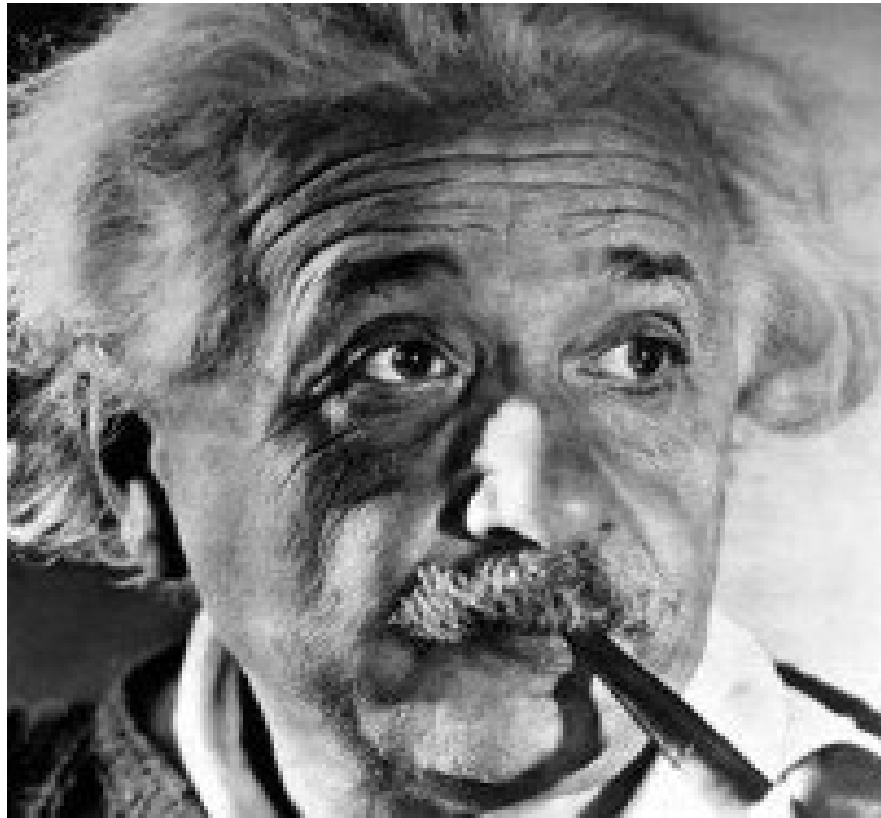


Review of Special Relativity



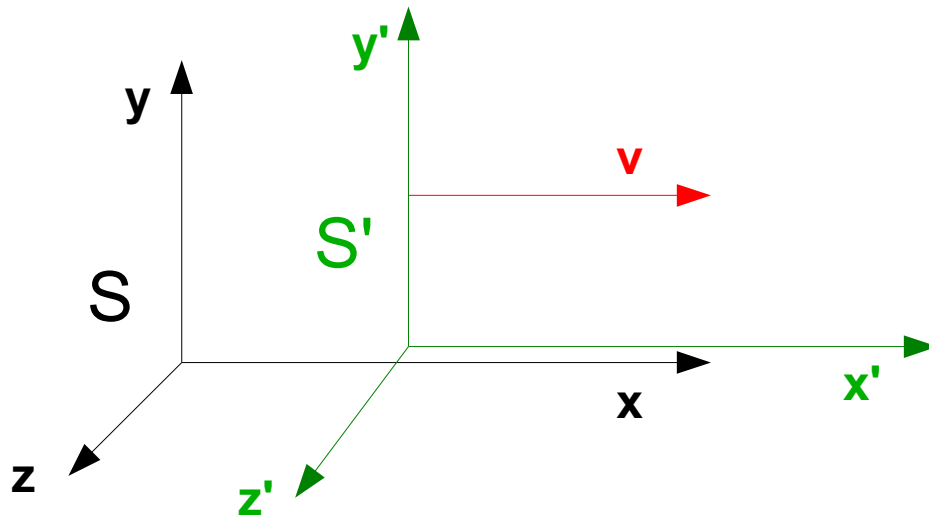
This review is not meant to teach the subject, but to repeat and to refresh, at least partially, what you have learnt at university.

Why was „Special Relativity“ needed?

Mechanical laws (Newton's laws) are the same for all inertial systems.

They are invariant under a Galilean transformation (G-T):

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$



Example: Man walking in train, observer at rest.

El.-mag. laws are not invariant under a G-T.

Take the wave equation $\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \Phi = 0$

it transforms to

$$\left[\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} - \frac{v^2}{c^2} \frac{\partial^2}{\partial x'^2} - 2 \frac{v}{c^2} \frac{\partial^2}{\partial x' \partial t'} \right] \Phi = 0$$

Moreover, it contains the speed of light as a constant, independent of a reference system.

This contradicted the deep belief in a supporting media (ether) for the waves.

The ether would be the absolute reference system.

Many experiments tried to prove el.-mag. theory wrong.

They all failed!

Michelson-Morley interferometer experiment (1887)
showed that c is a constant and that there exists no „ether“.

→ The Newton-Galileo concept of space and time
had to be modified

Relativistic Kinematics

Einstein based his theory on two postulates:

1. All inertial frames are equivalent w.r.t. all laws of physics.
2. The speed of light is equal in all reference frames.

Consequence of 1st postulate:

Space is isotropic (all directions are equivalent)

Space is homogeneous (all points are equivalent)

Lorentz Transformation

Homogeneity of space and form-invariance of laws under transformation require a linear transformation.

$$\begin{aligned}
 ct' &= a_{00} ct + a_{01} x + a_{02} y + a_{03} z \\
 x' &= a_{10} ct + a_{11} x + a_{12} y + a_{13} z \\
 y' &= a_{20} ct + a_{21} x + a_{22} y + a_{23} z \\
 z' &= a_{30} ct + a_{31} x + a_{32} y + a_{33} z
 \end{aligned}$$

Successive use of homogeneity, isotropy and the speed of light determines the constants, e.g.

$$a_{02} = a_{03} = 0 \quad \text{events } y = \pm y_0, z = \pm z_0 \text{ at same time } t'$$

$$a_{20} = a_{30} = 0 \quad \text{origin stays on } x\text{-axis}$$

$$a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$$

$$(x-, y-, z\text{-axis}) \parallel (x'-, x'-, z'\text{-axis})$$

etc.

Result is the **Lorentz-Transformation (L-T)** :

$$\begin{aligned} ct' &= \gamma (ct - \beta x) & y' &= y \\ x' &= \gamma (x - \beta ct) & z' &= z \end{aligned}$$

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

Inverse transformation:

Replace primed variables by unprimed,
unprimed by primed
 β by $-\beta$

$$\begin{aligned} ct &= \gamma (ct' + \beta x') & y &= y' \\ x &= \gamma (x' + \beta ct') & z &= z' \end{aligned}$$

We write the L-T as

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

L-T is an affine transformation. It preserves the rectilinearity and parallelism of straight lines.

$$\underline{L} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \underline{L}^{-1} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \underline{L}\underline{L}^{-1} = \underline{1}$$

Time dilatation:

Two events in S' at t_1' , t_2' and at location $x_1' = x_2'$

$$c(t_2 - t_1) = \gamma c(t_2' - t_1') + \beta(x_2' - x_1')$$

$$\rightarrow \Delta t = \gamma \Delta t'$$

Two events in S at t_1 , t_2 and at location $x_1 = x_2$

$$c(t_2' - t_1') = \gamma c(t_2 - t_1) - \beta(x_2 - x_1)$$

$$\rightarrow \Delta t' = \gamma \Delta t$$

Length contraction:

A meter in S' from x_1' to x_2' measured in S at $t_1 = t_2$

$$x_2' - x_1' = \gamma (x_2 - x_1) - \beta \gamma c (t_2 - t_1)$$

$$\rightarrow \Delta x = \frac{1}{\gamma} \Delta x'$$

A meter in S from x_1 to x_2 measured in S' at $t_1' = t_2'$

$$x_2 - x_1 = \gamma (x_2' - x_1') + \beta \gamma c (t_2' - t_1')$$

$$\rightarrow \Delta x' = \frac{1}{\gamma} \Delta x$$

Time intervals and distances depend on the motion of the observer.

$$\Delta t \leftarrow \gamma \Delta t' \quad \text{and} \quad \Delta x \leftarrow \frac{1}{\gamma} \Delta x'$$

are not standard equations !!

Perpendicular dimensions remain: $\Delta y = \Delta y'$, $\Delta z = \Delta z'$

Experiment length contraction: Michelson-Morley

Experiment time dilatation: Muons created in upper atmosphere

$$T_{1/2} = 1.5 \mu s, \quad v = 0.994c \quad \rightarrow \quad l = 450 \text{m}$$

$$\gamma = 9, \quad T_{1/2} = 14 \mu s \quad \rightarrow \quad l = 4 \text{km}$$

The space-time interval is invariant under L-T

$$ds^2 = (cdt)^2 - dx^2 - dy^2 - dz^2 = ds'^2$$

We write the space-time interval as

$$\begin{aligned} ds &= cdt \sqrt{1 - \frac{1}{c^2} \left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} \right)} = c dt \sqrt{1 - \left(\frac{v}{c} \right)^2} = \\ &= c \frac{dt}{\gamma} = c d\tau \end{aligned}$$

and identify $d\tau$ as the time interval a particle moving with v would measure. τ is called **proper time** and is Lorentz invariant.

4-Vectors

Let us define **contra- and covariant 4-vectors**

$$\text{contravariant } X^\mu = (X^0, X^1, X^2, X^3) = (ct, x, y, z)$$

$$\text{covariant } X_\mu = (X_0, X_1, X_2, X_3) = (ct, -x, -y, -z)$$

The scalar product is the L-T invariant space-time interval

$$\begin{aligned} X^\mu X_\mu &= X^0 X_0 + X^1 X_1 + X^2 X_2 + X^3 X_3 = \\ &= (ct)^2 - x^2 - y^2 - z^2 = X'^\mu X'_\mu \end{aligned}$$

In general:

- Any quadruple which transforms like (ct, x, y, z) is a 4-vector.
- The scalar product of two 4-vectors is Lorentz invariant.

$$A^\mu B_\mu = A'^\mu B'_\mu$$

Transformation of velocity

A particle moving with velocity u' in S' has velocity u in S

$$u_x = \frac{dx}{dt} = \frac{dx}{dt'} \frac{dt'}{dt} = \gamma \left(\frac{dx'}{dt'} + \beta c \right) \frac{dt'}{dt} = \gamma (u'_x + v) \frac{dt'}{dt}$$

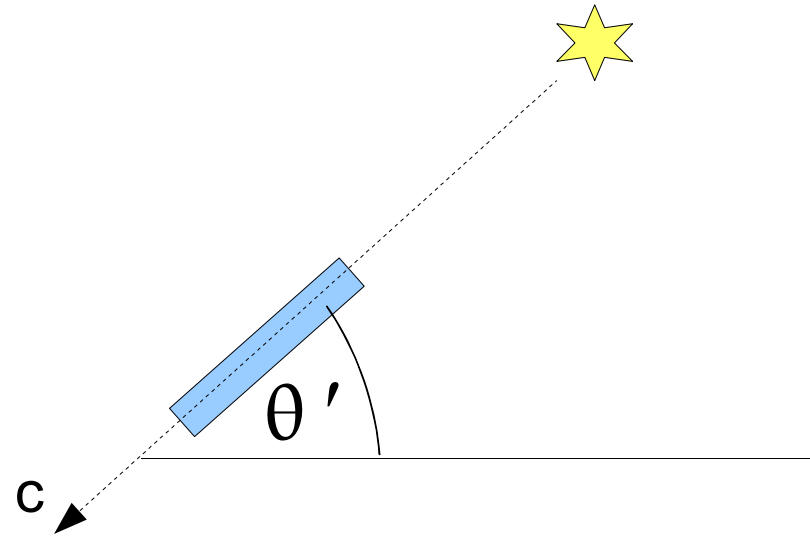
$$\frac{dt}{dt'} = \gamma \left(1 + \frac{\beta}{c} \frac{dx'}{dt'} \right) = \gamma \left(1 + \frac{v}{c^2} u'_x \right)$$

$$u_x = \frac{u'_x + v}{1 + vu'_x/c^2}, \quad u_y = \frac{u'_y}{\gamma(1 + vu'_x/c^2)}, \quad u_z = \frac{u'_z}{\gamma(1 + vu'_x/c^2)}$$

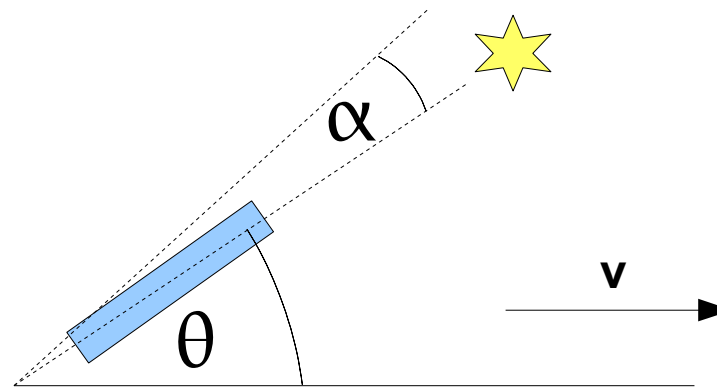
Example: Light aberration

A star appears on earth under an angle different than its real position.

Earth at rest:



Earth moving with v :



In rest system of earth the star moves with $-v$ and u' is c , then

$$u'_x = -c \cos(\vartheta'), \quad u'_y = -c \sin(\vartheta')$$
$$\frac{u_x}{c} = \frac{-\cos(\vartheta') + \beta}{1 + \beta \cos(\vartheta')}, \quad \frac{u_y}{c} = -\frac{\sin(\vartheta')}{\gamma(1 + \beta \cos(\vartheta'))}$$

$$\tan(\vartheta) = \frac{u_x}{u_y} = \frac{\sin(\vartheta')}{\gamma(\cos(\vartheta') + \beta)}$$

with

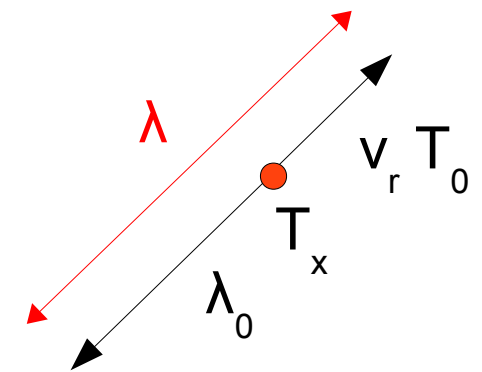
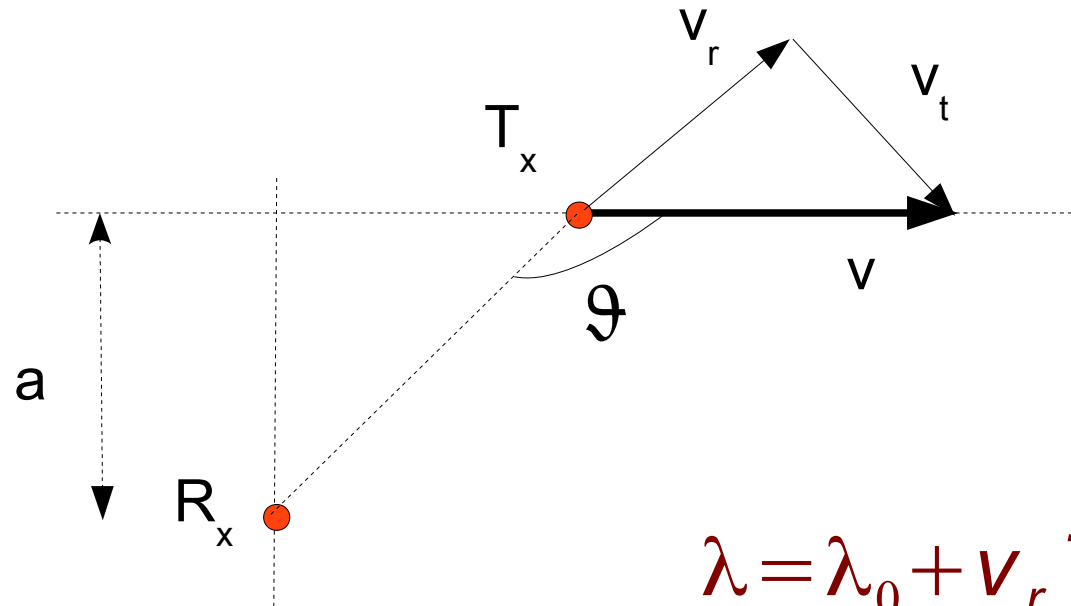
$$\tan\left(\frac{\vartheta}{2}\right) = \frac{\sin(\vartheta)}{1 + \cos(\vartheta)}$$

we get

$$\tan\left(\frac{\vartheta}{2}\right) = \sqrt{\frac{1 - \beta}{1 + \beta}} \tan\left(\frac{\vartheta'}{2}\right)$$

Example: Doppler effect

$$v_r = v \cos(\pi - \vartheta)$$



$$\lambda = \lambda_0 + v_r T_0$$

$$f_0 = 1/T_0 \rightarrow f_0/\gamma$$

$$\frac{f}{f_0/\gamma} = \frac{\lambda_0}{\lambda} \rightarrow \frac{f}{f_0} = \frac{\sqrt{1-\beta^2}}{1-\beta \cos(\vartheta)}$$

$$\vartheta \rightarrow \pi: \quad \frac{f}{f_0} \rightarrow \sqrt{\frac{1-\beta}{1+\beta}} \quad \textit{longitudinal Doppler effect}$$

$$\vartheta = \frac{\pi}{2}: \quad \frac{f}{f_0} = \sqrt{1-\beta^2} \quad \textit{transverse Doppler effect}$$

Astronomy: Transverse Doppler « longitudinal Doppler

$$\beta = \frac{1 - (\lambda_0/\lambda)^2}{1 + (\lambda_0/\lambda)^2}$$

Transformation of acceleration

A particle moving with u' in S' and experiencing an acceleration a' has a in S .

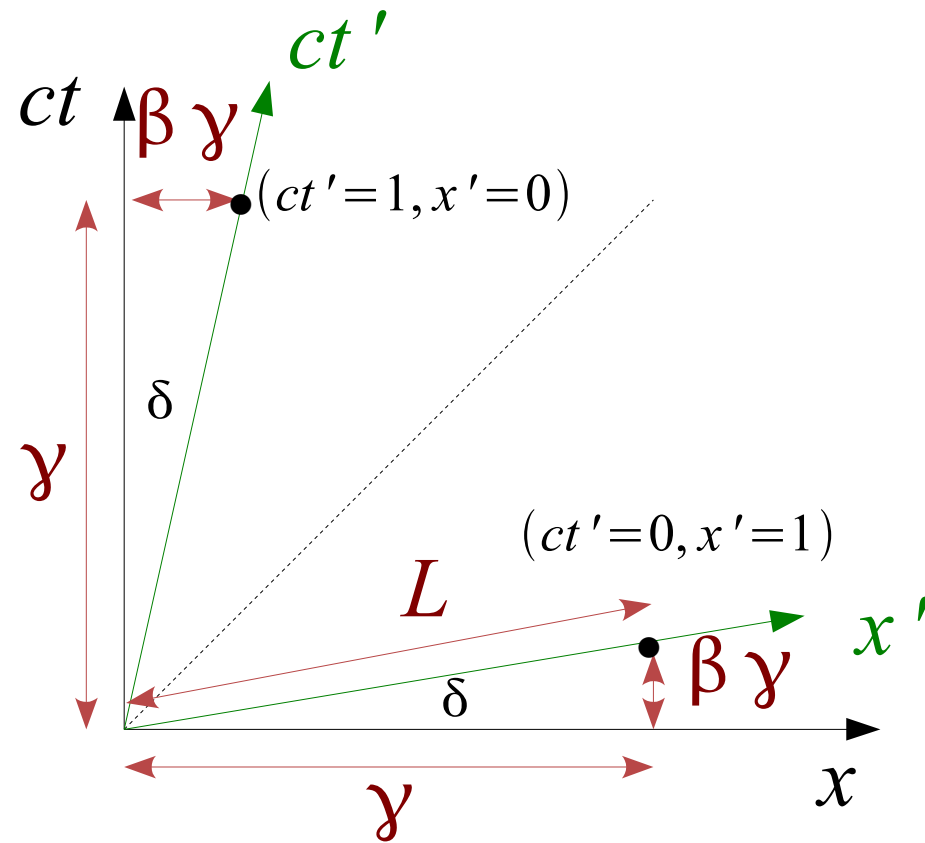
$$\begin{aligned} a_x &= \frac{du_x}{dt} = \frac{du_x}{dt'} \frac{dt'}{dt} = \frac{d}{dt'} \frac{u'_x + v}{1 + vu'_x/c^2} \frac{dt'}{dt} = \\ &= \frac{a'_x}{\gamma^3 (1 + vu'_x/c^2)^3} \\ a_y &= \frac{a'_y}{\gamma^2 (1 + vu'_x/c^2)^2} - \frac{(vu'_y/c^2) a'_x}{\gamma^2 (1 + vu'_x/c^2)^3} \\ a_z &= \frac{a'_z}{\gamma^2 (1 + vu'_x/c^2)^2} - \frac{(vu'_z/c^2) a'_x}{\gamma^2 (1 + vu'_x/c^2)^3} \end{aligned}$$

Acceleration in an inertial system is possible !!

Minkowski diagram

$$ct = \gamma(ct' + \beta x')$$

$$x = \gamma(x' + \beta ct')$$

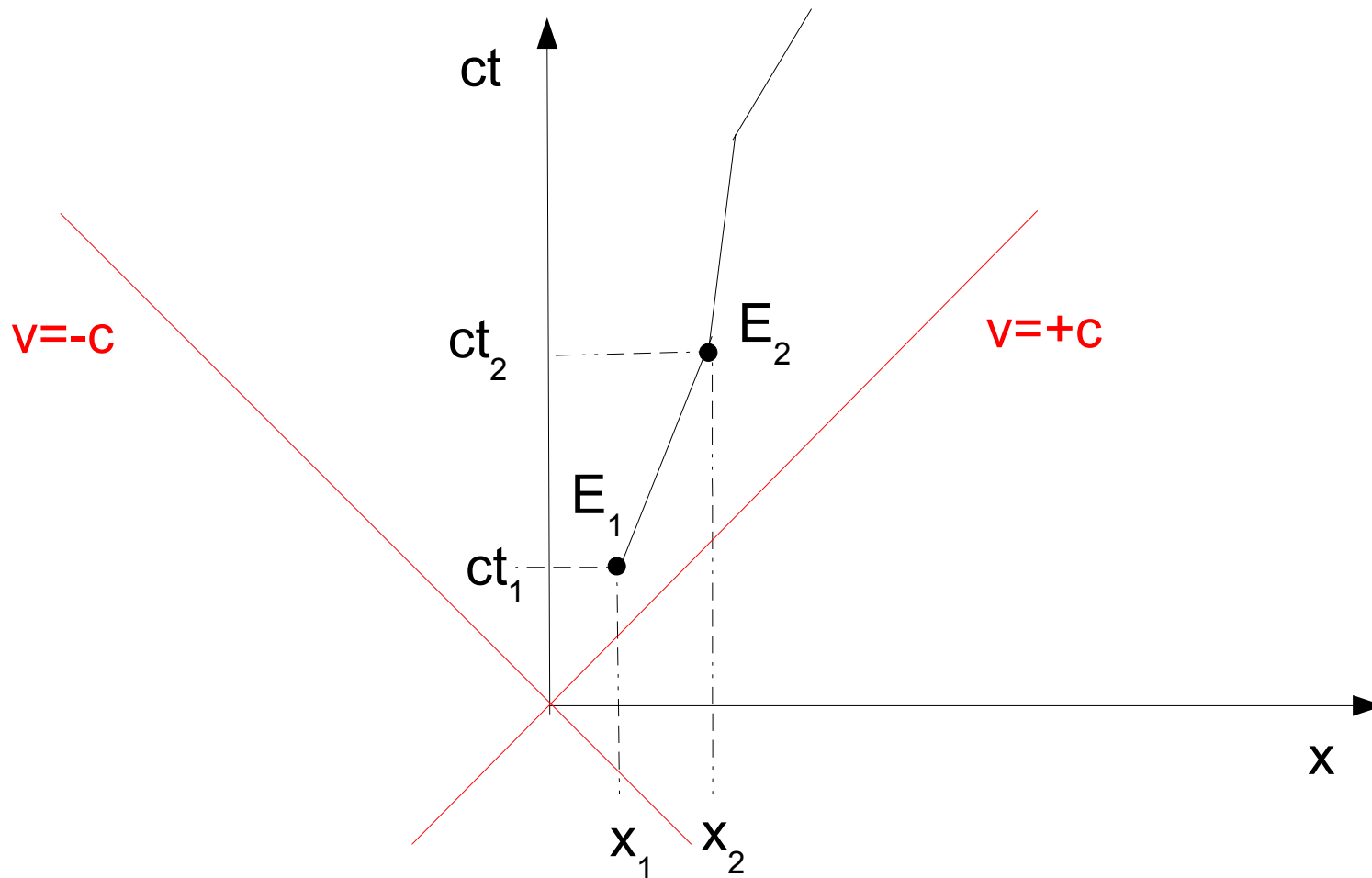


$$\tan(\delta) = \beta$$

Scale in S' :

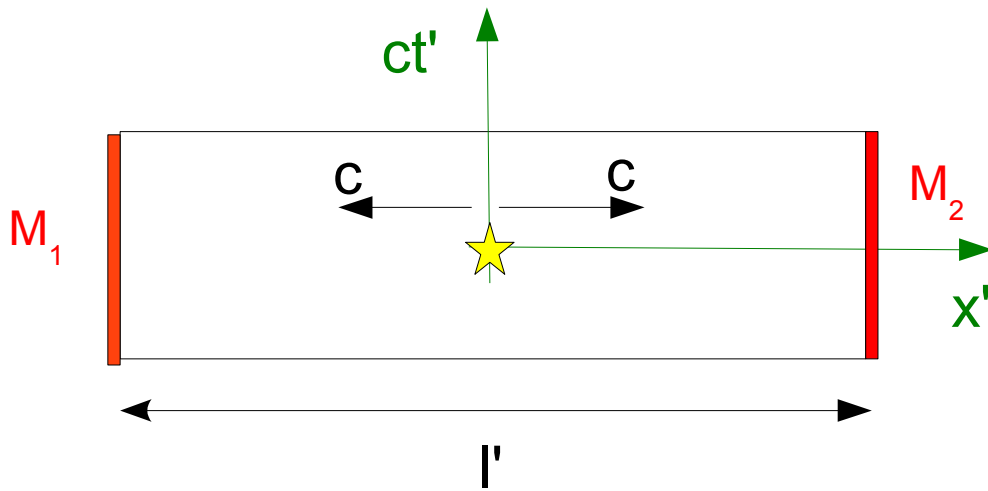
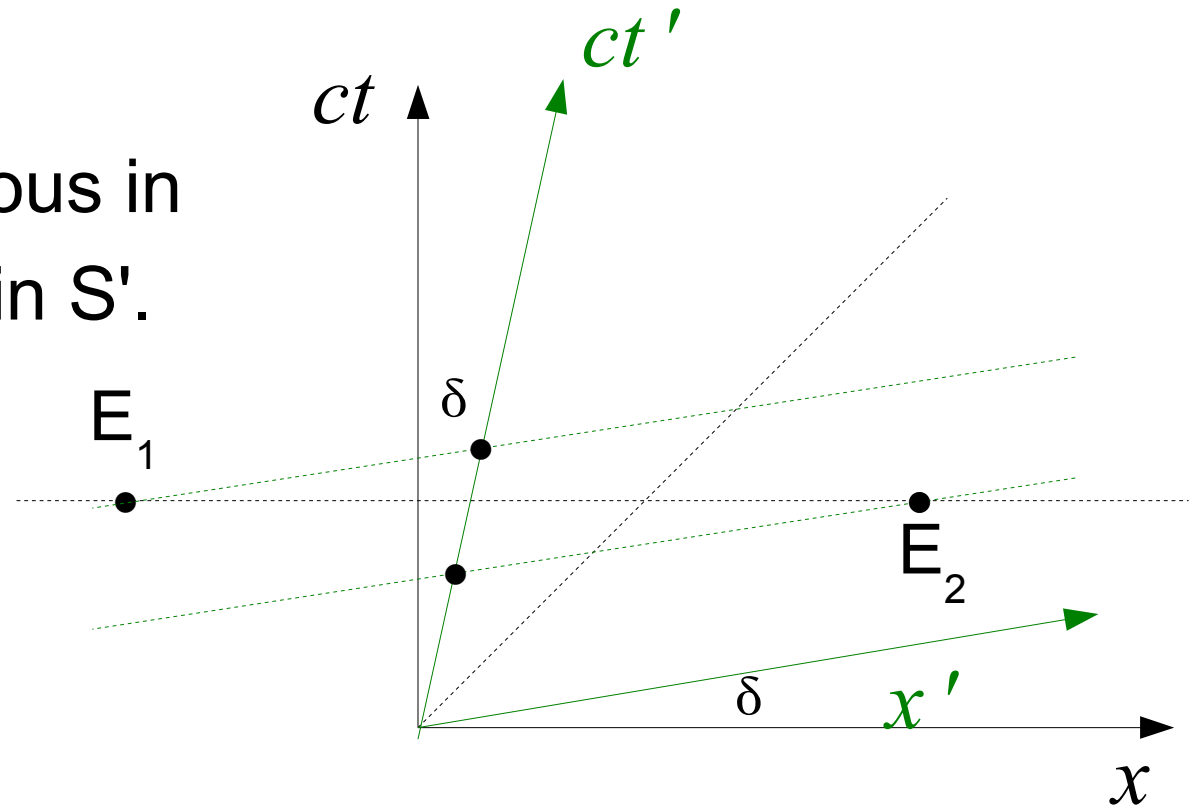
$$L = \sqrt{\gamma^2 + \beta^2 \gamma^2} = \sqrt{\frac{1 + \beta^2}{1 - \beta^2}}$$

World-line (path-time diagram)



Simultaneity

Events E_1, E_2 simultaneous in S are not simultaneous in S' .

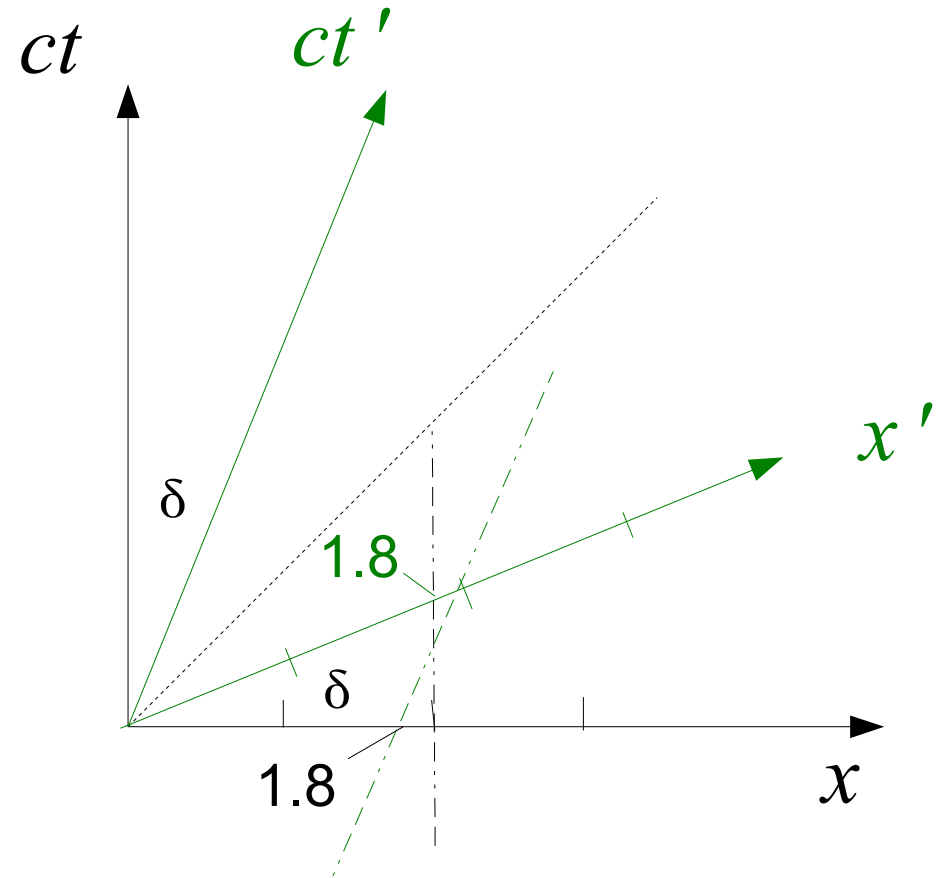
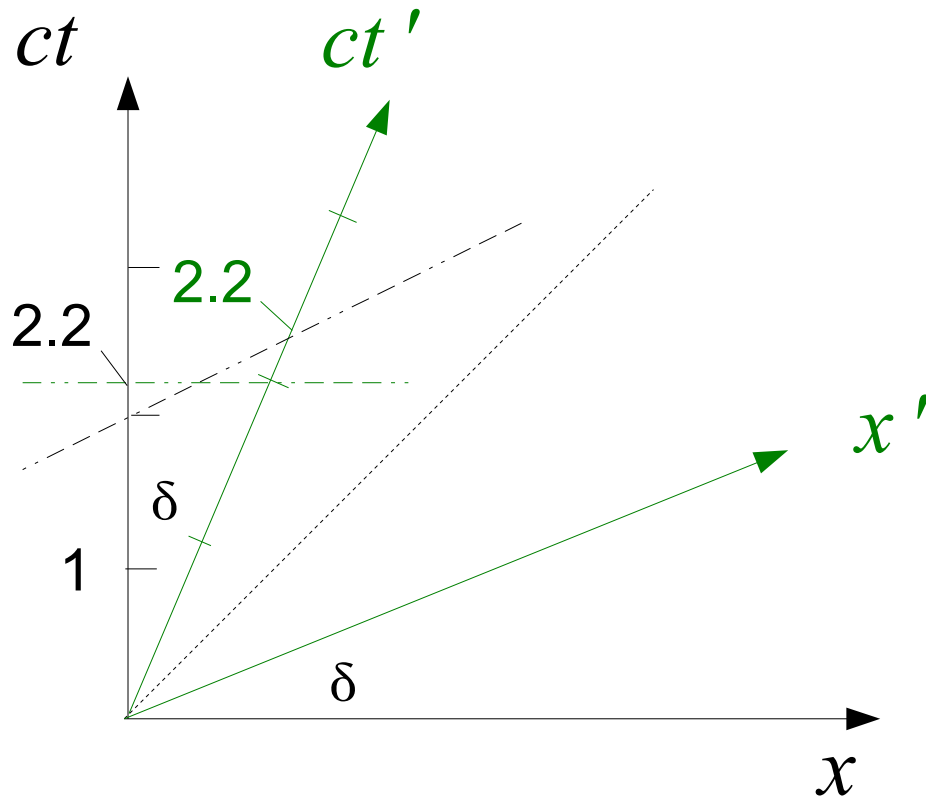


$$ct_1 = \frac{l}{2} - vt_1, \quad ct_2 = \frac{l}{2} + vt_2$$

$$t_2 - t_1 = \gamma^2 \frac{\beta}{c} l = \gamma \frac{\beta}{c} l'$$

Time dilatation:
 $\beta=0.42$, $\delta=22.8^\circ$, $\gamma=1.1$, $L=1.2$

Length contraction:
 $\beta=0.42$, $\delta=22.8^\circ$, $\gamma=1.1$, $L=1.2$



Relativistic Dynamics

Based on two principles:

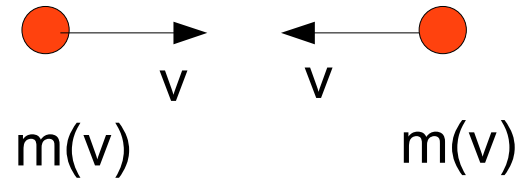
1. Conservation of linear momentum ($p=mu$)
2. Conservation of energy ($E=mc^2$)

Derivation of moving mass

Because of $E=mc^2$ we choose as ansatz $m=m(v)$

Inelastic collision between 2 identical particles:

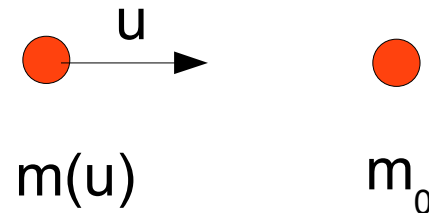
Laboratory frame



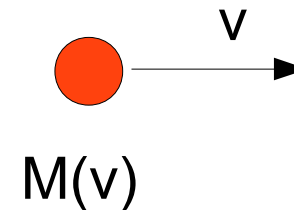
Composite particle at rest after collision



Rest frame of right particle:



Composite particle moving after collision (center of mass frame $\Sigma p=0$)



Conservation of momentum $m(u)u = M(v)v$ (1)

$$u_x = \frac{u'_x + v}{1 + vu'_x/c^2} \rightarrow u = \frac{2v}{1 + (v/c)^2} \quad (2)$$

Conservation of energy $m(u)c^2 + m_0c^2 = M(v)c^2$ (3)

From (1), (2), (3) after eliminating M

$$m(u) = \frac{m_0}{\sqrt{1 - (u/c)^2}} = \gamma_u m_0 \quad (4)$$

Mass

$$m(u) = \gamma_u m_0$$

From (1), (2), (4)

$$M(v) = \frac{2m_0}{1 - (v/c)^2} = \gamma M_0, \quad M_0 = 2\gamma m_0$$

$$M_0 - 2m_0 = 2m_0(\gamma - 1) > 0$$

$$M_0 c^2 - 2m_0 c^2 = 2(\gamma m_0 c^2 - m_0 c^2) = 2(E - E_0) = 2E_{kin}$$

rest mass is not conserved, E_{kin} completely converted in mass.

Momentum $\vec{p}(u) = m(u)\vec{u}$

Force

$$\vec{f} = \frac{d\vec{p}}{dt} = m_0 \frac{d\gamma_u}{dt} \vec{u} + m_0 \gamma_u \frac{d\vec{u}}{dt} = \gamma_u^3 \frac{m_0}{c^2} (\vec{u} \cdot \vec{a}) \vec{u} + \gamma_u m_0 \vec{a}$$

\vec{f} and \vec{a} are not parallel!

Using $\vec{f} \cdot \vec{u} = \gamma_u^3 m_0 (\vec{u} \cdot \vec{a})$ one can solve for \vec{a}

$$\vec{a} = \frac{1}{\gamma_u m_0} \left(\vec{f} - \frac{1}{c^2} (\vec{f} \cdot \vec{u}) \vec{u} \right)$$

For $\vec{u} = (u, 0, 0)$:

$$\vec{f} = (\gamma^3 m_0 a_x, \gamma m_0 a_y, \gamma m_0 a_z)$$

longitudinal mass ($\parallel \vec{u}$) $\gamma^3 m_0$

transverse mass ($\perp \vec{u}$) γm_0

Energy

A particle moves with $\vec{u} = (u, 0, 0)$ and experiences a force f_x .

Work done at path dx is

$$dE_{kin} = f_x dx = \gamma_u^3 m_0 a_x dx = \gamma_u^3 m_0 u du$$

$$E_{kin} = m_0 c^2 \int_0^{\beta_u} \frac{\beta_u d\beta_u}{(1 - \beta_u^2)^{3/2}} = \gamma_u m_0 c^2 - m_0 c^2 = E - E_0$$

$$P = \frac{dE_{kin}}{dt} = \frac{dm}{dt} c^2$$

$$\vec{f} = \frac{d\vec{p}}{dt} = \frac{dm}{dt} \vec{u} + m \frac{d\vec{u}}{dt} = \frac{1}{c^2} \frac{dE_{kin}}{dt} \vec{u} + \gamma_u m_0 \vec{a}$$

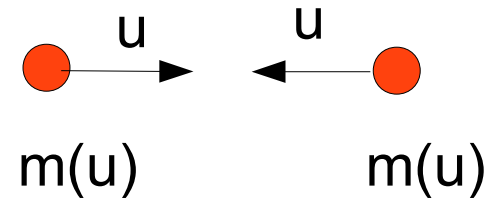
$$\gamma_0 m_0 \vec{a} = \vec{f} - \frac{1}{c^2} (\vec{f} \cdot \vec{u}) \vec{u}$$

$$\rightarrow P = \frac{dE_{kin}}{dt} = \vec{f} \cdot \vec{u}$$

The temporal change of E_{kin} of a body , or the power it absorbs , is the scalar product of \vec{f} and \vec{u} .

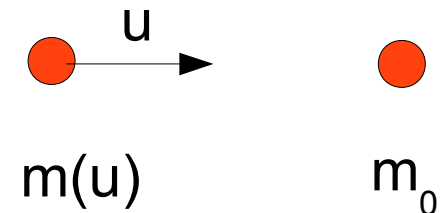
Example: Collider

1) 3.5 TeV head-on p-p collider

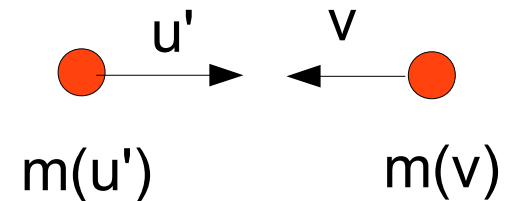


$$M_0 = 2 \gamma_u m_0 \rightarrow E_{CM} = M_0 c^2 = 2 E = 7 \text{ TeV}$$

2) 3.5 TeV fixed target p-machine



Center of mass frame ($\Sigma p=0$)
moves with +v



$$u' = \frac{u-v}{1-uv/c^2} \rightarrow \gamma_{u'} = \gamma_u \gamma_v (1 - \beta_u \beta_v)$$

$$\Sigma p = \gamma_{u'} m_0 u' - \gamma_v m_0 v = 0 \rightarrow \beta_v = \frac{\beta_u}{1 + \sqrt{1 - \beta_u^2}}$$

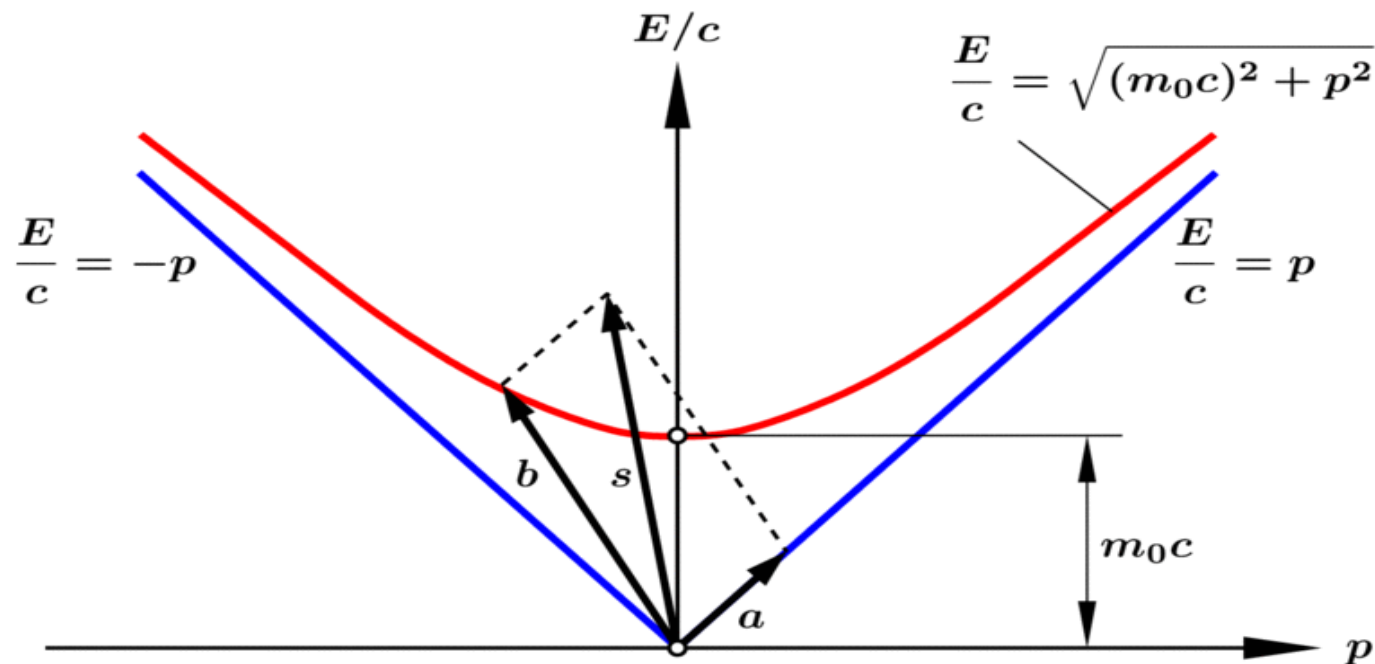
$$\gamma_v = \gamma_{u'} = \sqrt{(1 + \gamma_u)/2}$$

$$\begin{aligned} E_{CM} &= \gamma_{u'} m_0 c^2 + \gamma_v m_0 c^2 = \sqrt{2(1 + \gamma_u)} E_0 = \\ &= \sqrt{2(E_0 + E)} E_0 = 81 \text{ GeV} \quad (E_{p0} = 938 \text{ MeV}) \end{aligned}$$

Energy-momentum equation and diagram

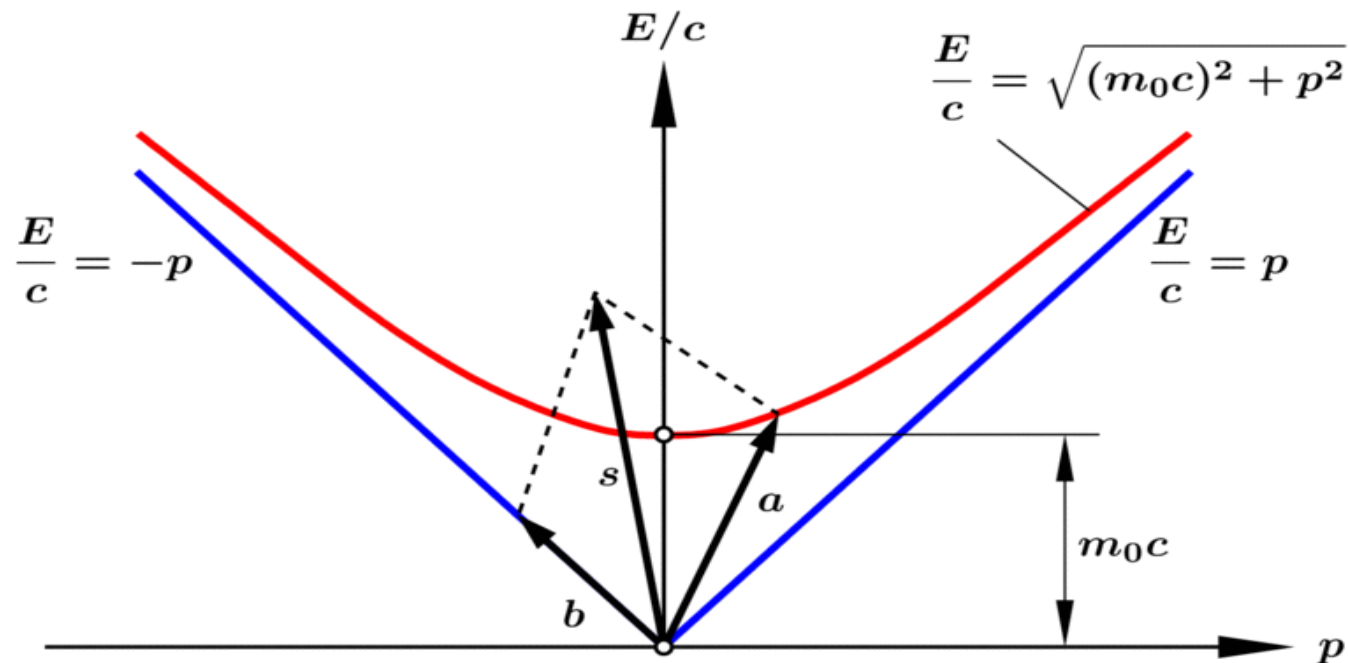
$$E^2 = (mc^2)^2 = (m_0 c^2)^2 \frac{(1 - \beta^2) + \beta^2}{1 - \beta^2} = E_0^2 + (pc)^2$$

$$\frac{E}{c} = \sqrt{(m_0 c)^2 + p^2}, \quad \text{mass-less particle: } E = |p|c$$

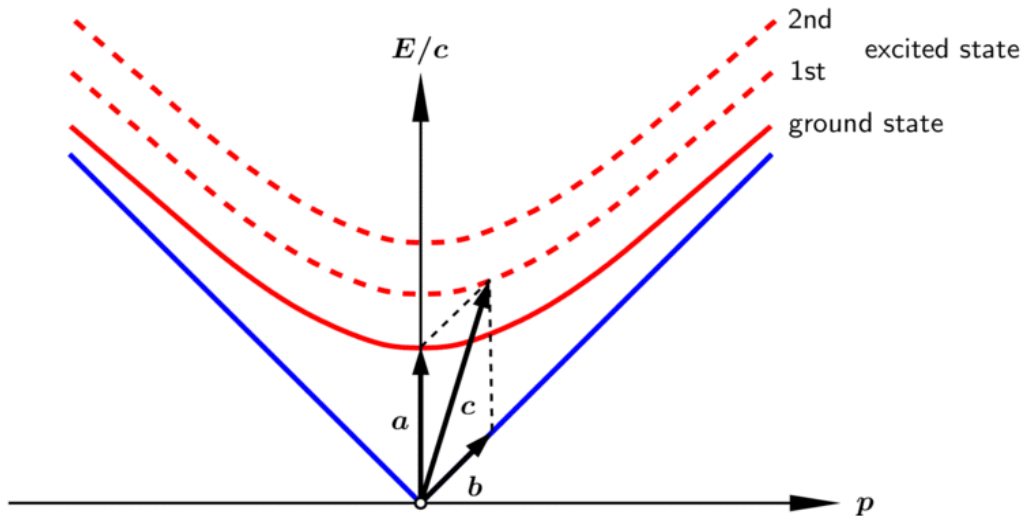


Since conserved quantities are plotted, arrows can be added like vectors.

All interactions are allowed in which energy-momentum vectors c , d after interaction add to vector s .

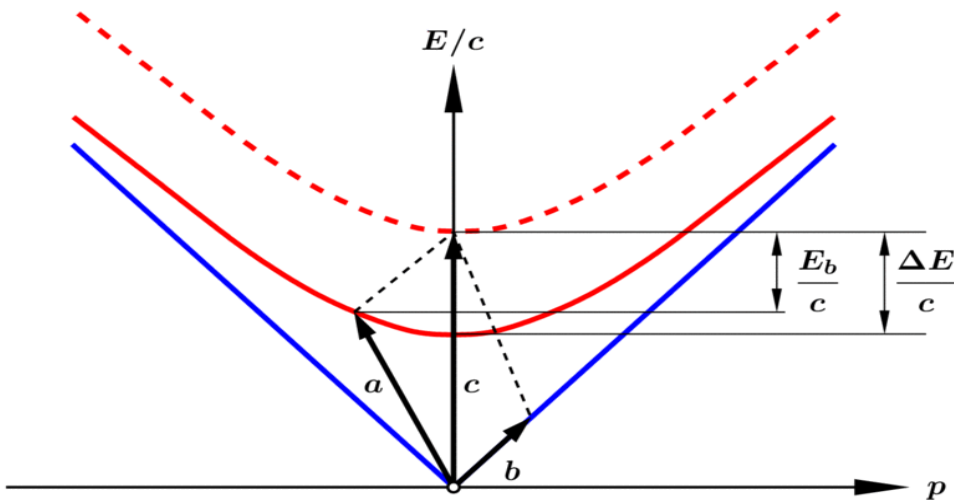


Example: Photon absorption by a particle at rest



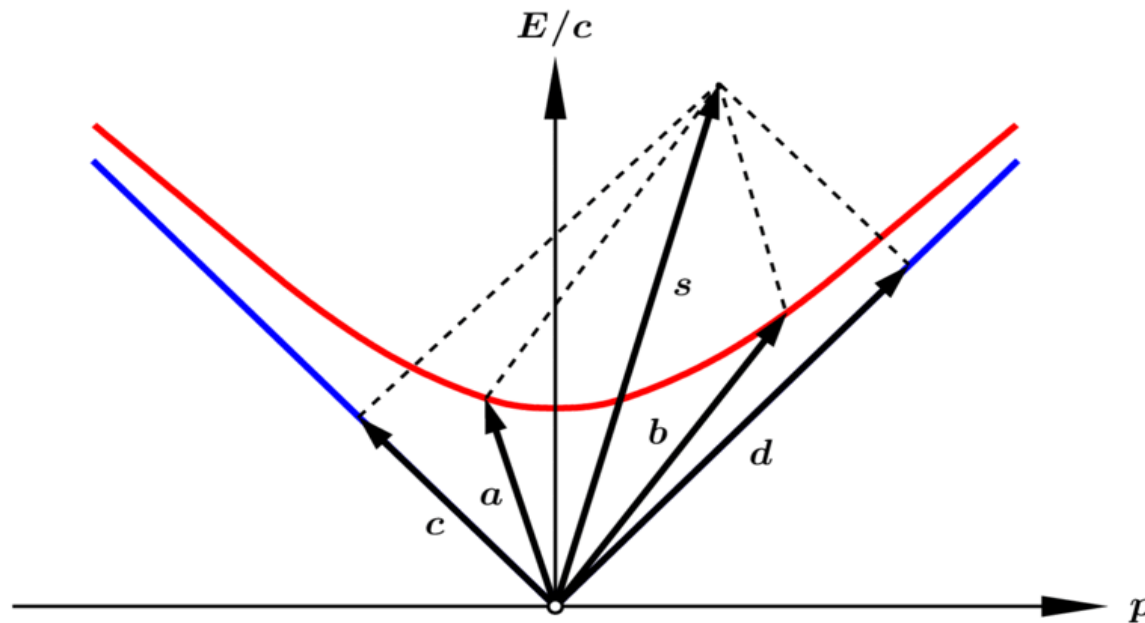
Absorption only for composite particles with excited states.

Example: Photon emission by a composite particle at rest



$\Delta E > E_b$
difference in recoil of particle

Example: Pair annihilation



Exercise 1

An object has a velocity of 30km/s in system S' moving also with 30km/s. What is its velocity in the lab system? Repeat the calculation with 270 000km/s.

Exercise 2

Prove that the scalar product of any two 4-vectors is Lorentz invariant.

Exercise 3

A space craft travels away from earth with $\beta=0.8$. At a distance $d= 2.16 \cdot 10^8$ km a radio signal from earth is transmitted to the space craft.
How long does the signal need in the system of the earth?

Exercise 4

Which mass has to be lifted by 1m in order to provide the energy corresponding to 1mg of mass?

Exercise 5

A charge q is at rest. At $t=0$ an electric field E_x is turned on. Calculate the velocity as a function of time.

L-T of energy and momentum

A particle with m_0 moves in S with $\vec{u}=(u,0,0)$.

$$E = \gamma_u m_0 c^2, \quad p_x = \gamma_u m_0 u = E \frac{u}{c^2}$$

In S' it's velocity, energy and momentum is

$$u' = \frac{u-v}{1-uv/c^2} \rightarrow \gamma_{u'} = \gamma_u \gamma (1 - \beta_u \beta)$$

$$\frac{E'}{c} = \gamma_{u'} m_0 c = \gamma \left(\frac{E}{c} - \beta p_x \right)$$

$$p_x' = \gamma_{u'} m_0 u' = \gamma \left(p_x - \beta \frac{E}{c} \right), \quad p_y' = p_y, \quad p_z' = p_z$$

E/c transforms like ct and \vec{p} transforms like \vec{r}

Derivation of Planck's hypothesis $E=h\nu$

A photon with energy E' in S' travels in $-x'$ direction

$$p_x' = -\frac{E'}{c}$$

$$E = \gamma (E' + v p_x') = \sqrt{\frac{1-\beta}{1+\beta}} E'$$

Frequency Doppler shift

$$\nu = \sqrt{\frac{1-\beta}{1+\beta}} \nu' \quad \rightarrow \quad \frac{E}{\nu} = \frac{E'}{\nu'} = h$$

Position 4-vector

$$X^\mu = (ct, x, y, z)$$

Energy-momentum 4-vector

$$P^\mu = \left(\frac{E}{c}, p_x, p_y, p_z \right)$$

1. Derivation of energy momentum equation:

In a frame where momentum does not vanish:

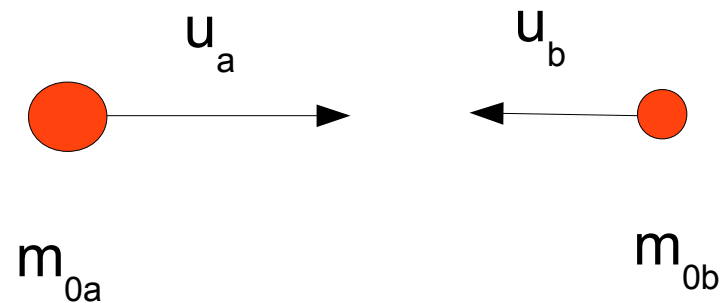
$$P^\mu P_\mu = \left(\frac{E}{c} \right)^2 - p_x^2 - p_y^2 - p_z^2 = \left(\frac{E}{c} \right)^2 - p^2$$

If momentum vanishes: $P'^\mu P'_\mu = \left(\frac{E_0}{c} \right)^2$

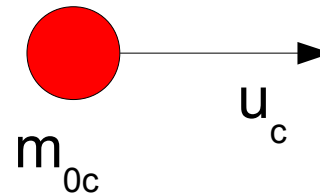
$$P^\mu P_\mu = P'^\mu P'_\mu \quad \rightarrow \quad E^2 = E_0^2 + (pc)^2$$

2. Inelastic collision:

before collision



after collision



$$E_a + E_b = E_c, \quad \vec{p}_a + \vec{p}_b = \vec{p}_c \quad \rightarrow$$

$$P_a^\mu + P_b^\mu = P_c^\mu \quad \cdot (P_{a\mu} + P_{b\mu} = P_{c\mu})$$

$$P_a^\mu P_{a\mu} + 2P_a^\mu P_{b\mu} + P_b^\mu P_{b\mu} = P_c^\mu P_{c\mu} \quad (i)$$

rest frames for a, b, c

$$P_a^\mu P_{a\mu} = (m_{0a} c)^2, \quad P_b^\mu P_{b\mu} = (m_{0b} c)^2, \quad P_c^\mu P_{c\mu} = (m_{0c} c)^2$$

laboratory frame

$$P_a^\mu = (\gamma_a m_{0a} c, \gamma_a m_{0a} u_a, 0, 0)$$

$$P_b^\mu = (\gamma_b m_{0b} c, -\gamma_b m_{0b} u_b, 0, 0)$$

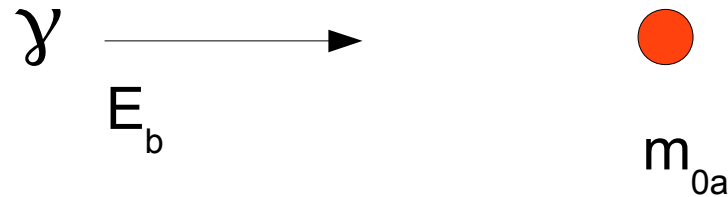
$$2P_a^\mu P_{b\mu} = 2\gamma_a \gamma_b m_{0a} m_{0b} (c^2 + u_a u_b)$$

substituted in (i)

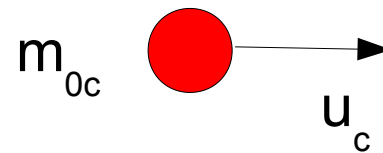
$$m_{0c} = \sqrt{m_{0a}^2 + m_{0b}^2 + 2m_{0a} m_{0b} \gamma_a \gamma_b \left(1 + \frac{u_a u_b}{c^2}\right)} \geq m_{0a} + m_{0b}$$

3. Absorption of a photon by an atom at rest

before absorption



after absorption



$$P_a^\mu P_{a\mu} + 2P_a^\mu P_{b\mu} + P_b^\mu P_{b\mu} = P_c^\mu P_{c\mu} \quad (i)$$

rest frame of (a) before absorption

$$P_a^\mu = (m_{0a} c, 0, 0, 0), \quad P_b^\mu = \left(\frac{E_b}{c}, p_{bx}, 0, 0 \right) = \left(\frac{h\nu}{c}, \frac{h\nu}{c}, 0 \right)$$

rest frame of (c) after absorption

$$P_c^\mu = (m_{0c} c, 0, 0, 0)$$

scalar product for photons

$$P_b^\mu P_{b\mu} = \left(\frac{E}{c}\right)^2 - p_{bx}^2 = \left(\frac{h\nu}{c}\right)^2 - \left(\frac{h\nu}{c}\right)^2 = 0$$

substituted in (i)

$$(m_{0a} c)^2 + 2m_{0a} c \frac{h\nu}{c} + 0 = (m_{0c} c)^2$$

$$m_{0c} = \sqrt{m_{0a}^2 + 2m_{0a} \frac{h\nu}{c^2}} = m_{0a} \sqrt{1 + 2 \frac{h\nu}{m_{0a} c^2}}$$

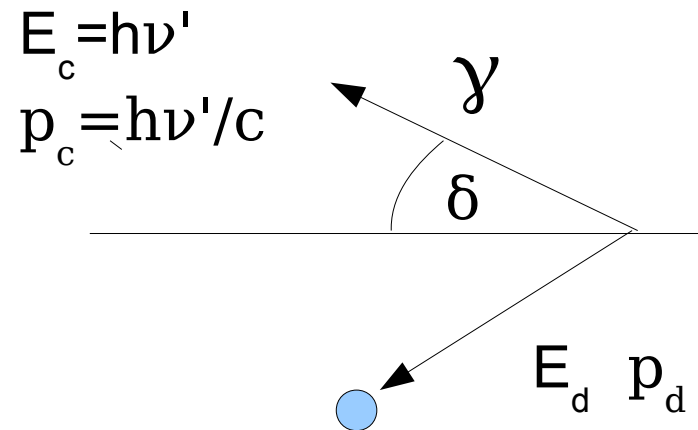
$$\text{If } m_{0a} c^2 \gg h\nu \quad \rightarrow \quad m_{0c} c^2 \approx m_{0a} c^2 + h\nu$$

4. Compton effect (photon scattered at electron)

before collision



after collision



$$P_a^\mu + P_b^\mu = P_c^\mu + P_d^\mu \quad (i)$$

scalar product with itself \rightarrow

$$P_a^\mu P_{a\mu} + 2P_a^\mu P_{b\mu} + P_b^\mu P_{b\mu} = P_c^\mu P_{c\mu} + 2P_c^\mu P_{d\mu} + P_d^\mu P_{d\mu}$$

for photons

$$P_a^\mu P_{a\mu} = P_c^\mu P_{c\mu} = 0$$

in restframes b,d

$$P_b^\mu P_{b\mu} = P_d^\mu P_{d\mu}$$

$$\rightarrow P_a^\mu P_{b\mu} = P_c^\mu P_{d\mu}$$

multiplication of (i) with $P_{c\mu}$

$$P_a^\mu P_{c\mu} + P_b^\mu P_{c\mu} = P_c^\mu P_{c\mu} + P_d^\mu P_{c\mu} = P_a^\mu P_{b\mu} \quad (ii)$$

$$P_a^\mu = \left(\frac{h\nu}{c}, \frac{h\nu}{c}, 0, 0 \right)$$

$$P_b^\mu = \left(\gamma m_0 c, -\gamma m_0 v, 0, 0 \right)$$

$$P_c^\mu = \left(\frac{h\nu'}{c}, -\frac{h\nu'}{c} \cos(\vartheta), \frac{h\nu'}{c} \sin(\vartheta), 0 \right)$$

substituted in (ii):

$$\left(\frac{h}{c}\right)^2 \nu \nu' (1 + \cos(\vartheta)) + \gamma m_0 h \nu' (1 - \beta \cos(\vartheta)) = \\ = \gamma m_0 h \nu (1 + \beta)$$

$$\frac{\nu'}{\nu} = \frac{1 + \beta}{1 - \beta \cos(\vartheta) + (1 + \cos(\vartheta)) E_\gamma / E_e}$$

Astronomy: Microwave background radiation with $E \approx 10^{-3}$ eV is scattered at high energy electrons $\gamma \gg 10^8$

$$\vartheta = 0, \quad 1 + \beta \approx 2, \quad 1 - \beta = \frac{1}{2} (1 - \beta)(1 + \beta) = \frac{1}{2} (1 - \beta^2) = \frac{1}{2 \gamma^2}$$

$$\frac{4 \gamma E}{E_{0e}} \gg 1, \quad E'_\gamma \approx \gamma m_0 c^2 = \gamma 511 \text{ keV}$$

Dramatic increase of photon energy !

Velocity 4-vector

(meaningless, dX^μ , $d\tau$ in different frames)

$$U^\mu = \frac{dX^\mu}{d\tau} = \frac{dX^\mu}{dt} \frac{dt}{d\tau} = \left(c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) \frac{dt}{d\tau} = \\ = \gamma_u (c, u_x, u_y, u_z)$$

$$U^\mu U_\mu = U'^\mu U'_\mu = c^2$$

Acceleration 4-vector

$$A^\mu = \frac{dU^\mu}{dt} \frac{dt}{d\tau} = \gamma_u \left[\frac{d\gamma_u}{dt} (c, u_x, u_y, u_z) + \gamma_u \frac{d}{dt} (c, u_x, u_y, u_z) \right]$$

$$A^\mu = \frac{\gamma_u^4}{c^2} (\vec{u} \cdot \vec{a}) (c, \vec{u}) + \gamma_u^2 (0, \vec{a}) \quad U^\mu A_\mu = 0$$

$$A^\mu A_\mu = -\frac{\gamma_u^6}{c^2} (\vec{u} \cdot \vec{a})^2 - \gamma_u^4 a^2 \quad (i)$$

In instantaneous rest frame S' of a particle

$$u' = 0, \quad \gamma_{u'} = 1 \quad \rightarrow \quad A'^{\mu} A'_{\mu} = -\alpha^2$$

α is proper acceleration

Linear acceleration, $\vec{u} \parallel \vec{a}$ and use of (i):

$$\alpha^2 = \gamma_u^6 \beta_u^2 a^2 + \gamma_u^4 a^2 = \gamma_u^6 a^2 \quad \rightarrow \quad \alpha = \gamma_u^3 a$$

Radial acceleration, $\vec{u} \perp \vec{a}$ and use of (i):

$$\alpha^2 = \gamma_u^4 a^2 \quad \rightarrow \quad \alpha = \gamma_u^2 a = \gamma_u^2 \frac{u^2}{r}$$

Frequency-wavenumber 4-vector

Plane wave: $\vec{E} = \vec{E}_0 \sin(\omega t - \vec{k} \cdot \vec{r}), \quad |\vec{k}| = \frac{2\pi}{\lambda} = \frac{\omega}{c}$

Phase at a fixed position must be the same for all reference systems.

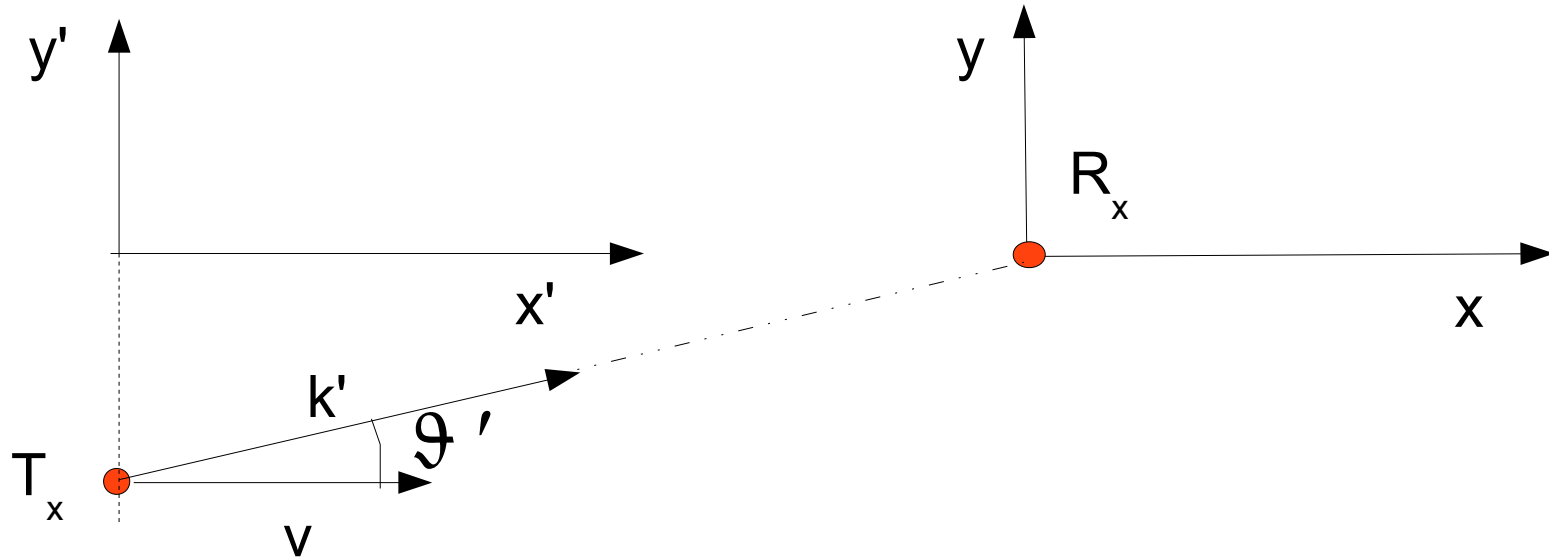
$$\begin{aligned}\Phi &= \omega t - \vec{k} \cdot \vec{r} = \omega t - (k_x x + k_y y + k_z z) = \\ &= K^\mu X_\mu = K'^\mu X'_\mu = \Phi'\end{aligned}$$

where $K^\mu = \left(\frac{\omega}{c}, k_x, k_y, k_z\right)$

Since $E = h\nu = \hbar\omega$ and $E = pc$ for photons, it is $p = \hbar\omega/c = \hbar k$ and

$$P^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right) = \hbar K^\mu$$

6. Doppler effect



$$K'^{\mu} = \left(\frac{\omega'}{c}, k'_x, k'_y, k'_z \right)$$

A L-T of K'^{μ} yields the frequency shift

$$\frac{\omega}{c} = \gamma \left(\frac{\omega'}{c} + \beta k'_x \right) = \gamma (1 + \beta \cos(\theta')) \frac{\omega'}{c}$$

and the aberration

$$k_x = \frac{\omega}{c} \cos(\vartheta) = \gamma (\beta + \cos \vartheta') \frac{\omega'}{c}$$

$$k_y = \frac{\omega}{c} \sin(\vartheta) = \frac{\omega'}{c} \sin \vartheta', \quad k_z = 0$$

$$\tan(\vartheta) = \frac{k_y}{k_x} = \frac{\sin(\vartheta')}{\gamma (\beta + \cos(\vartheta'))} \quad (i)$$

with $\sin(\vartheta)$, $\cos(\vartheta)$, $\tan\left(\frac{\vartheta}{2}\right) = \sin(\vartheta)/(1 + \cos(\vartheta))$

we transform (i)

$$\tan\left(\frac{\vartheta}{2}\right) = \sqrt{\frac{1-\beta}{1+\beta}} \tan\left(\frac{\vartheta'}{2}\right)$$

Charge-current 4-vector

charge density

$$\rho = \gamma_u \rho_0$$

current density

$$\vec{j} = \rho \vec{u} = \gamma_u \rho_0 \vec{u}$$

$$J^\mu = (\rho c, j_x, j_y, j_z) = \gamma_u \rho_0 (c, u_x, u_y, u_z)$$

$$\left[P^\mu = \left(\frac{E}{c}, p_x, p_y, p_z \right) = \gamma_u m_0 (c, u_x, u_y, u_z) \right]$$

Power-force 4-vector (Minkowski force)

$$\begin{aligned} F^\mu &= \frac{dP^\mu}{d\tau} = \frac{dP^\mu}{dt} \frac{dt}{d\tau} = \gamma_u \left(\frac{1}{c} \frac{dE}{dt}, \frac{dp_x}{dt}, \frac{dp_y}{dt}, \frac{dp_z}{dt} \right) = \\ &= \gamma_u \left(\frac{1}{c} \vec{f} \cdot \vec{u}, f_x, f_y, f_z \right) \end{aligned}$$

Relativistic Newton's 2nd law:

$$F^\mu = m_0 A^\mu$$

$$F^0 = m_0 A^0 = m_0 \frac{\gamma_u^4}{c} \vec{u} \cdot \vec{a} \quad ?$$

$$m_0 \frac{\gamma_u^4}{c} \vec{u} \cdot \vec{a} = m_0 \frac{\gamma_u^4}{c} \frac{1}{\gamma_u m_0} \vec{f} \cdot \vec{u} \left(1 - \left(\frac{u}{c} \right)^2 \right) = \frac{\gamma_u}{c} \vec{f} \cdot \vec{u}$$

Transformation of electromagnetic fields

Force $\vec{f} = q(\vec{E} + \vec{u} \times \vec{B})$

Power-force 4-vector

$$F^\mu = \gamma_u \left(\frac{1}{c} \vec{f} \cdot \vec{u}, f_x, f_y, f_z \right) = \gamma_u q \left(\frac{1}{c} \vec{E} \cdot \vec{u}, \vec{E} + \vec{u} \times \vec{B} \right)$$

$$\begin{bmatrix} F^0 \\ F^1 \\ F^2 \\ F^3 \end{bmatrix} = \frac{q}{c} \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{bmatrix} \begin{bmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{bmatrix} = \frac{q}{c} \underline{T} U^\mu$$

With the Lorentz-transformation from S to S'
and the inverse transformation

$$\underline{L} = \begin{bmatrix} \gamma_v & -\beta_v \gamma_v & 0 & 0 \\ -\beta_v \gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \underline{L}^{-1} = \begin{bmatrix} \gamma_v & \beta_v \gamma_v & 0 & 0 \\ \beta_v \gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we get

$$\underline{L} \underline{L}^{-1} = \underline{L}^{-1} \underline{L} = \underline{1}$$

$$F'^{\mu} = \underline{L} F^{\mu}, \quad F^{\mu} = \underline{L}^{-1} F'^{\mu}$$

$$F^\mu = \underline{L}^{-1} F'^\mu = \frac{q}{c} \underline{T} U^\mu = \frac{q}{c} \underline{T} \underline{L}^{-1} U'^\mu$$

$$\rightarrow F'^\mu = \frac{q}{c} \underline{L} \underline{T} \underline{L}^{-1} U'^\mu = \frac{q}{c} \underline{T}^{-1} U'^\mu$$

$$E'_x = E_x$$

$$B'_x = B_x$$

$$E'_y = \gamma_v (E_y - v B_z)$$

$$B'_y = \gamma_v (B_y + \frac{v}{c^2} E_z)$$

$$E'_z = \gamma_v (E_z + v B_y)$$

$$B'_z = \gamma_v (B_z - \frac{v}{c^2} E_y)$$

Transformation in case of $\mathbf{v}=(v_x, v_y, v_z)$

For $\mathbf{v}=(v, 0, 0)$ we write the transformation as

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

$$\vec{E}'_{\perp} = \gamma_v (\vec{E}_{\perp} + \vec{v} \times \vec{B})$$

$$\vec{B}'_{\perp} = \gamma_v (\vec{B}_{\perp} - \frac{1}{c^2} \vec{v} \times \vec{E})$$

\vec{E}_{\parallel} and E_{\perp} follow through projection on \vec{v}

$$\vec{E}_{\parallel} = \frac{1}{v^2} (\vec{v} \cdot \vec{E}) \vec{v}$$

$$\vec{E}_{\perp} = \vec{E} - \vec{E}_{\parallel} = \vec{E} - \frac{1}{v^2} (\vec{v} \cdot \vec{E}) \vec{v}$$

for \vec{B} correspondingly.

The general transformation is then

$$\vec{E}' = \gamma_v (\vec{E} + \vec{v} \times \vec{B}) - (\gamma_v - 1) \frac{(\vec{v} \cdot \vec{E}) \vec{v}}{v^2}$$

$$\vec{B}' = \gamma_v \left(\vec{B} - \frac{1}{c^2} \vec{v} \times \vec{E} \right) - (\gamma_v - 1) \frac{(\vec{v} \cdot \vec{B}) \vec{v}}{v^2}$$

7. Uniformly moving charge

Point charge at rest in origin of S'

$$\vec{E}' = \frac{q}{4\pi\epsilon_0(x'^2 + y'^2 + z'^2)^{3/2}}(x', y', z'), \quad \vec{B}' = 0$$

As a function of time the field at a point P=(0,a,0) in S is

$$\vec{E}'_P(t') = \frac{q}{4\pi\epsilon_0(v^2 t'^2 + a^2)^{3/2}}(-vt', a, 0)$$

Transformation of t'

$$t' = \gamma \left(t - \frac{v}{c^2} x \right) = \gamma t$$

Transformation of fields

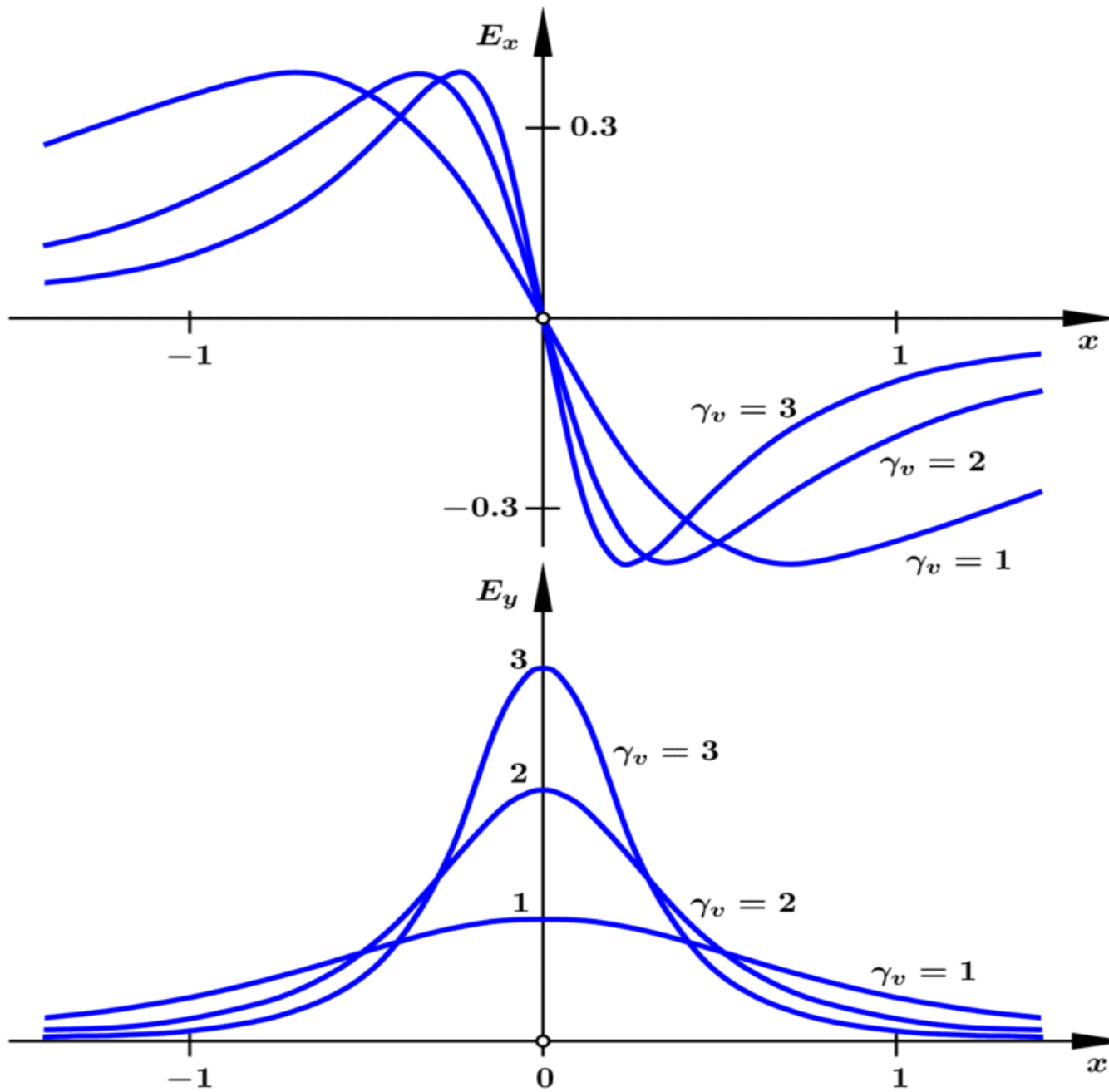
$$E_{Px} = E_{Px}' , \quad B_{Px} = 0$$

$$E_{Py} = \gamma E_{Py}' , \quad B_{Py} = -\gamma \frac{v}{c^2} E_{Pz}'$$

$$E_{Pz} = \gamma E_{Pz}' , \quad B_{Pz} = +\gamma \frac{v}{c^2} E_{Py}'$$

$$\vec{E}_P(t) = \frac{q}{4\pi\epsilon_0(\gamma^2 v^2 t^2 + a^2)^{3/2}} (-\gamma vt, \gamma a, 0)$$

$$\vec{B}_P(t) = \frac{q}{4\pi\epsilon_0(\gamma^2 v^2 t^2 + a^2)^{3/2}} (0, 0, \gamma \frac{v}{c^2} a)$$



Literature:

- R. P. Feynman, R. B. Leighton, M. Sands: Lectures on physics. Vol. I. Addison & Wesley, 1963
- A. P. French: Special relativity. W. W. Norton & Company, 1966
- J. Freund: Special relativity for beginners. World Scientific, 2008

Exercise 6

Consider the charge of exercise 5 in its instantaneous rest frame S' where it experiences a constant acceleration $\alpha = q E_x / m_0$.

Exercise 7

Particles a and b with u_a and u_b are in frame S. Use the velocity 4-vector to derive u_b' in S' , the rest frame of a.

Exercise 8

A particle moves in S with velocity \vec{u} and experiences a force \vec{f} . What is the force in S' ?

Exercise 9

A point charge q moves parallel to a current carrying wire. By transforming the e.-m. fields calculate the force in its rest frame.