\ B_s \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} v_x \\ 0 \\ v_s \end{pmatrix}$$

We get

$$\dot{\vec{v}} = \frac{e}{m_0\gamma}\begin{pmatrix} -v_s B_z \\ -v_x B_s \\ v_x B_z \end{pmatrix}$$

The velocity component in z-direction is very small and can be neglected. With $\dot{x} = v_x$ and $\dot{s} = v_s$ we have the motion in the s-x-plane

$$\ddot{x} = -\dot{s}\frac{e}{m_0\gamma}B_z(s) \qquad \ddot{s} = \dot{x}\frac{e}{m_0\gamma}B_z(s) \qquad (7.3)$$

This is a coupled set of equations. The influence of the horizontal motion on the longitudinal velocity is very small

$$\dot{x} = v_x << c \quad \text{and} \quad \dot{s} = v_s = \beta c = \text{const.}$$

In this case only the first equation of (7.3) is important and we get

$$\ddot{x} = -\frac{\beta c e\tilde{B}}{m_0\gamma}\cos(k_u s)$$

We replace with

$$\dot{x} = x'\beta c \quad \text{and} \quad \ddot{x} = x''\beta^2 c^2$$

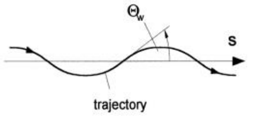
the time derivative by a spatial one and get

$$x'' = -\frac{e\tilde{B}}{m_0\beta c\gamma}\cos(k_u s) = -\frac{e\tilde{B}}{m_0\beta c\gamma}\cos\left(2\pi\frac{s}{\lambda_u}\right)$$

With $\beta = 1$ we can write

$$x'(s) = \frac{\lambda_u e\tilde{B}}{2\pi m_0\gamma c}\sin(k_u s) \qquad x(s) = \frac{\lambda_u^2 e\tilde{B}}{4\pi^2 m_0\gamma c}\cos(k_u s) \qquad (7.4)$$

The maximum angle is at $\sin(k_u s) = 1$



$$\Theta_w = x'_{max} = \frac{1}{\gamma}\frac{\lambda_u e\tilde{B}}{2\pi m_0 c}$$

We get the wiggler- or undulator parameter

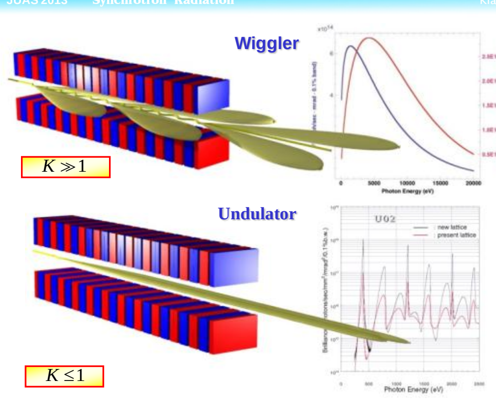
$$K = \frac{\lambda_u e\tilde{B}}{2\pi m_0 c} \qquad (7.5)$$

The maximum trajectory angle is

$$\Theta_w = \frac{K}{\gamma}$$

This is the natural opening angle of the synchrotron radiation. With the parameter K we can now distinguish between wiggler and undulator:

undulator	if	$K \le 1$	i.e.	$\Theta_w \le 1/\gamma$
wiggler	if	$K > 1$	i.e.	$\Theta_w > 1/\gamma$



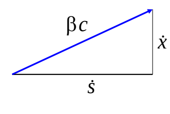
Now we go back to the system of coupled equations (7.3). We assume that the horizontal motion is only determined by a constant average velocity $\bar{v}_s = \langle\dot{s}\rangle$. From (7.4) and (7. 5) we get

$$x'(s) = \frac{K}{\gamma}\sin(k_u s) = \Theta_w\sin(k_u s)$$

With $\dot{x} = \beta c x'$, $s = \beta c t$ and $\omega_u = k_u\beta c$ we can write

$$\dot{x}(t) = \beta c\Theta_w\sin(\omega_u t) = \beta c\frac{K}{\gamma}\sin(\omega_u t) \qquad (7.6)$$

For the velocity holds



$$\dot{s}^2 = (\beta c)^2 - \dot{x}^2$$

and with

$$\beta^2 = 1-\frac{1}{\gamma^2}$$

we get

$$\dot s(t) = c\sqrt{1-\left(\frac{1}{\gamma^2}+\frac{\dot x^2}{c^2}\right)}$$

Since the expression in the brackets is very small, the root can be expand in the way

$$\dot s(t) = c\left[1-\frac{1}{2}\left(\frac{1}{\gamma^2}+\frac{\dot x^2}{c^2}\right)\right] = c\left[1-\frac{1}{2\gamma^2}\left(1+\frac{\gamma^2}{c^2}\dot x^2\right)\right]$$

Inserting the horizontal velocity (7.6) and using the relation

$$\sin^2(x) = \frac{1-\cos 2x}{2}$$

we get

$$\dot s(t) = c\left\{1-\frac{1}{2\gamma^2}\left[1+\frac{\beta^2K^2}{2}\left(1-\cos(2\omega_u t)\right)\right]\right\}$$

This can be written in the form

$$\dot s(t) = \langle\dot s\rangle + \Delta\dot s(t)$$

with the average velocity

$$\langle\dot s\rangle = c\left\{1-\frac{1}{2\gamma^2}\left[1+\frac{\beta^2K^2}{2}\right]\right\} \qquad (7.7)$$

and the oscillation

$$\Delta\dot s(t) = \frac{c\beta^2K^2}{4\gamma^2}\cos(2\omega_u t)$$

From (8.7) we derive the relative velocity with $\beta = 1$

$$\beta^* = \frac{\langle\dot s\rangle}{c} = 1-\frac{1}{2\gamma^2}\left[1+\frac{K^2}{2}\right] \qquad (7.8)$$

With (8.6) and (8.7) to (8.8) we get

$$\dot x(t) = \beta c\frac{K}{\gamma}\sin(\omega_u t) \qquad \dot s(t) = \beta^* c + \frac{c\beta^2K^2}{4\gamma^2}\cos(2\omega_u t)$$

Using $\omega_u = k_u\beta c$ and $\beta = 1$ one can evaluate the velocity simply by integration. In the laboratory frame we have

$$x(t) = -\frac{K}{k_u\gamma}\cos(\omega_u t) \qquad s(t) = \beta^* ct + \frac{K^2}{8k_u\gamma^2}\sin(2\omega_u t)$$

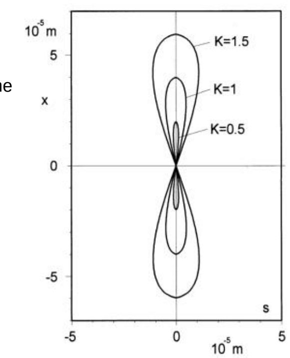
We get an impressive form of motion in the center of mass system K^*, which moves with the velocity β^* with respect to the laboratory sys-tem. With the transformation

$$x^* = x \quad \text{and} \quad s^* = \gamma\left(s-\beta^* ct\right)$$

we get

$$x^*(t) = -\frac{K}{k_u\gamma}\cos(\omega_u t)$$

$$s^*(t) = \frac{K^2}{8k_u\gamma}\sin(2\omega_u t)$$



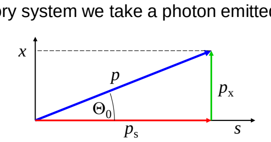
Because of periodic motion in the undulator radiation is emitted in the laboratory frame with a well defined frequency

$$\Omega_w = \frac{2\pi}{T} = \frac{2\pi\beta c}{\lambda_u} = k_u\beta c$$

In the moving frame with the average velocity β^* the frequency is transformed according to

$$\omega^* = \gamma^*\Omega_w \qquad (7.9)$$

The system emits monochromatic radiation. To transform a photon into the laboratory system we take a photon emitted under the angle Θ_0



Energy and momentum of the photon are

$$E = \hbar\omega, \qquad p = \frac{\hbar\omega}{c}$$

and the 4-vector becomes

$$P_\mu = \begin{pmatrix}E/c\\ p_x\\ p_z\\ p_s\end{pmatrix} = \begin{pmatrix}E/c\\ p\sin\Theta_0\\ 0\\ p\cos\Theta_0\end{pmatrix}$$

Transformation into the System K^* is then

$$\begin{pmatrix}E^*/c\\ p_x^*\\ p_z^*\\ p_s^*\end{pmatrix} = \begin{pmatrix}\gamma^* & 0 & 0 & -\beta^*\gamma^*\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\beta^*\gamma^* & 0 & 0 & \gamma^*\end{pmatrix} \cdot \begin{pmatrix}E/c\\ p\sin\Theta_0\\ 0\\ p\cos\Theta_0\end{pmatrix}$$

The energy of the photon becomes

$$\frac{E^*}{c} = \gamma^* \frac{E}{c} - \beta^*\gamma^* p\cos\Theta_0 = \gamma^* \frac{\hbar\omega_w}{c}\left(1-\beta^*\cos\Theta_0\right)$$

With $E^* = \hbar\omega^*$ we get

$$\frac{\hbar\omega^*}{c} = \gamma^* \frac{\hbar\omega_w}{c}\left(1-\beta^*\cos\Theta_0\right)$$

and

$$\omega_w = \frac{\omega^*}{\gamma^*\left(1-\beta^*\cos\Theta_0\right)}$$

Using (8.9) we can write

$$\omega_w = \frac{\Omega_w}{1-\beta^*\cos\Theta_0}$$

and find

$$\frac{\omega_w}{\Omega_w} = \frac{\lambda_u}{\lambda_w} = \frac{1}{1-\beta^*\cos\Theta_0} \qquad (7.10)$$

with

$$\lambda_w = \lambda_u\left(1-\beta^*\cos\Theta_0\right)$$

Now we replace β^* by (7.8) and expand

$$\cos\Theta_0 \approx 1-\frac{\Theta_0^2}{2} \quad \text{since} \quad \Theta_0 \approx \frac{1}{\gamma} \ll 1$$

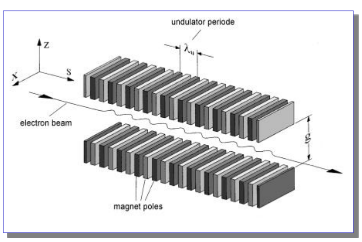
After this manipulations we find

$$\lambda_u\left(1-\beta^*\cos\Theta_0\right) = \lambda_u\left[1-\left(1-\frac{1+K^2/2}{2\gamma^2}\right)\left(1-\frac{\Theta_0^2}{2}\right)\right]$$

$$= \lambda_u\left[1-\left(1-\frac{\Theta_0^2}{2}-\frac{1+K^2/2}{2\gamma^2}\right)+\ldots\right] \approx \lambda_u\left(\frac{\Theta_0^2}{2}+\frac{1+K^2/2}{2\gamma^2}\right)$$

Using equation (7.10) we get the important "coherence condition for undulator radiation"

$$\lambda_w = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}+\gamma^2\Theta_0^2\right) \qquad (7.11)$$



The undulator radiation

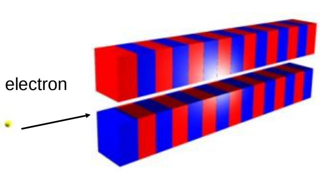
Periode length: $\lambda_u = 0.25$ m

Simply we expect a radiation with a wavelength $\lambda_u = 0.25$ m.

But: actually the radiation from the magnet is **blue light** !

In the laboratory frame the magnet has the periode length $\lambda_u = 0.25$ m.

electron



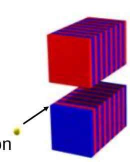
The electron has the energy $E = 450$ MeV, i.e..

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = 880$$

In the electron system the undulator appeares shorter:

$$\lambda' = \frac{\lambda_u}{\gamma} = 2.84\cdot 10^{-4}\,\text{m}$$

Undulator seen by the fast moving electron



electron

In the frame of the electron a wave with the wavelength λ' is generated.

But we can only observe the radiation in the laboratory frame. It is therefore again shortened by a factor $1/\gamma$. The resulting wavelength is finally

$$\lambda_{\text{Undulator}} = \frac{\lambda'}{\gamma} = \frac{\lambda_u}{\gamma^2} = 323\,\text{nm}$$

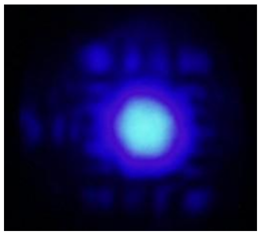
The wavelength is shortened by the factor $\gamma^2 = 774400$!

The exact calculation gives the important coherence condition:

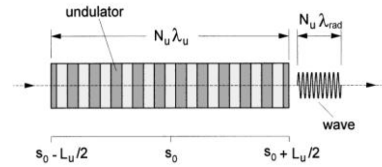
$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$$

With $K = 2$ we get the exact wavelength $\lambda = 480$ nm

This is blue light !



The wavelength of the radiation is mainly determined by , γ, and K.
With increasing angle Θ_0 also the wavelength increases.



The total length of the undulator is $L_u = N_u\lambda_u$

If s_0 marks the center of the undulator, the emitted wave has the time dependent function

$$u(\omega_w, t) = \begin{cases} a\exp i\omega_w t & \text{if } -\frac{T}{2} \le t \le \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \qquad (7.12)$$

The wave has the duration

$$T = N_u\lambda_w/c \quad \Rightarrow \quad \omega_w T = 2\pi N_u \qquad (7.13)$$

Such limited wave generates a continuous spectrum of partial waves. Their amplitudes are given by the Fourier integral

$$A(\omega) = \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{+\infty} u(\omega_w, t)\exp(-i\omega t)dt$$

Insertion into (8.12) gives

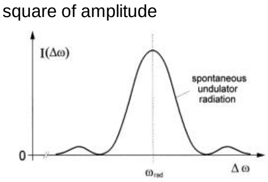
$$A(\omega) = \frac{a}{\sqrt{2\pi T}}\int_{-T/2}^{+T/2}\exp[-i(\omega-\omega_w)t]dt = \frac{2a}{\sqrt{2\pi T}}\,\frac{\sin(\omega-\omega_w)T}{2(\omega-\omega_w)}$$

With $\Delta\omega = \omega - \omega_w$ and (7.13) we get

$$A(\omega) = \frac{a}{\sqrt{2\pi}}\sin\left(\pi N_u\frac{\Delta\omega}{\omega_w}\right)\Big/\pi N_u\frac{\Delta\omega}{\omega_w}$$

The intensity is proportional to the square of amplitude

$$I(\Delta\omega) \propto \left[\frac{\sin\left(\pi N_u\frac{\Delta\omega}{\omega_w}\right)}{\pi N_u\frac{\Delta\omega}{\omega_w}}\right]^2$$



We get the half width of maximum from

$$\left(\frac{\sin x}{x}\right)^2 = \frac{1}{2} \quad \text{with} \quad x = \pi N_u\frac{\Delta\omega}{\omega_w} = 1.392$$

and find

$$\frac{2\Delta\omega}{\omega_w} = \frac{2x}{\pi N_u} = \frac{0.886}{N_u} \approx \frac{1}{N_u}$$

i.e. an undulator with $N_u = 100$ periods gives a line width of $\approx 1\%$.

The spectrum of an undulator is