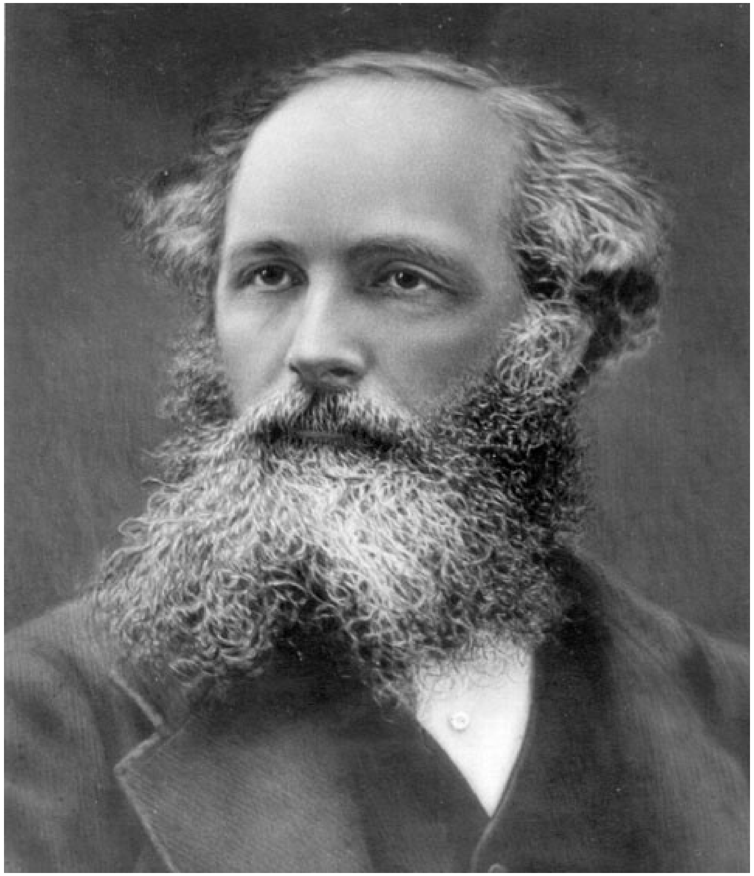
Review of Electromagnetism



This review is not meant to teach the subject, but to repeat and to refresh, at least partially, what you have learnt at university.

Maxwell's equations

(in integral form)

$$\oint \vec{H}(\vec{r},t)\cdot d\vec{s} = \iint \vec{J}(\vec{r},t)\cdot d\vec{A} + \frac{d}{dt}\iint \vec{D}(\vec{r},t)\cdot d\vec{A}$$

$$\oint \vec{E}(\vec{r},t)\cdot d\vec{s} = -\frac{d}{dt}\iint \vec{B}(\vec{r},t)\cdot d\vec{A}$$

$$\oint\!\!\oint \vec{D}(\vec{r},t)\cdot d\vec{A} = \iiint \rho(\vec{r},t)\,dV$$

$$\oint\!\!\oint \vec{B}(\vec{r},t)\cdot d\vec{A} = 0$$

$\vec{E}, \vec{H}$ electric and magnetic field

$\vec{D}, \vec{B}$ electric displacement and magnetic induction

$\vec{J}$ electric current density

ρ electric charge density

$\iint \vec{J}(\vec{r},t)\cdot d\vec{A}$ stands for all currents going through the area A. It may consist of 3 parts

$$\vec{J}(\vec{r},t)=\vec{J}_c(\vec{r},t)+\vec{J}_{cv}(\vec{r},t)+\vec{J}_i(\vec{r},t)$$

$\vec{J}_c(\vec{r},t)=\kappa\vec{E}(\vec{r},t)$ conduction current

$\vec{J}_{cv}(\vec{r},t)=\rho(\vec{r},t)\vec{v}(\vec{r},t)$ convection current

$\vec{J}_i(\vec{r},t)$ impressed current

$\iiint \rho(\vec{r},t)\,dV$ stands for all charges in the volume V

Maxwell's equations (in differential form)

With Stokes' theorem:

$$\oint \vec{E}\cdot d\vec{s} = \iint (\vec{\nabla}\times\vec{E})\cdot d\vec{A} = -\frac{d}{dt}\iint \vec{B}\cdot d\vec{A} = -\iint \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

$$\iint \left[\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t}\right]\cdot d\vec{A} = 0 \quad \rightarrow \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (2)$$

correspondingly $\rightarrow \quad \vec{\nabla}\times\vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} \qquad (1)$

With Gauss' theorem:

$$\oiint \vec{D} \cdot d\vec{A} = \iiint \vec{\nabla} \cdot \vec{D}\, dV = \iiint \rho\, dV$$

$$\iiint [\vec{\nabla} \cdot \vec{D} - \rho]\, dV = 0 \quad \rightarrow \quad \vec{\nabla} \cdot \vec{D} = \rho \qquad (3)$$

correspondingly $\rightarrow \quad \vec{\nabla} \cdot \vec{B} = 0 \qquad (4)$

Time-harmonic fields

Time-harmonic fields can be written as complex quantities

$$\vec{e}(\vec{r},t)=\vec{e}(\vec{r})\cos(\omega t+\varphi)=$$
$$=\Re[\vec{e}(\vec{r})\mathrm{e}^{i\varphi}\mathrm{e}^{i\omega t}]=\Re[\tilde{\vec{E}}(\vec{r})\mathrm{e}^{i\omega t}]=\Re\,\vec{E}(\vec{r},t)$$

$\tilde{\vec{E}}(\vec{r})$ is called phasor.

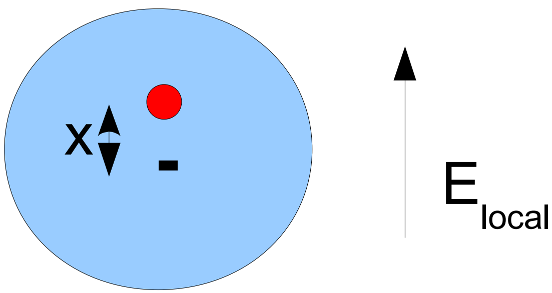
Advantages are:

- $\frac{\partial}{\partial t} \rightarrow i\omega$,
- phasors are vectors in a coordinate system rotating with ωt,
- $\mathrm{e}^{i\omega t}$ cancels out in the equations.

Fields in matter (time-harmonic case)

The effect of electric fields on matter can be described by a polarization P,
the effect of magnetic fields by a magnetization M.

P and M result from averaging over atomic / molecular electric and magnetic dipoles induced by the fields, e.g.



$$p_e = q x \quad \rightarrow \quad \vec{P} = n\,\vec{p}_e = \epsilon_0 \chi_e \vec{E}$$

Linear materials:
$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon \vec{E}$$
$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M} = \mu_0 \vec{H} + \mu_0 \chi_m \vec{H} = \mu \vec{H}$$

There are losses due to changing polarisation

$$\epsilon = \epsilon' - i\epsilon'' = \epsilon'(1 - i\tan(\delta_\epsilon))$$

$$\tan(\delta_\epsilon) = \frac{\epsilon''}{\epsilon'}, \qquad \delta_\epsilon \quad \text{electric loss angle}$$

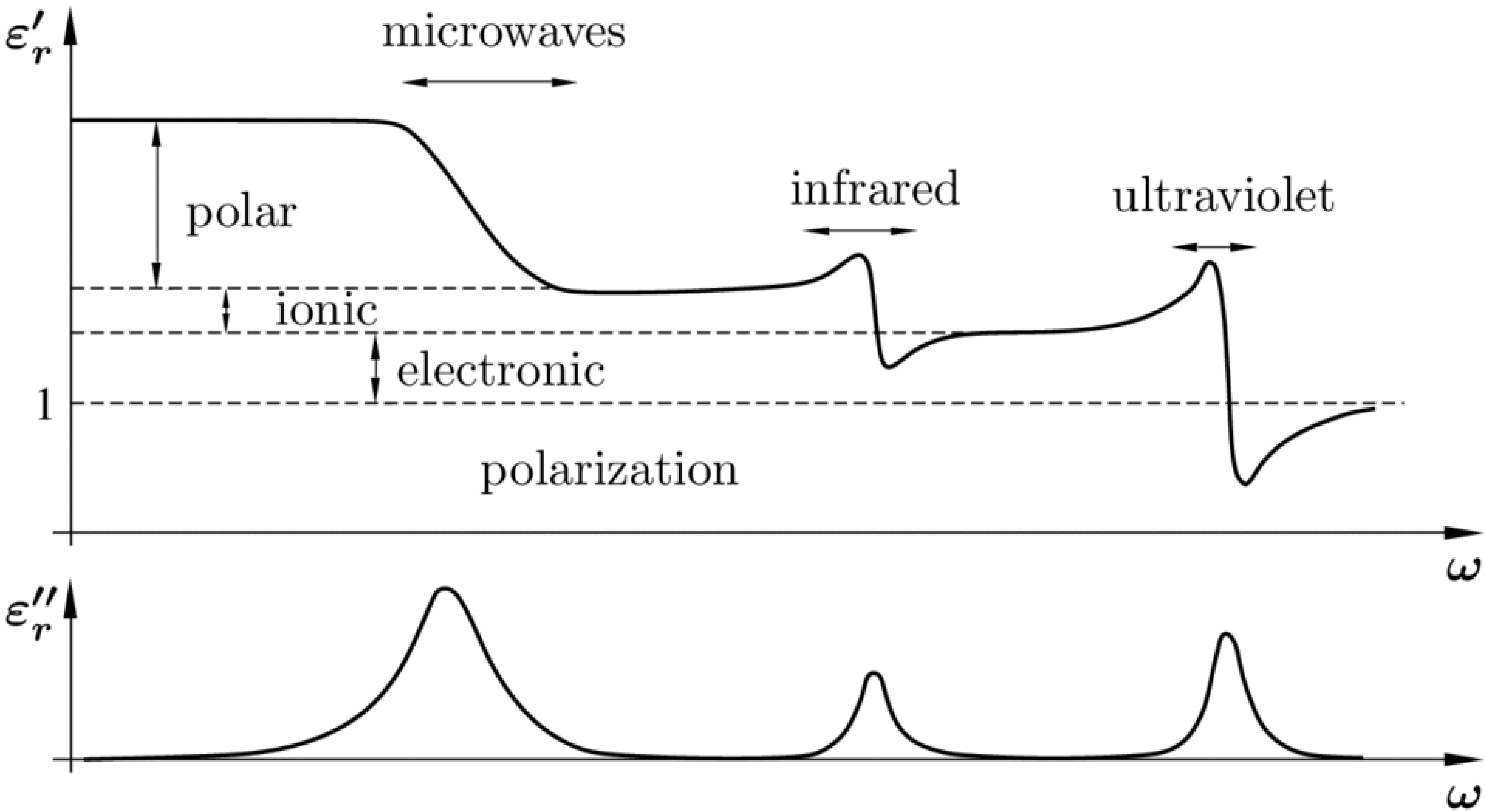
and losses due to free charges

$$\vec{\nabla} \times \vec{H} = \vec{J} + i\omega\epsilon\vec{E} = \kappa\vec{E} + i\omega\epsilon\vec{E} = i\omega\left(\epsilon - i\frac{\kappa}{\omega}\right)\vec{E}$$

$$\epsilon_c = \epsilon - i\frac{\kappa}{\omega}$$

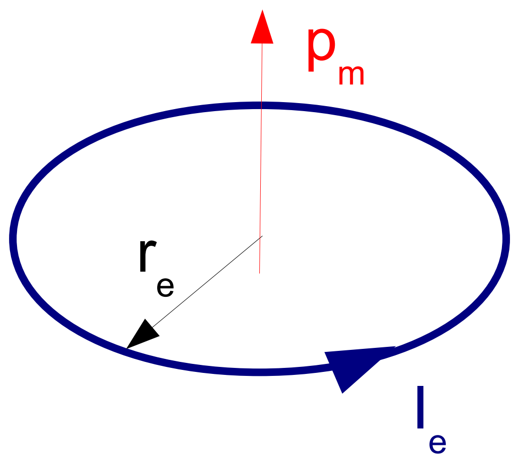
In most dielectrics is $\tan(\delta_\varepsilon) \ll 1$.

In good conductors is $\kappa/\omega \gg \varepsilon \;\rightarrow\; \varepsilon_c \approx \kappa/i\omega$.



ε_r'
microwaves
polar
infrared
ultraviolet
ionic
electronic
1
polarization
ω
ε_r''
ω

Magnetic reaction of material is due to circulating electrons and due to particle spins. It can be described by means of magnetic dipoles, i.e. by circulating elementary currents:

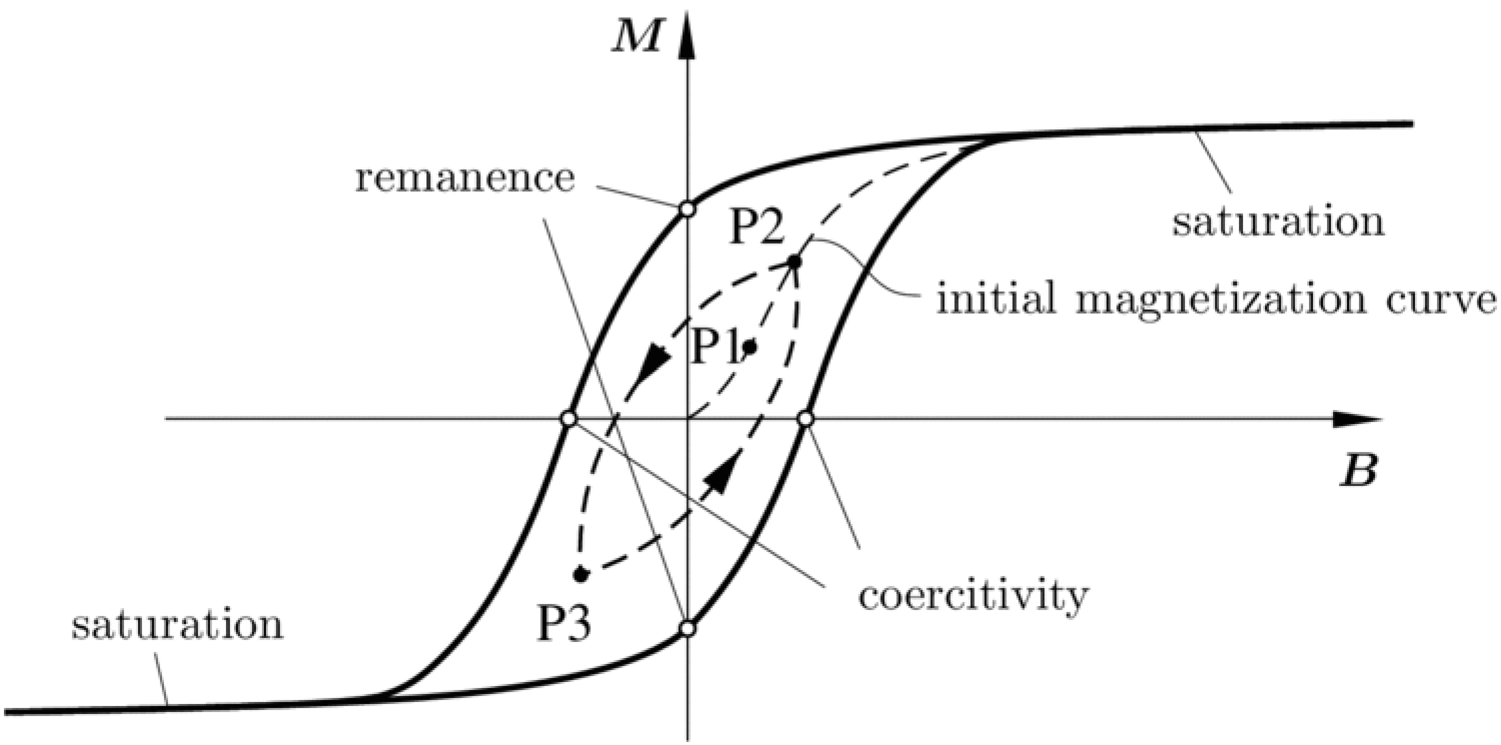


$$p_m = \pi r_e^2 I_e \quad \rightarrow \quad \vec{M} = n\,\vec{p}_m = \chi_m \vec{H}$$

Like P, the magnetization M is a dynamic process with losses due to rotating dipoles

$$\mu = \mu' - i\mu'' = \mu'(1 - i\tan(\delta_\mu)), \quad \delta_\mu \ \textit{magnetic loss angle}$$

For ferromagnetic materials the relation between the external field and the magnetization is non-linear and depends typically on the history of the material (hysteresis).



Boundary / continuity conditions

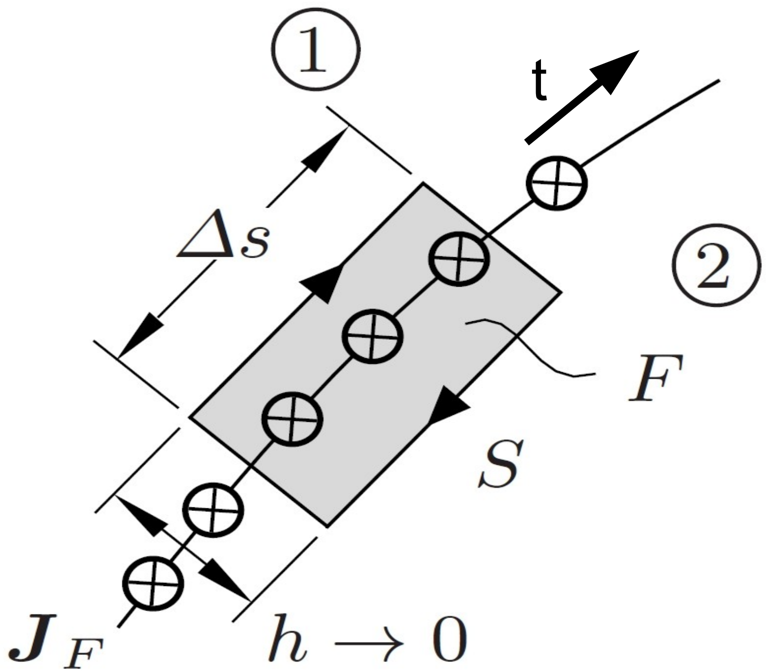
Maxwell's theory is a continuums theory. It reqires continuously double differentiable functions.

Solutions in different media have to be matched at the interface.

Make an intelligent choice for the integration area:

$$\oint \vec{H}(\vec{r},t)\cdot d\vec{s} = \iint \vec{J}(\vec{r},t)\cdot d\vec{A} + \frac{d}{dt}\iint \vec{D}(\vec{r},t)\cdot d\vec{A}$$

$$\oint \vec{E}(\vec{r},t)\cdot d\vec{s} = -\frac{d}{dt}\iint \vec{B}(\vec{r},t)\cdot d\vec{A}$$



Δs is finite but small, such that the fields are constant,
h → 0 :

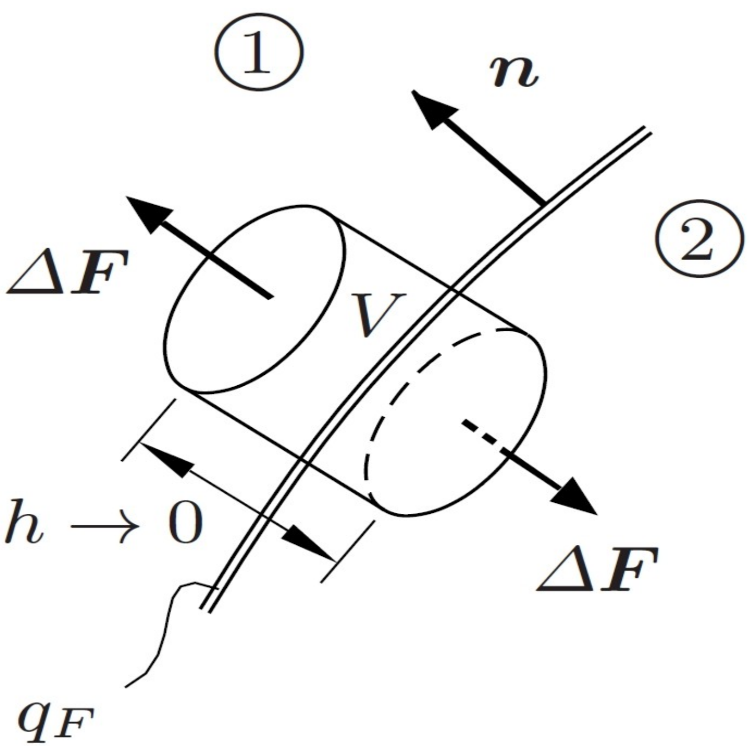
$$H_{t1} - H_{t2} = J_A$$
$$E_{t1} - E_{t2} = 0$$

If medium 2 is perfectly electric conducting (pec) :

$$E_{t1} = 0, \quad H_{t1} = J_A$$

An intelligent choice of the integration volume:

$$\oiint \vec{D}(\vec{r},t)\cdot d\vec{A} = \iiint \rho(\vec{r},t)\,dV$$
$$\oiint \vec{B}(\vec{r},t)\cdot d\vec{A} = 0$$



$$B_{n1} - B_{n2} = 0, \quad D_{n1} - D_{n2} = \rho_A$$

$$B_{n1} = 0, \quad D_{n1} = \rho_a \quad \text{if 2 is pec}$$

Electrostatic fields

(δ/δt=0, ε=const.)

$$\vec{\nabla}\times\vec{E}=0 \qquad \rightarrow \qquad \vec{E}=-\vec{\nabla}\Phi$$

Poisson equation:

$$\vec{\nabla}\cdot\vec{D}=\vec{\nabla}\cdot(\epsilon\vec{E})=\rho \qquad \rightarrow \qquad \vec{\nabla}^2\Phi=-\frac{\rho}{\epsilon}$$

Example: Solution of Laplace equation

$$\vec{\nabla}^2\Phi=\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

Bernoulli ansatz

$$\Phi(x,y,z)=X(x)Y(y)Z(z)$$

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-k_x^2}+\underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-k_y^2}+\underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{-k_z^2}=0$$

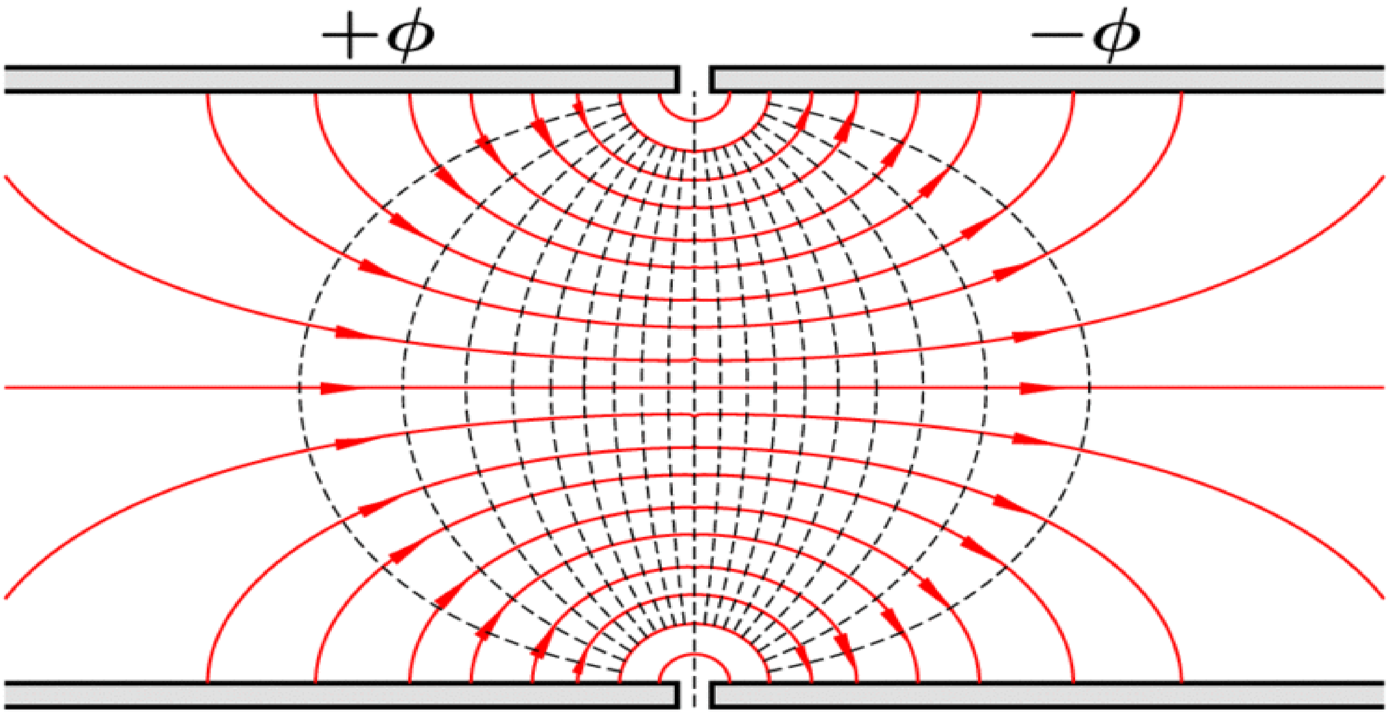
$$X(x)=\begin{cases} A_0+B_0x & \text{for } k_x=0 \\ A\cos(k_x x)+B\sin(k_x x) & \text{for } k_x\neq 0 \end{cases}$$

$Y(y)$ *correspondingly*

$$Z(z)=\begin{cases} E_0+F_0z & \text{for } k_x=k_y=0 \\ E\,\mathrm{e}^{k_z z}+F\,\mathrm{e}^{-k_z z} & \text{for } k_x\vee k_y\neq 0 \end{cases}$$

$$k_z^2=-\left(k_x^2+k_y^2\right)$$

Example: Circular electrostatic lense



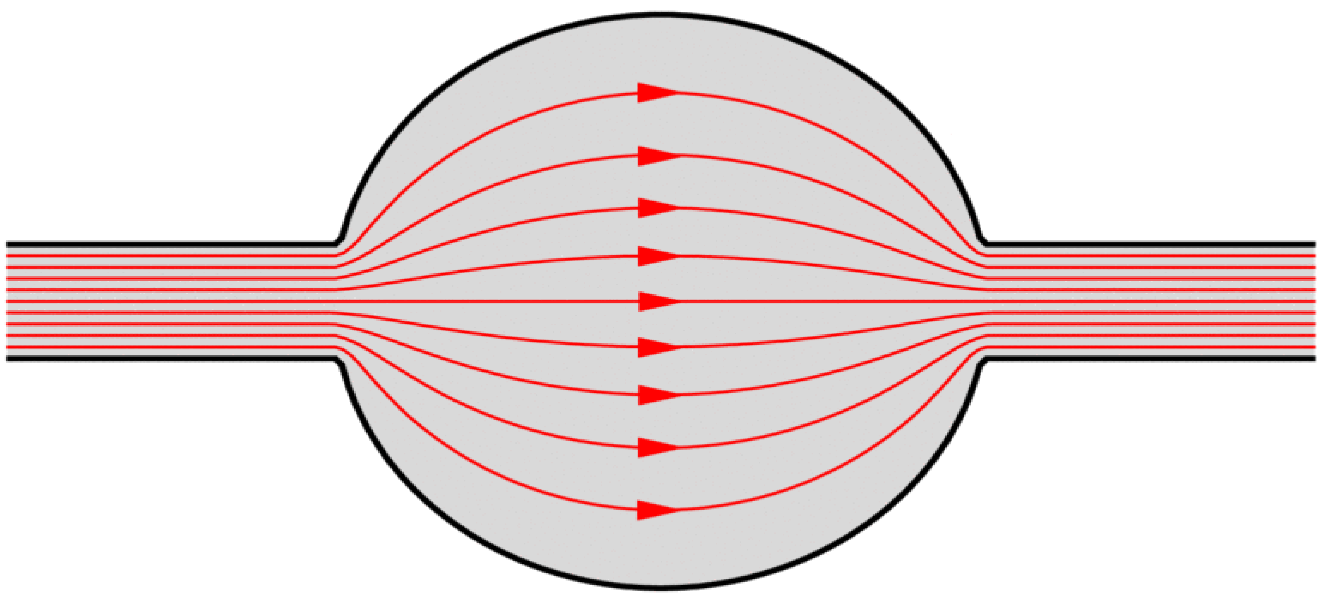
E-field pattern

Stationary currents

$(\delta/\delta t=0, \ \kappa=\text{const.})$

$$\vec{\nabla}\times\vec{E}=0 \quad \rightarrow \quad \vec{E}=-\vec{\nabla}\Phi$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H})=0=\vec{\nabla}\cdot\vec{J}=\vec{\nabla}\cdot(\kappa\vec{E}) \quad \rightarrow \quad \vec{\nabla}^2\Phi=0$$



J-field lines

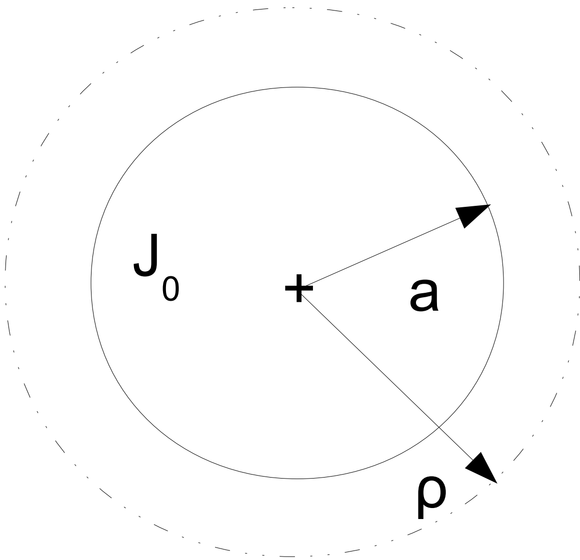
Magnetostatic fields

$(\delta/\delta t=0,\ \mu=\text{const.})$

$$\vec{\nabla}\cdot\vec{B}=0 \quad\rightarrow\quad \vec{B}=\vec{\nabla}\times\vec{A}=\mu\,\vec{H}$$

$$\vec{\nabla}\times\vec{H}=\vec{J} \quad\rightarrow\quad \vec{\nabla}^2\vec{A}=\mu\,\vec{J}$$

Example: Wire carrying a constant current



$$\vec{\nabla}^2 A_z=\frac{\partial^2 A_z}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial A_z}{\partial\rho}=$$

$$=\begin{cases}0 & \text{for } \rho\geq a\\ \mu J_0 & \text{for } \rho\leq a\end{cases}$$

$\rho \geq a:$

$$\frac{d}{d\rho}\frac{dA_z^{(1)}}{d\rho} = -\frac{1}{\rho}\frac{dA_z^{(1)}}{d\rho} \quad \rightarrow \quad \frac{dA_z^{(1)}}{d\rho} = \frac{C}{\rho}$$

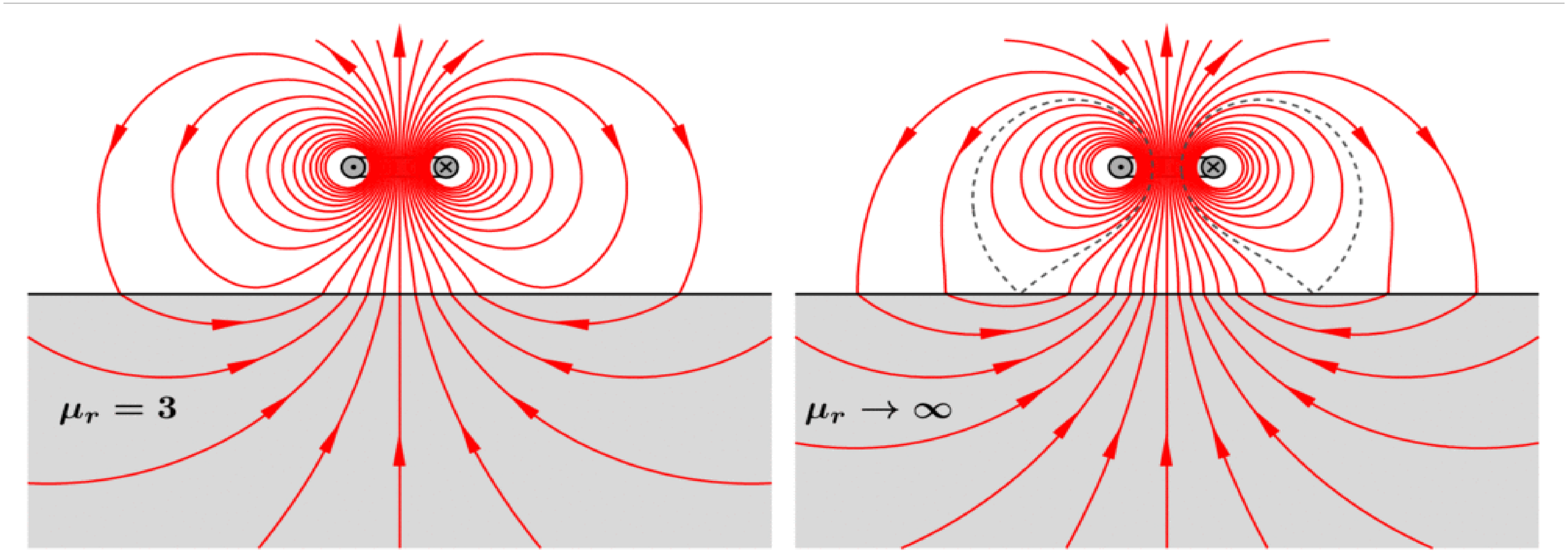
$\rho \leq a:$

$$\frac{d}{d\rho}\frac{dA_z^{(2)}}{d\rho} + \frac{1}{\rho}\frac{dA_z^{(2)}}{d\rho} = \mu J_0 \quad \rightarrow \quad A_z^{(2)} = \frac{\mu}{4} J_0 \rho^2$$

$at\,\rho = a: \quad B_\varphi^{(1)} = B_\varphi^{(2)}$

$$\vec{B} = \vec{\nabla} \times (A_z \vec{e}_z) = -\frac{dA_z}{d\rho}\vec{e}_\varphi \quad \rightarrow \quad B_\varphi^{(1)} = -\frac{\mu a^2}{2\rho} J_0$$

Example:
Planar constant current loop above permeable half-space



B-field pattern

Quasi-stationary fields

(|δD/δt|<<|J|, ε, μ, κ=const.)

$$\vec{\nabla}\cdot\vec{B}=0 \quad\rightarrow\quad \vec{B}=\vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \quad\rightarrow\quad \vec{E}=-\vec{\nabla}\Phi-\frac{\partial\vec{A}}{\partial t}$$

good conductors: $\rho=0$

no impressed voltages: $\Phi=0$

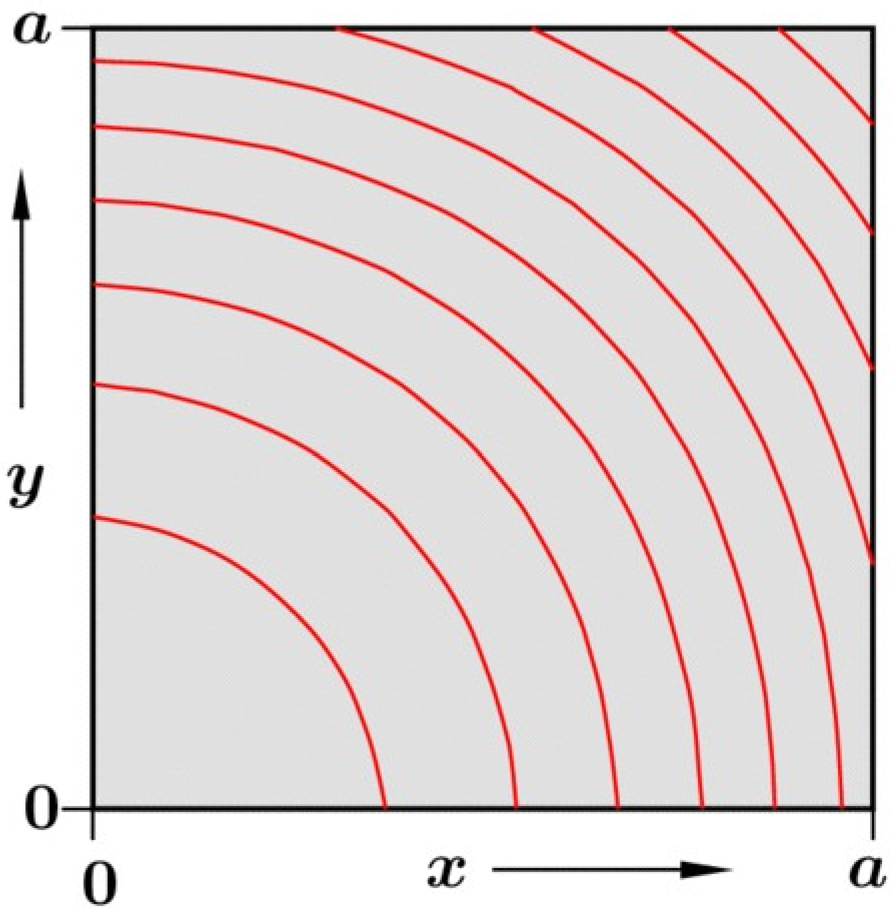
$$\vec{\nabla}\cdot\vec{D}=-\epsilon\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A}=0 \quad\rightarrow\quad \vec{\nabla}\cdot\vec{A}=0$$

$$\vec{\nabla}\times\vec{H}=\frac{1}{\mu}\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=-\frac{1}{\mu}\vec{\nabla}^2\vec{A}=\vec{J}=\kappa\vec{E}$$

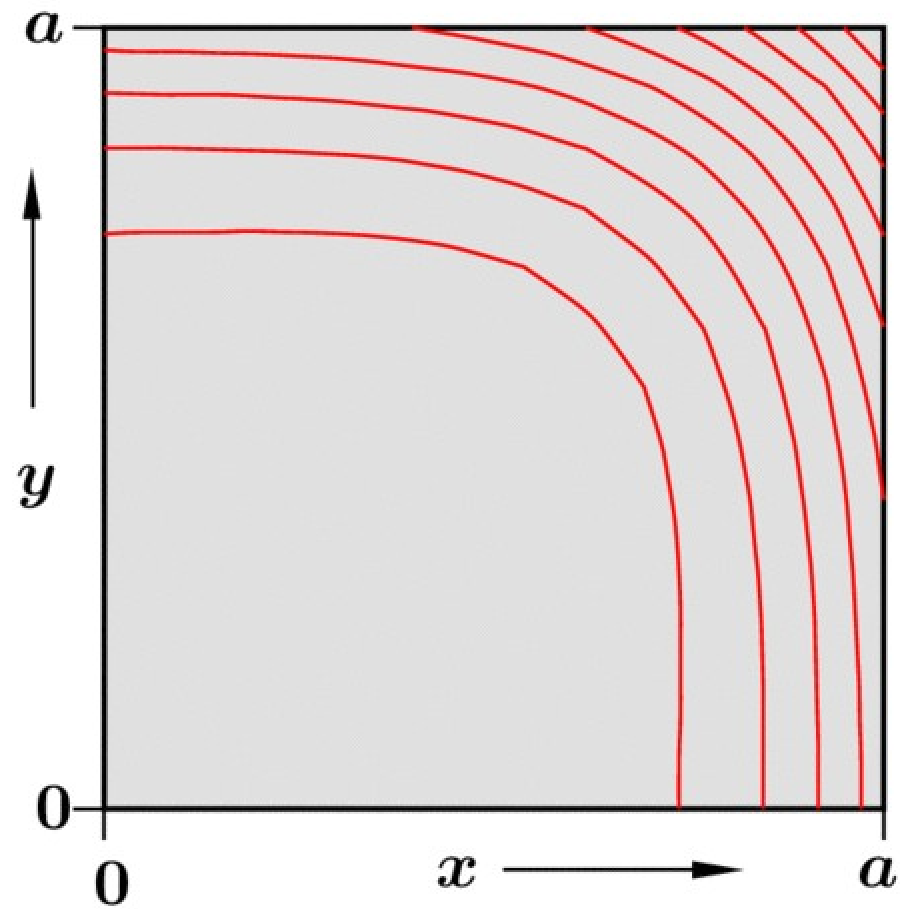
$$\rightarrow\quad \vec{\nabla}^2\vec{A}-\mu\kappa\frac{\partial\vec{A}}{\partial t}=0$$ diffusion equation

Current distribution in aluminum bar, κ=17 10^6 $\Omega^{-1}m^{-1}$, a=1cm

f=50 Hz



f=5 kHz



Poynting's theorem

(ε, μ, κ=const., J=κE,
full set of Maxwell's equations)

Work done by the fields on charges ρ dV

$$d\frac{\delta W}{\delta t}=d\vec{f}\cdot\frac{\delta\vec{s}}{\delta t}=\rho\frac{\delta\vec{s}}{\delta t}\cdot(\vec{E}+\vec{v}\times\vec{B})\,dV=\vec{J}\cdot\vec{E}\,dV$$

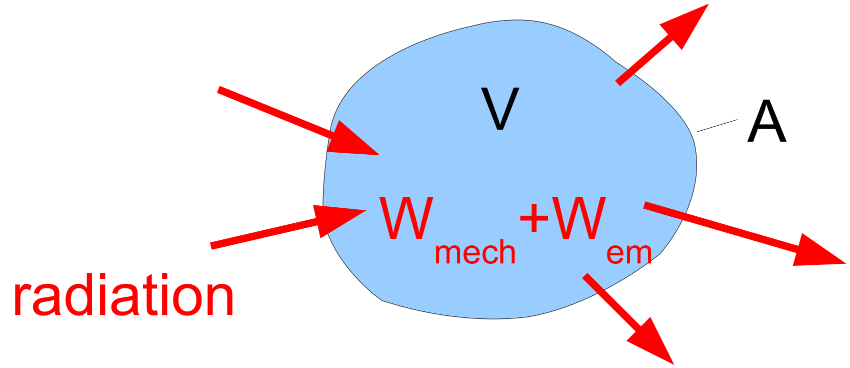
using Maxwell's equations

$$\vec{E}\cdot\quad \vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

$$\vec{H}\cdot\quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\rightarrow\quad \vec{E}\cdot\vec{J}=-\vec{\nabla}\cdot(\vec{E}\times\vec{H})-\frac{\partial}{\partial t}[\frac{1}{2}\vec{H}\cdot\vec{B}+\frac{1}{2}\vec{E}\cdot\vec{D}]$$

After integration:



$$-\oint\!\!\!\oint (\vec{E}\times\vec{H})\cdot d\vec{A} =$$

$$= \iiint \vec{E}\cdot\vec{J}\,dV + \frac{\partial}{\partial t}\iiint \left(\frac{1}{2}\vec{E}\cdot\vec{D}+\frac{1}{2}\vec{H}\cdot\vec{B}\right)dV$$

poynting vector (radiation flux) $\vec{S}=\vec{E}\times\vec{H}$

dissipated power density $p_d=\vec{E}\cdot\vec{J}$

electric energy density $w_e=\frac{1}{2}\vec{E}\cdot\vec{D}$

magnetic energy density $w_m=\frac{1}{2}\vec{H}\cdot\vec{B}$

Poynting's theorem for time-harmonic fields

decompose e.g. $$\vec{E} = \Re[\tilde{\vec{E}}\, \mathrm{e}^{i\omega t}] = \frac{1}{2}[\tilde{\vec{E}}\, \mathrm{e}^{i\omega t} + \tilde{\vec{E}}^{*}\, \mathrm{e}^{-i\omega t}]$$

$$w_E = \frac{1}{2}\vec{E}\cdot\vec{D} = \frac{1}{8}[\tilde{\vec{E}}\cdot\tilde{\vec{D}}\,\mathrm{e}^{i2\omega t} + \tilde{\vec{E}}^{*}\cdot\tilde{\vec{D}}^{*}\,\mathrm{e}^{-i2\omega t}] + \frac{1}{8}[\tilde{\vec{E}}\cdot\tilde{\vec{D}}^{*} + \tilde{\vec{E}}^{*}\cdot\tilde{\vec{D}}]$$

$$= \frac{1}{4}\Re[\tilde{\vec{E}}\cdot\tilde{\vec{D}}\,\mathrm{e}^{i2\omega t}] + \frac{1}{4}\Re[\tilde{\vec{E}}\cdot\tilde{\vec{D}}^{*}] \quad \rightarrow \quad \bar{w}_e = \frac{1}{4}\vec{E}\cdot\vec{D}^{*}$$

time-averaged quantities

$$\bar{w}_e = \frac{1}{4}\vec{E}\cdot\vec{D}^{*} \qquad \bar{w}_m = \frac{1}{4}\vec{H}\cdot\vec{B}^{*} \qquad \bar{p}_d = \frac{1}{2}\vec{E}\cdot\vec{J}^{*}$$

$$\vec{S}_c = \frac{1}{2}\vec{E}\times\vec{H}^{*} \quad \rightarrow \quad \vec{S} = \frac{1}{2}\Re[\vec{E}\times\vec{H}^{*}]$$

using Maxwell's equations

$$\vec{E}\cdot \quad \vec{\nabla}\times\vec{H}^* = \vec{J}^* - j\omega\vec{D}^*$$

$$\vec{H}^*\cdot \quad \vec{\nabla}\times\vec{E} = -j\omega\vec{B}$$

$$\rightarrow \quad \frac{1}{2}\vec{E}\cdot\vec{J}^* = -\vec{\nabla}\cdot\left(\frac{1}{2}\vec{E}\times\vec{H}^*\right) - j2\omega\left(\frac{1}{4}\vec{H}\cdot\vec{B}^* - \frac{1}{4}\vec{E}\cdot\vec{D}^*\right)$$

conservation of energy flow

$$-\oiint \vec{S}_c\cdot d\vec{A} = \iiint \bar{p}_d\, dV + i2\omega\iiint(\bar{w}_m - \bar{w}_e)\, dV$$

active power (time-averaged Joulean heat)

$$\bar{P}_{act} = \bar{P}_d = -\oiint \Re[\vec{S}_c]\cdot d\vec{A} = -\oiint \vec{S}\cdot d\vec{A}$$

reactive power

$$\bar{P}_{react} = -\oiint \Im[\vec{S}_c]\cdot d\vec{A} = 2\omega\iiint(\bar{w}_m - \bar{w}_e)\, dV$$

In good conductors is $W_m \gg W_e$ $(|E| \ll |H|)$

$$-\oiint \vec{S}_c \cdot d\vec{A} = \bar{P}_c = \bar{P}_d + i2\omega \bar{W}_m$$

Now if we define

$$\bar{I}^* = \iint \vec{\bar{J}}^* \cdot d\vec{A}$$

$$\bar{U} = \int_1^2 \vec{\bar{E}} \cdot d\vec{s} = \bar{I}(R + i\omega L_i)$$

we obtain the resistance and *internal inductance*

$$\bar{P}_c = \frac{1}{2}\bar{U}\bar{I}^* = \frac{1}{2}|\bar{I}^2|(R + i\omega L_i) = \bar{P}_d + i2\omega \bar{W}_m$$

Tut-Ex 1

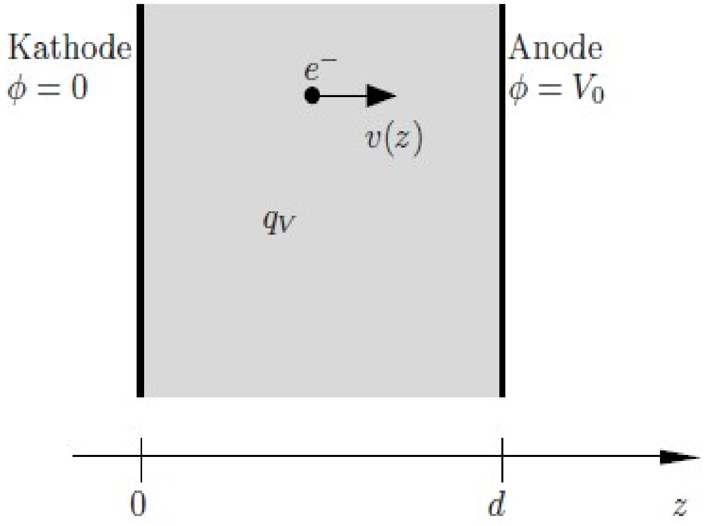
Given is a conducting hollow sphere carrying a charge Q. What is the field inside and outside and what is the electrostatic field energy?

Tut-Ex 2

A capacotor is filled with a lossy dielectric and charged to a voltage V. What is the time constant for discharge?

Tut-Ex 3

Given is a 1-dimensional planar diode. What is the current density at saturation?

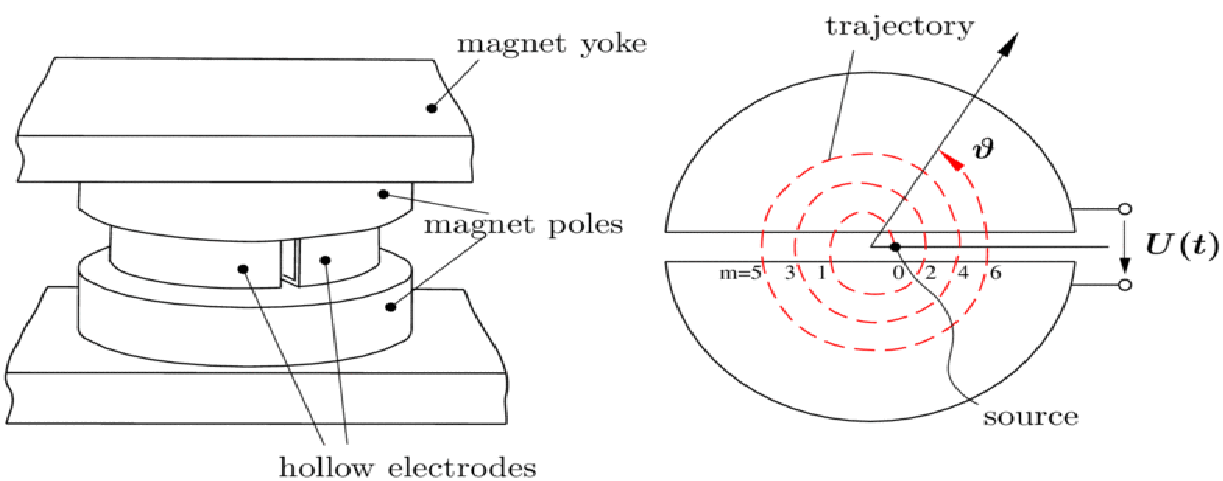


Tut-Ex 4

Derive the magnetic vectorpotential for a given current density.

Tut-Ex 5

Given is a non-relativistic cyclotron with a constant magnetic induction B and maximum radius R. What is the end energy?



Tut-Ex 6

A long dipole magnet is excited by a coil with n windings and current I_0. Calculate the magnetic field in the air gap.

Electromagnetic waves

(ε, μ=const., ρ=0, J=κE)

The simplest electromagnetic wave is a plane wave.
It depends only on one space variable (direction of propagation) and on the time.

$$\vec{E}=\vec{E}(z,t), \qquad \vec{H}=\vec{H}(z,t):$$

$$\textit{Maxwells equs. yield two sets of uncoupled equations:}$$

$$-\frac{\partial H_y}{\partial z}=\epsilon\frac{\partial E_x}{\partial t} \qquad \frac{\partial E_x}{\partial z}=-\mu\frac{\partial H_y}{\partial t}$$

$$\frac{\partial H_x}{\partial z}=\epsilon\frac{\partial E_y}{\partial t} \qquad -\frac{\partial E_y}{\partial z}=-\mu\frac{\partial H_x}{\partial t}$$

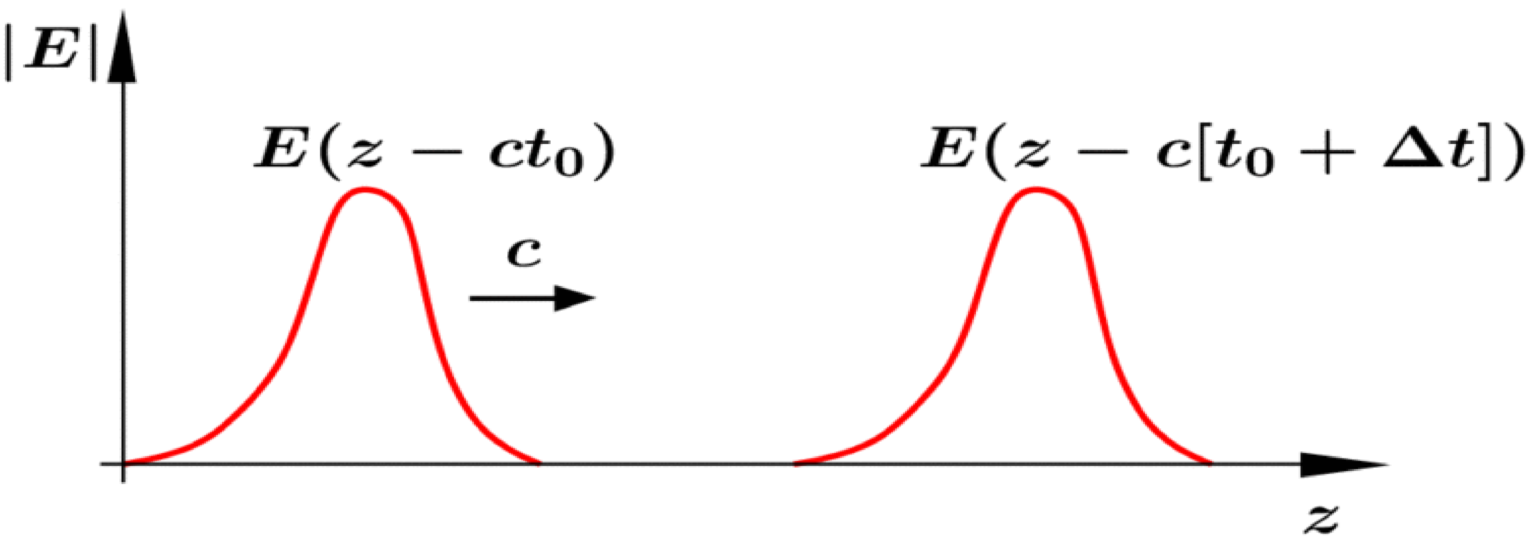
The red set gives the wave equation

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = 0, \qquad c = \frac{1}{\sqrt{\mu\epsilon}}$$

with d'Alembert's solution

$$E_x = f(z-ct) + g(z+ct) \quad \rightarrow$$

$$H_y = \frac{1}{Z}[f(z-ct) - g(z+ct)], \qquad Z = \sqrt{\frac{\mu}{\epsilon}}$$



$$\textit{velocity of light}: \qquad c = \frac{1}{\sqrt{\mu\,\epsilon}}$$

$$\textit{wave impedance}: \qquad Z = \sqrt{\frac{\mu}{\epsilon}}$$

$$\approx 377\,\Omega \quad \textit{in free space}$$

field properties:

$$\vec{E} \perp \vec{H}, \quad \vec{E} \times \vec{H} \rightarrow \textit{direction of propagation}$$

$$\vec{E}, \vec{H} \perp \textit{direction of propagation}$$

$$E^{+}/H^{+} = -E^{-}/H^{-} = Z$$

Time-harmonic plane wave

$$\left(\frac{\partial}{\partial t}=i\omega\right)$$

$$\frac{\partial^2 E_x}{\partial z^2}+k^2 E_x=0, \qquad k=\frac{\omega}{c}=\omega\sqrt{\mu\epsilon}=\frac{2\pi}{\lambda}$$

$$E_x=A\,e^{i(\omega t-kz)}+B\,e^{i(\omega t+kz)}$$

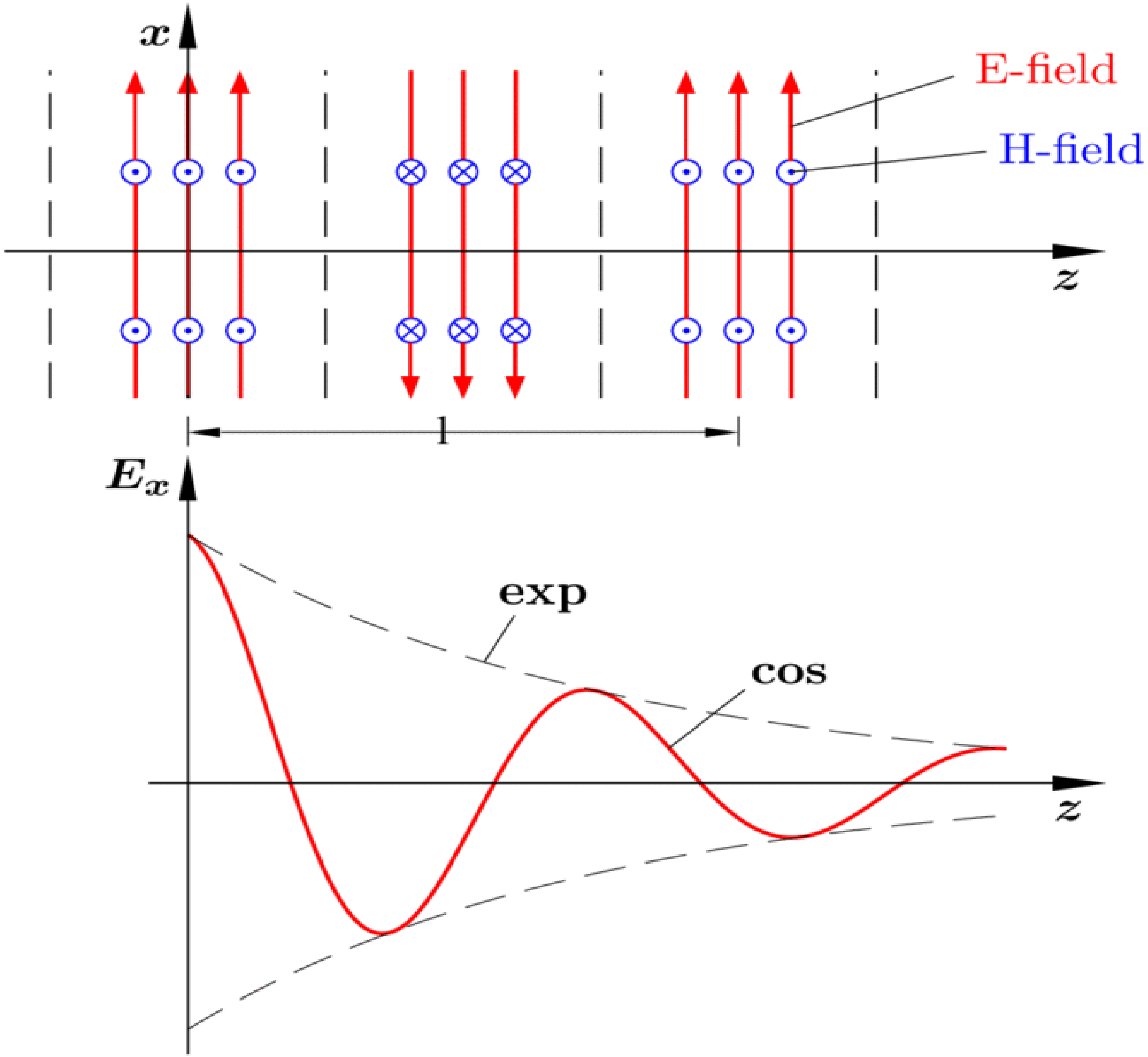
$$H_y=\frac{1}{Z}\left(A\,e^{i(\omega t-kz)}-B\,e^{i(\omega t+kz)}\right)$$

lossy material: $\epsilon_c=\epsilon_r\epsilon_0\left(1-i\frac{\kappa}{\omega\epsilon_r\epsilon_0}\right)=\epsilon_0\left(\epsilon_r'-i\epsilon_r''\right)$

$$k=\omega\sqrt{\mu\epsilon_c}=\beta-i\alpha$$

$$\frac{\beta}{k_0}=\sqrt{\frac{\epsilon_r'}{2}+\frac{\epsilon_r'}{2}\sqrt{1+\left(\frac{\epsilon_r''}{\epsilon_r'}\right)^2}}, \quad \frac{\alpha}{k_0}=\sqrt{-\frac{\epsilon_r'}{2}+\frac{\epsilon_r'}{2}\sqrt{1+\left(\frac{\epsilon_r''}{\epsilon_r'}\right)^2}}$$

$$E_x = \Re\, A e^{i(\omega t - kz)} = A \cos(\omega t - \beta z) e^{-\alpha z}$$



Phase velocity

$$\phi = \omega t - \beta z = const. \quad \rightarrow \quad \frac{d\phi}{dt} = \omega - \beta \frac{dz}{dt} = \omega - \beta v_{ph} = 0$$

$$v_{ph} = \frac{\omega}{\beta}$$

Group velocity

(two plane waves with ω_1 and ω_2)

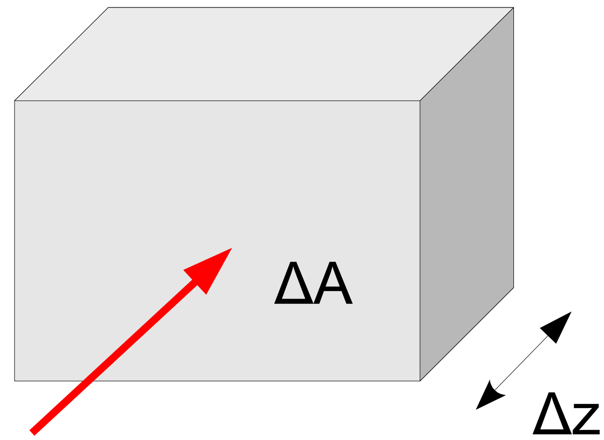
$$\omega_1 = \omega_0 + \delta\omega, \qquad \omega_2 = \omega_0 - \delta\omega$$

$$\beta_1 = \beta_0 + \delta\beta, \qquad \beta_2 = \beta_0 - \delta\beta$$

$$\Re\left[e^{i(\omega_1 t - \beta_1 z)} + e^{i(\omega_2 t - \beta_2 z)}\right] = 2\cos(\delta\omega t - \delta\beta z)\cos(\omega_0 t - \beta_0 z)$$

$$v_g = \frac{\delta\omega}{\delta\beta} \quad \rightarrow \quad v_g = \frac{d\omega}{d\beta}$$

Energy velocity



$$\frac{\bar{w}\,\Delta A\,\Delta z}{\Delta t} = \bar{S}_z\,\Delta A \quad \rightarrow \quad v_e = \frac{\Delta z}{\Delta t} = \frac{\bar{S}_z}{\bar{w}}$$

for plane waves

$$\bar{S}_z = \frac{1}{2}\left(\vec{E}\times\vec{H}^*\right)_z = \frac{|E_0{}^2|}{2Z}, \qquad \bar{w} = \frac{1}{4}\vec{E}\cdot\vec{D}^* + \frac{1}{4}\vec{H}\cdot\vec{B}^* = \frac{1}{2}\epsilon|E_0{}^2|$$

$$v_e = \frac{1}{Z\epsilon} = \frac{1}{\sqrt{\mu\epsilon}} = c$$

Low-loss dielectrics: ε''«ε'

$$\beta \approx \sqrt{\epsilon_r'}\,k_0, \qquad \alpha \approx \frac{1}{2}\frac{\epsilon_r''}{\sqrt{\epsilon_r'}}k_0, \qquad Z \approx \frac{Z_0}{\sqrt{\epsilon_r'}}\left(1+\frac{i}{2}\frac{\epsilon_r''}{\epsilon_r'}\right)$$

Example: Polyamide (nylon), $\kappa=10^{-8}\,\Omega^{-1}m^{-1}$, $\varepsilon_r=3$, f=10MHz

11% attenuation in 100km, arc $Z \approx 10^{-4}\,°$

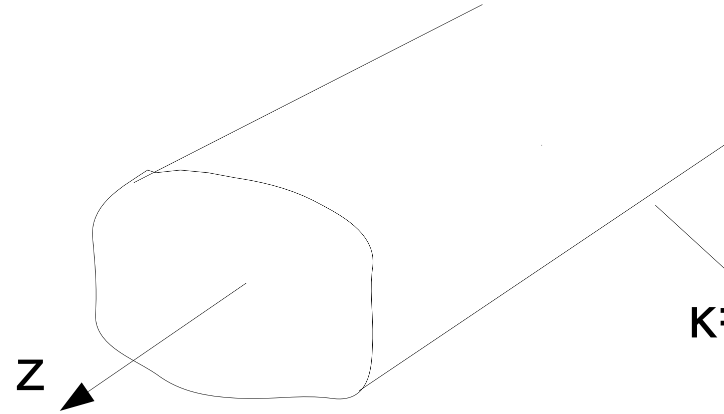
Very good conductors (metallic): ε''≈-iκ/ω » ε'

$$\beta \approx \alpha \approx \sqrt{\frac{\omega\mu\kappa}{2}}, \qquad Z \approx (1+i)\frac{\alpha}{\kappa}, \qquad arc\,Z = 45\,°$$

Skin depth:

$$\alpha\,\delta_S = 1 \quad \rightarrow \quad \delta_S = \sqrt{\frac{2}{\omega\mu\kappa}}$$

Cylindrical, ideal conducting waveguides



$$\vec{\nabla}\cdot\vec{E}=0 \quad \rightarrow \quad \vec{E}^{TE}=\vec{\nabla}\times\vec{A}^{TE}$$

$$\vec{\nabla}\cdot\vec{H}=0 \quad \rightarrow \quad \vec{H}^{TM}=\vec{\nabla}\times\vec{A}^{TM}$$

$e.g.\ \ TE\ waves:$

$$Using\ \vec{\nabla}\times\vec{H}=\epsilon\frac{\partial\vec{E}}{\partial t}\ \ yields \qquad \vec{H}=\vec{\nabla}\Phi+\epsilon\frac{\partial\vec{A}}{\partial t}$$

$$\text{Next using} \quad \vec{\nabla}\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \quad \text{gives}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A} = -\mu\vec{\nabla}\frac{\partial\Phi}{\partial t} - \mu\epsilon\frac{\partial^2\vec{A}}{\partial t^2}$$

Because $\vec{A},\Phi$ *are not fully determined, use Lorenz' gauge*

$$\vec{\nabla}\cdot\vec{A} = -\mu\frac{\partial\Phi}{\partial t}$$

yielding a vectorial wave equation

$$\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0$$

Similarily, we proceed for the TM − case and obtain the same equation.

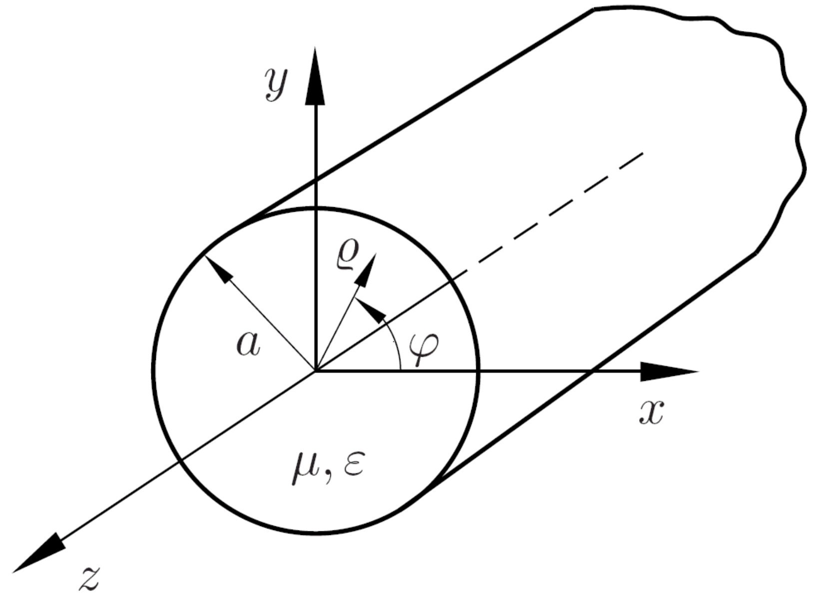
Since only two independent functions are needed, we choose

$$\vec{A}^{TE} = A^{TE}\,\vec{e}_z\,, \qquad \vec{A}^{TM} = A^{TM}\,\vec{e}_z$$

which for time − harmonic fields result in a scalar Helmholtz equation

$$\vec{\nabla}^2 A^p + k^2 A^p = 0\,, \quad k = \frac{\omega}{c} = \omega\sqrt{\mu\,\epsilon}\,, \qquad p = \begin{Bmatrix} TE \\ TM \end{Bmatrix}$$

Circular waveguide



Helmholtz equation:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 A}{\partial\varphi^2}+\frac{\partial^2 A}{\partial z^2}+k^2 A=0$$

Bernoulli ansatz: $A=R(\rho)\Phi(\varphi)Z(z)$

$$\frac{1}{\rho R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+\frac{1}{\rho^2\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}+\underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{-k_z^2}+k^2=0$$

$$\frac{d^2 Z}{dz^2}+k_z^2 Z=0 \;\rightarrow\; Z=A\,e^{-ik_z z}+B\,e^{ik_z z} \;\rightarrow\; e^{-ik_z z}$$

for waves propagating in +z-direction

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+\underbrace{\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}}_{-k_\mu^2}+\rho^2\left(k^2-k_z^2\right)=0$$

$$\frac{d^2\Phi}{d\varphi^2}+k_\mu^2\Phi=0 \;\rightarrow\; \Phi=C\cos\left(k_\mu\varphi\right)+D\sin\left(k_\mu\varphi\right)$$

$$\rightarrow\; C\cos\left(m\varphi\right)$$

because of rotational symmetry and 2π-periodicity

$$\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} + [k^2 - k_z^2 - \frac{m^2}{\rho^2}]R = 0$$

$$R = E J_\mu(K\rho) + F N_\mu(K\rho) \quad \rightarrow \quad J_\mu(K\rho), \quad K = \sqrt{k^2 - k_z^2}$$

because Neumann function is infinite at ρ=0

Vector potential:

$$A = C\cos(m\varphi) J_m(K\rho) e^{-ik_z z}$$

$TE - waves:$ $\quad \vec{E} = \vec{\nabla} \times A\vec{e}_z$

$$E_\varphi = -\frac{\partial A}{\partial \rho} \sim J_m{}'(K\rho)$$

$$E_\varphi(\rho = a) = 0 \quad \rightarrow \quad K_{mn} a = j'_{mn}$$

$$E_\rho = -\frac{m}{\rho} C_{mn} \sin(m\varphi) J_m(j'_{mn}\frac{\rho}{a}) e^{-ik_z z}$$

$$E_\varphi = -\frac{j'_{mn}}{a} C_{mn} \cos(m\varphi) J'_m(j'_{mn}\frac{\rho}{a}) e^{-ik_z z}, \qquad E_z = 0$$

$$\vec{\nabla} \times \vec{E} = -i\omega\mu \vec{H}:$$

$$H_\rho = \frac{k_z}{\omega\mu} \frac{j'_{mn}}{a} C_{mn} \cos(m\varphi) J'_m(j'_{mn}\frac{\rho}{a}) e^{-ik_z z}$$

$$H_\varphi = -\frac{k_z}{\omega\mu} \frac{m}{\rho} C_{mn} \sin(m\varphi) J_m(j'_{mn}\frac{\rho}{a}) e^{-ik_z z}$$

$$H_z = -\frac{j'^2_{mn}/a^2}{i\omega\mu} C_{mn} \cos(m\varphi) J_m(j'_{mn}\frac{\rho}{a}) e^{-ik_z z}$$

$$TM-waves: \quad \vec{H}=\vec{\nabla}\times A\,\vec{e}_z$$

$$E_z=\frac{K^2}{i\omega\epsilon}A\sim J_m(K\rho), \qquad E_z(\rho=a)=0 \;\rightarrow\; K_{mn}a=j_{mn}$$

$$H_\rho=-\frac{m}{\rho}D_{mn}\sin(m\varphi)J_m\left(j_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-ik_z z}$$

$$H_\varphi=-\frac{j_{mn}}{a}D_{mn}\cos(m\varphi)J'_m\left(j_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-ik_z z}, \qquad H_z=0$$

$$\vec{\nabla}\times\vec{H}=i\omega\epsilon\vec{E}:$$

$$E_\rho=-\frac{k_z}{\omega\epsilon}\frac{j_{mn}}{a}D_{mn}\cos(m\varphi)J'_m\left(j_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-ik_z z}$$

$$E_\varphi=\frac{k_z}{\omega\epsilon}\frac{m}{\rho}D_{mn}\sin(m\varphi)J_m\left(j_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-ik_z z}$$

$$E_z=\frac{j_{mn}^2/a^2}{i\omega\epsilon}D_{mn}\cos(m\varphi)J_m\left(j_{mn}\frac{\rho}{a}\right)\mathrm{e}^{-ik_z z}$$

Wave impedance

$$Z_F = \begin{Bmatrix} Z_F^{TE} = \dfrac{E_\rho}{H_\varphi} = -\dfrac{E_\varphi}{H_\rho} = \dfrac{\omega\mu}{k_z} \\ Z_F^{TM} = \dfrac{E_\rho}{H_\varphi} = -\dfrac{E_\varphi}{H_\rho} = \dfrac{k_z}{\omega\epsilon} \end{Bmatrix}$$

Dispersion relation

$$K_{mn}^2 = k^2 - k_{zmn}^2$$

$$k_{zmn} = \sqrt{k^2 - K_{mn}^2} = \sqrt{k^2 - k_{cmn}^2}$$

$$k_{zmn} = \begin{Bmatrix} real & & k > k_{cmn} \\ 0 & for & k = k_{cmn} \\ imaginary & & k < k_{cmn} \end{Bmatrix}$$

$$critical\ wavenumber: \quad k_{cmn} = K_{mn} = \begin{cases} j_{mn}'/a \\ j_{mn}/a \end{cases} \quad for \quad \begin{cases} TE \\ TM \end{cases}$$

$$cutoff\ \ frequency: \quad f_{cmn} = c\, k_{cmn}/2\pi$$

$$cutoff\ wavelength: \quad \lambda_{cmn} = 2\pi/k_{cmn}$$

$$guide\ wavelength: \quad \lambda_{zmn} = 2\pi/k_{zmn} = \frac{\lambda}{\sqrt{1-(\lambda/\lambda_{cmn})^2}}$$

energy flux density:

$$\bar{S}_{cz} = \frac{1}{2}(\vec{E}\times\vec{H}^*)_z = \frac{1}{2} Z_{Fmn}\left[|H_{xmn}|^2 + |H_{ymn}|^2\right]$$

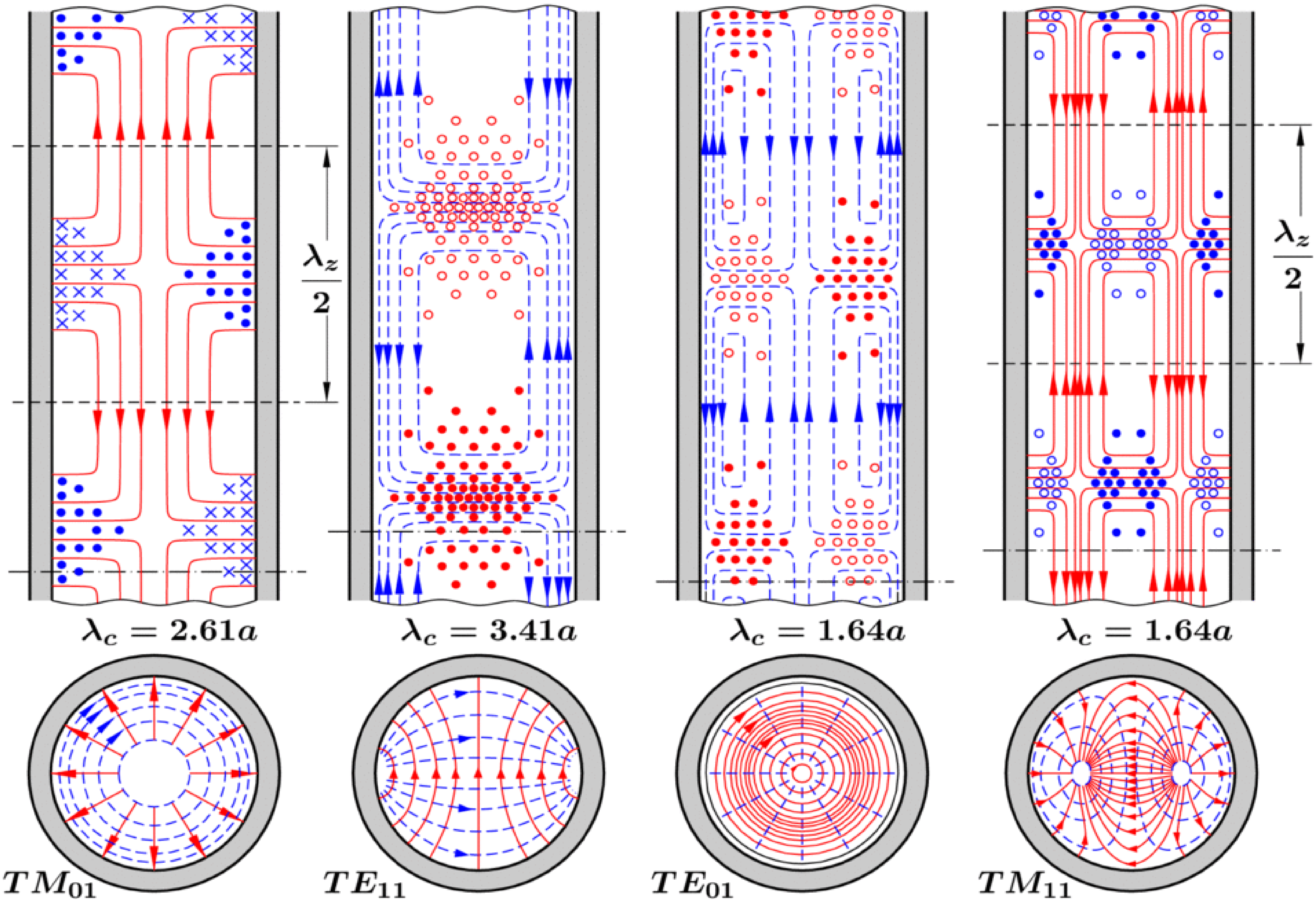
$$= \begin{cases} imaginary \\ 0 \\ real \end{cases} \quad for \quad \begin{cases} k < k_c \\ k = k_c \\ k > k_c \end{cases}$$

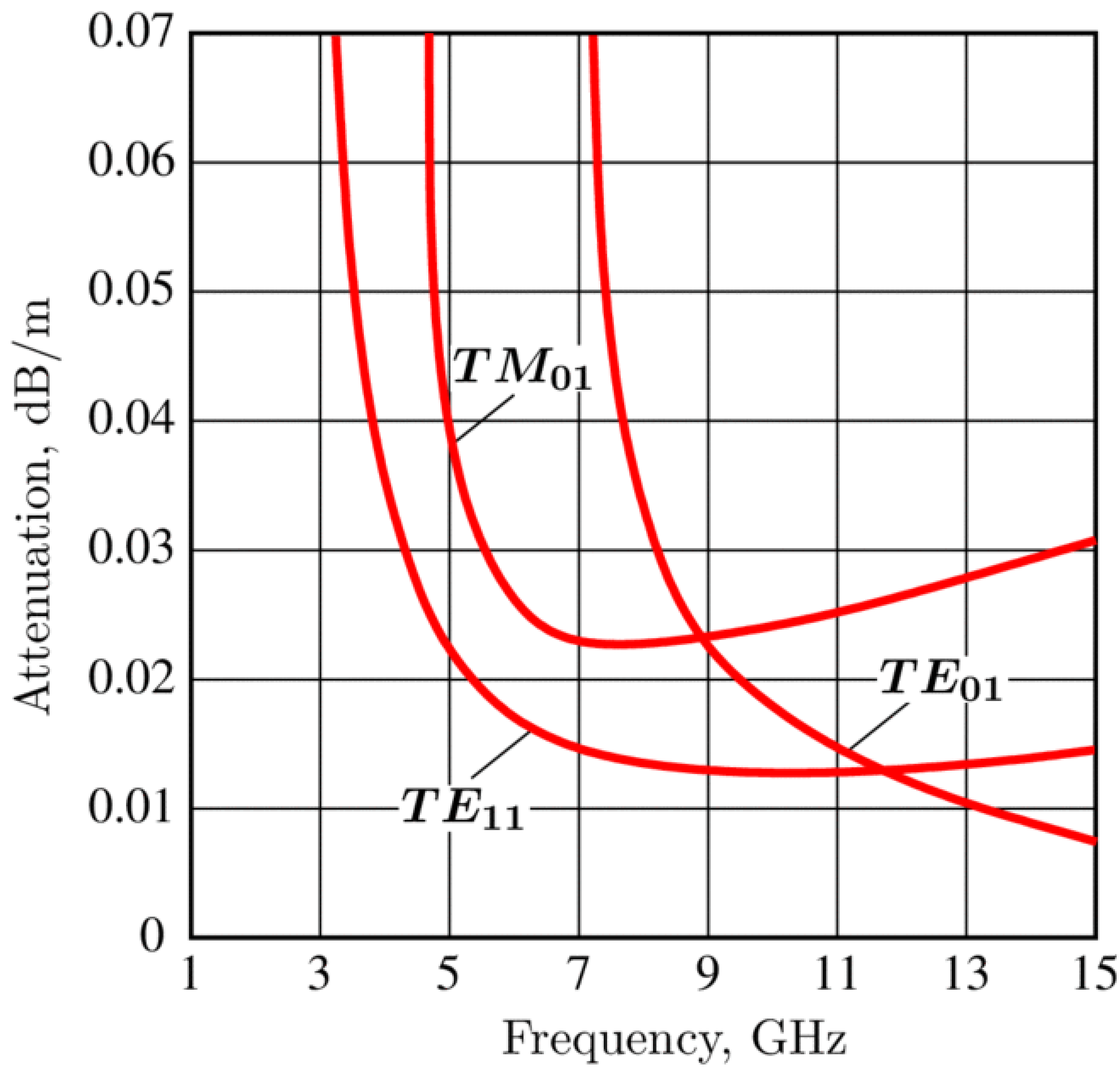
Each mn defines a certain (eigen-) mode. The general solution is the linear combination of all modes

$$\vec{E}=\sum\left(\vec{E}_{mn}^{TE}+\vec{E}_{mn}^{TM}\right),\qquad \vec{H}=\sum\left(\vec{H}_{mn}^{TE}+\vec{H}_{mn}^{TM}\right)$$

Modes are normally sorted referring to their cutoff frequency:

type	m	n	f_c / GHz
TE	1	1	1.76
TM	0	1	2.30
TE	2	1	2.92
TE/TM	0/1	2/1	3.66
TM	3	1	4.01


λ_z
2
$\lambda_c = 2.61a$
$\lambda_c = 3.41a$
$\lambda_c = 1.64a$
$\lambda_c = 1.64a$
TM_{01}
TE_{11}
TE_{01}
TM_{11}


0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
Attenuation, dB/m
TM_{01}
TE_{01}
TE_{11}
1
3
5
7
9
11
13
15
Frequency, GHz

a=2.5 cm

Losses in very good conductors

Fields on metallic surfaces: E≈perp, H≈parallel

$$\vec{E}=\vec{E}_t+E_z\vec{e}_z,\qquad \vec{H}=\vec{H}_t+H_z\vec{e}_z,\qquad \vec{\nabla}=\vec{\nabla}_t+\vec{e}_z\frac{\partial}{\partial z}$$

$$\vec{\nabla}\times\vec{H}=\kappa\vec{E}:\qquad \vec{E}_t=-\frac{1}{\kappa}\vec{e}_z\times\vec{\nabla}_t H_z+\frac{1}{\kappa}\vec{e}_z\times\frac{\partial\vec{H}_t}{\partial z}$$

$$E_z\vec{e}_z=\frac{1}{\kappa}\vec{\nabla}_t\times\vec{H}_t$$

$$\vec{\nabla}\times\vec{E}=-i\omega\mu_0\vec{H}:\qquad \vec{H}_t=-\frac{i}{\omega\mu_0}\vec{e}_z\times\vec{\nabla}_t E_z+\frac{i}{\omega\mu_0}\vec{e}_z\times\frac{\partial\vec{E}_t}{\partial z}$$

$$H_z\vec{e}_z=\frac{i}{\omega\mu_0}\vec{\nabla}_t\times\vec{E}_t$$

Parallel to the surface the typical length of change is λ_0.

In the metal the typical length of change is $\delta_S \ll \lambda_0$.

Order of magnitude approximation: $\quad |\vec{\nabla}_t| \sim \frac{1}{\lambda_0}$

$$|E_z| = \left|\frac{1}{\kappa}\vec{\nabla}_t \times \vec{H}_t\right| \sim \frac{1}{\kappa\lambda_0}|\vec{H}_t| = \pi\left(\frac{\delta_s}{\lambda_0}\right)^2 Z_0 |\vec{H}_t|$$

$$Z_0|H_z| = \left|\frac{i}{\omega\mu_0}\vec{\nabla}_t \times \vec{E}_t\right| \sim \frac{1}{\omega\mu_0}\frac{Z_0}{\lambda_0}|\vec{E}_t| = \frac{1}{2\pi}|\vec{E}_t|$$

$$\left|\frac{1}{\kappa}\vec{e}_z \times \vec{\nabla}_t H_z\right| \sim \frac{1}{\kappa\lambda_0}|H_z| = \pi\left(\frac{\delta_s}{\lambda_0}\right)^2 Z_0|H_z| \sim \frac{1}{2}\left(\frac{\delta_s}{\lambda_0}\right)^2 |\vec{E}_t|$$

$$\left|\frac{i}{\omega\mu_0}\vec{e}_z \times \vec{\nabla}_t E_z\right| \sim \frac{1}{\omega\mu_0\lambda_0}|E_z| = \frac{1}{2\pi Z_0}|E_z| \sim \frac{1}{2}\left(\frac{\delta_s}{\lambda_0}\right)^2 |\vec{H}_t|$$

$$\kappa \vec{E}_t \approx \vec{e}_z \times \frac{\partial \vec{H}_t}{\partial z} = \frac{\partial}{\partial z}(\vec{e}_z \times \vec{H}_t)$$

$$i\omega\mu_0 \vec{H}_t \approx -\vec{e}_z \times \frac{\partial \vec{E}_t}{\partial z} \quad \rightarrow \quad i\omega\mu_0(\vec{e}_z \times \vec{H}_t) = \frac{\partial \vec{E}_t}{\partial z}$$

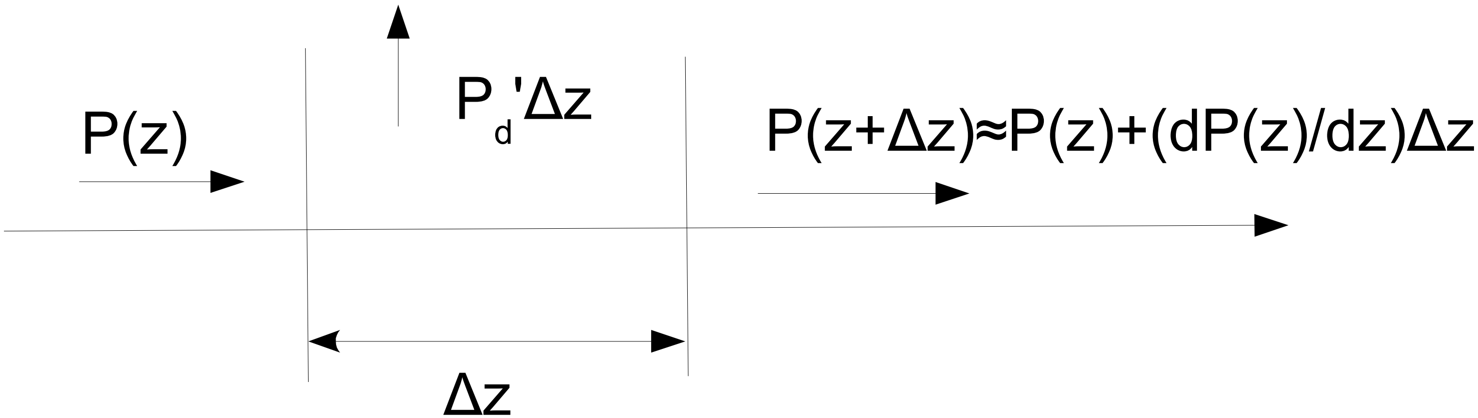
$$\kappa \frac{\partial \vec{E}_t}{\partial z} \approx \frac{\partial^2}{\partial z^2}(\vec{e}_z \times \vec{H}_t) \approx i\omega\mu_0\kappa(\vec{e}_z \times \vec{H}_t)$$

$$\frac{\partial^2 \vec{H}_t}{\partial z^2} - i\omega\mu_0\kappa \vec{H}_t = 0 \quad \rightarrow \quad \vec{H}_t = \vec{H}_{t0} \mathrm{e}^{-(1+i)z/\delta_s}$$

impedance boundary condition:

$$\vec{E}_{t0} \approx \frac{1}{\kappa} \vec{e}_z \times \frac{\partial \vec{H}_{t0}}{\partial z} = Z_W(\vec{n} \times \vec{H}_{t0}), \qquad Z_W = \frac{1+i}{\kappa\delta_s}$$

Attenuation in waveguides (power-loss method)



conservation of power: $$\frac{dP(z)}{dz} = -P_d' = -2\alpha P(z)$$

dissipation per surface area:

$$\frac{\Delta P_d}{\Delta A} = -\vec{n}\cdot\Re(\vec{S}_c) = -\frac{1}{2}\Re(\vec{n}\cdot(\vec{E}\times\vec{H}^*)) = \frac{1}{2}\Re(Z_W)\left|\vec{H}_{tan0}\right|$$

$$= \frac{1}{2\kappa\delta_s}\left|\vec{H}_{tan0}\right|^2$$

dissipation per length:

$$P_d' = \frac{1}{2\kappa\delta} \oint \left|\vec{H}_{tan0}\right|^2 ds$$

transported power:

$$P(z) = \iint \Re(\vec{S}_c) \cdot d\vec{A} = \frac{1}{2} \Re(Z_F) \iint \left|\vec{H}_t\right|^2 dA$$

$$= \frac{1}{2} Z_F \iint \left|\vec{H}_t\right|^2 dA$$

attenuation: $\alpha = \frac{1}{2} \frac{P_d'}{P(z)}$

Resonant cavities

Example: Cylindrical cavity, radius a, length g, TM-modes

Forward plus backward traveling wave from transparency 46

$$E_\varphi = \frac{k_z}{\omega \epsilon} \frac{m}{\rho} D_{mn} \sin(m\varphi) J_m(K_{mn}\rho) [e^{-ik_z z} - r\, e^{ik_z z}]$$

Boundary conditions

$$E_\varphi(z=0)=0 \quad \rightarrow \quad r_{mn}=1, \qquad E_\varphi \sim \sin(k_z z)$$

$$E_\varphi(z=g)=0 \quad \rightarrow \quad k_{zp}\, g = p\pi, \quad p=0,1,2,\ldots$$

Fields:

$$H_\rho = -2\frac{m}{\rho} D_{mnp} \sin(m\varphi)\cos(k_{zp} z) J_m(K_{mn}\rho)$$

$$H_\varphi = -2K_{mn} D_{mnp} \cos(m\varphi)\cos(k_{zp} z) J_m'(K_{mn}\rho), \quad H_z = 0$$

$$E_\rho = i2\frac{k_{zp}}{\omega\epsilon} K_{mn} D_{mnp} \cos(m\varphi)\sin(k_{zp} z) J_m'(K_{mn}\rho)$$

$$E_\varphi = -i2\frac{k_{zp}}{\omega\epsilon}\frac{m}{\rho} D_{mnp} \sin(m\varphi)\sin(k_{zp} z) J_m(K_{mn}\rho)$$

$$E_z = -i2\frac{K_{mn}^2}{\omega\epsilon} D_{mnp} \cos(m\varphi)\cos(k_{zp} z) J_m(K_{mn}\rho)$$

$$K_{mn} = \sqrt{k^2 - k_{zp}^2} = \frac{j_{mn}}{a}$$

Example: TM_{010}-resonator (m=0, n=1, p=0)

$$H_\varphi = 2\frac{j_{01}}{a} D_{010} J_1\left(j_{01}\frac{\rho}{a}\right)$$

$$E_z = -i\frac{2}{\omega\epsilon}\left(\frac{j_{01}}{a}\right)^2 D_{010} J_0\left(j_{01}\frac{\rho}{a}\right)$$

Resonance frequency $k_{010} = \frac{\omega_{010}}{c} = K_{01} = \frac{j_{01}}{a}$

$$f_{010} = \frac{j_{01} c_0}{2\pi a}$$

Stored energy

$$\bar{W} = \bar{W}_e + \bar{W}_m = 2\bar{W}_e = \frac{1}{2}\iiint \vec{E}\cdot\vec{D}^{\star}\, dV = \frac{\epsilon}{2}\iiint |E_z|^2\, dV$$

$$\bar{W} = \frac{2\pi g}{\omega^2 \epsilon} \frac{j_{01}^4}{a^2} |D_{010}|^2 J_1^2(j_{01})$$

Dissipation per unit area

$$\bar{P}_d'' = \frac{1}{2\kappa\delta_s} |\vec{H}_{\tan}|^2$$

$$\bar{P}_d = \frac{4\pi}{\kappa\delta_s} j_{01}^2 \left(1 + \frac{g}{a}\right) |D_{010}|^2 J_1^2(j_{01})$$

Quality factor (Q − value)

$$Q_0 = \frac{\omega_0 \bar{W}}{\bar{P}_d} = \frac{1}{\delta_s} \frac{g}{1 + g/a} \quad \rightarrow \quad \delta_s Q_0 = 2\frac{V}{S} \quad \sim \frac{V}{S}$$

Q_0 gives the decay rate of the stored energy

$$-\frac{d\bar{W}}{dt}=\bar{P}_d=\frac{\omega_0}{Q_0}\bar{W} \quad \rightarrow \quad \bar{W}=\bar{W}_0 e^{-2t/T_f}, \quad T_f=2\frac{Q_0}{\omega_0}$$

Example: 3 Ghz copper cavity, $g=\lambda_0/2=5$ cm

$j_{01}=2.405$, $J_1(j_{01})=0.5191$, $\kappa=58\ 10^6\ \Omega^{-1}m^{-1}$

$A=3.828$ cm, $\delta_S=1.207$ µm, $Q_0=17963$, $T_f=1.9$µs

Resonance behaviour of a cavity mode

Instead of lossy walls assume lossy dielectric filling. That preserves the ideal mode but allows for studying losses.

The cavity is driven by a current J passing through. J splits into a conduction current $J_c = \kappa E$, responsible for the losses in the dielectric, and in an enforced current J_0 as driving term.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = -\mu \frac{\partial}{\partial t} \vec{\nabla} \times \vec{H} =$$

$$= -\mu \frac{\partial}{\partial t} \left(\vec{J}_0 + \kappa \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \right)$$

$$\vec{\nabla}^2 \vec{E} - \mu \kappa \frac{\partial \vec{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \mu \frac{\partial \vec{J}_0}{\partial t} \qquad (1)$$

We expand E in modes

$$\vec{E}=\sum_n a_n(t)\vec{e}_n(x,y,z) \qquad (2)$$

$$\text{where} \quad \vec{\nabla}^2\vec{e}_n+k_n^2\vec{e}_n=0$$

$$\vec{\nabla}\cdot\vec{e}_n=0 \ \text{in volume,} \qquad \vec{n}\times\vec{e}_n=0 \ \text{on walls}$$

$$\iiint \vec{e}_n\cdot\vec{e}_m\,dV=\delta_m^n$$

Substituting (2) in (1) and deviding by με

$$\sum_n\left[\frac{\partial^2 a_n}{\partial t^2}+\frac{\kappa}{\epsilon}\frac{\partial a_n}{\partial t}+\frac{k_n^2}{\mu\epsilon}a_n\right]\vec{e}_n=\frac{1}{\epsilon}\frac{\partial\vec{J}_0}{\partial t} \qquad (3)$$

Multiplying (3) with e_m and integrating over V

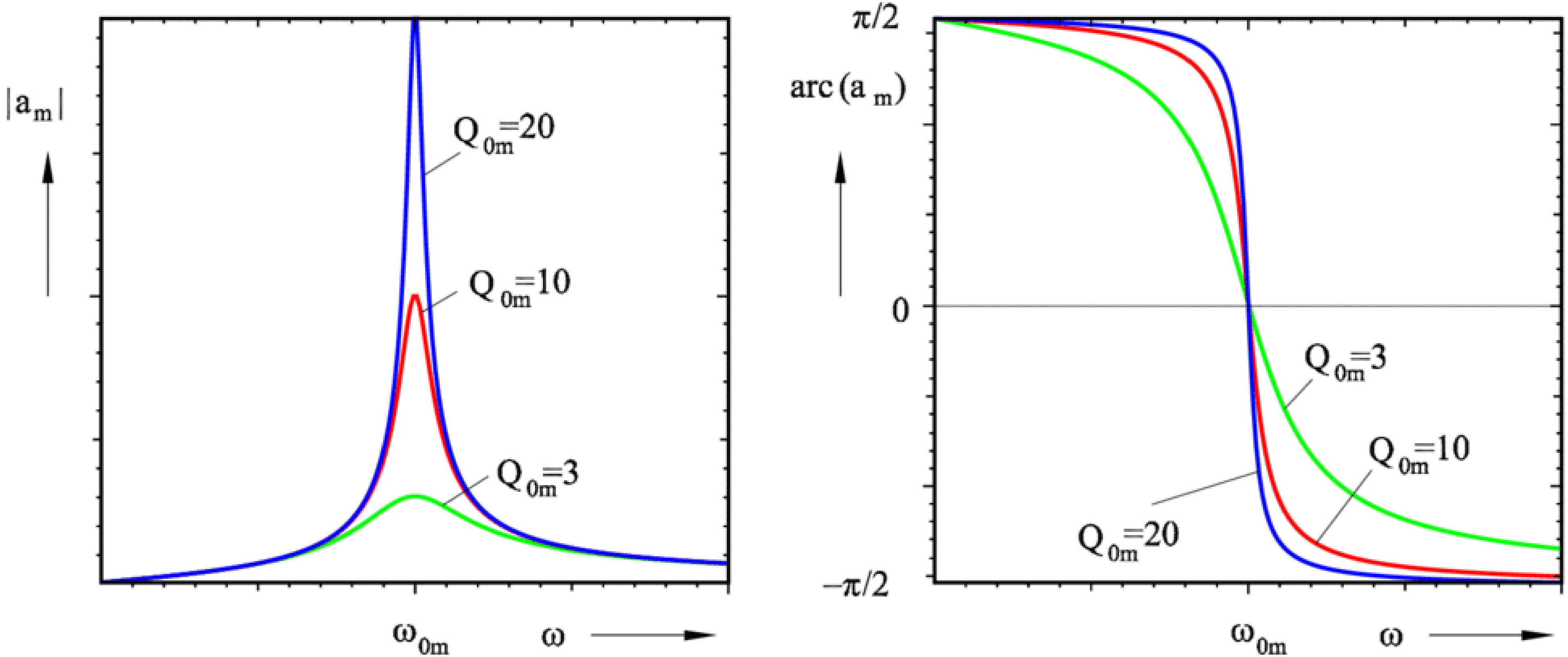
$$\frac{\partial^2 a_m}{\partial t^2} + \frac{\kappa}{\epsilon}\frac{\partial a_m}{\partial t} + \frac{k_n^2}{\mu\epsilon} a_m = \frac{1}{\epsilon}\iiint \frac{\partial \vec{J}_0}{\partial t}\cdot \vec{e}_m dV = \frac{\partial f_m}{\partial t}$$

Time-harmonic excitation

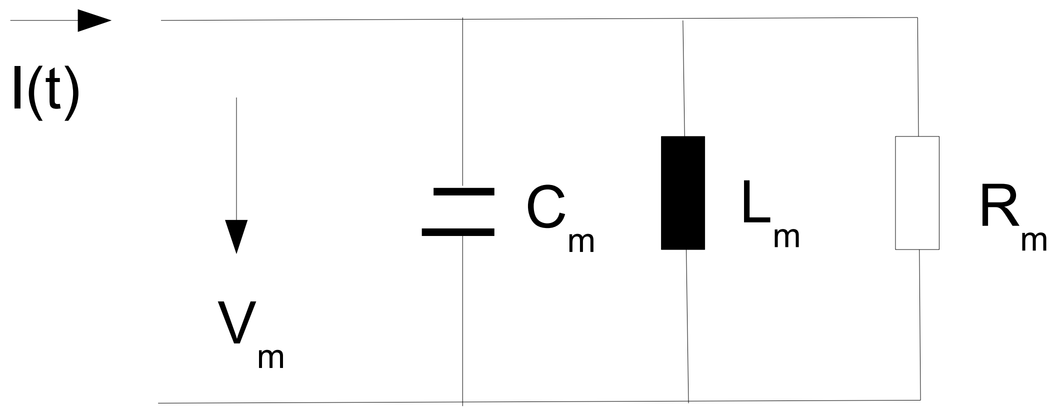
$$[-\omega^2 + i\frac{\kappa}{\epsilon}\omega + \frac{k_m^2}{\mu\epsilon}] a_m = i\omega f_m$$

$$a_m = \frac{Q_{0m}}{\omega_{0m}} \frac{f_m}{1 + iQ_{0m}[\frac{\omega}{\omega_{0m}} - \frac{\omega_{0m}}{\omega}]}$$

$$with \qquad \omega_{0m} = ck_m, \qquad Q_{0m} = \epsilon\,\omega_{0m}/\kappa$$


|a_m|
Q_0m=20
Q_0m=10
Q_0m=3
ω_0m
ω
π/2
arc (a_m)
0
−π/2
Q_0m=3
Q_0m=10
Q_0m=20
ω_0m
ω

Well seperated modes can be represented by a resonator



$$\omega_{0m} = \frac{1}{\sqrt{L_m C_m}}, \quad Q_{0m} = \frac{\omega_{0m} W_m}{P_{dm}} = \omega_{0m} R_m C_m$$

Bandwidth $$B_m = \frac{(\omega_{0m} + \delta\omega) - (\omega_{0m} - \delta\omega)}{\omega_{0m}} = 2\frac{\delta\omega}{\omega_{0m}} = \frac{1}{Q_{0m}}$$

Filling time $$T_{fm} = 2\frac{Q_{0m}}{\omega_{0m}} = \frac{1}{\delta\omega}$$

Accelerating voltage for a particle passing the cavity on-axis

$$V_m = \left| \int_0^g a_m \vec{e}_m \cdot \vec{e}_z \, \mathrm{e}^{i\omega t} \, dz \right|, \qquad z = vt$$

Shunt impedance (amplitude independent)

$$R_{shm} = \frac{V_m^2}{P_{dm}} = 2\mathrm{R}_m$$

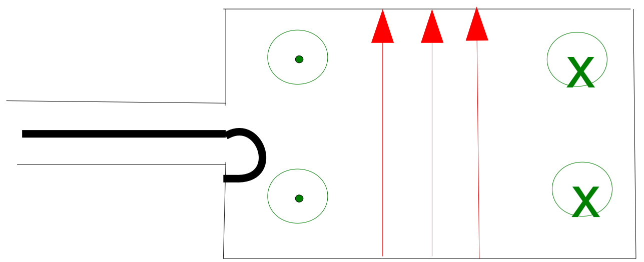
ω_{0m}, B_m and R_{shm} define R_m, L_m, C_m.

R-upon-Q (accelerating voltage for a given stored energy, loss idependent)

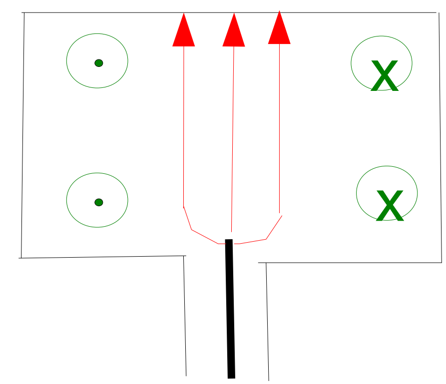
$$\frac{R_{shm}}{Q_{0m}} = \frac{V_m^2}{\omega_{0m} W_m} = \frac{2}{\omega_{0m} C_m}$$

Coupling to a cavity

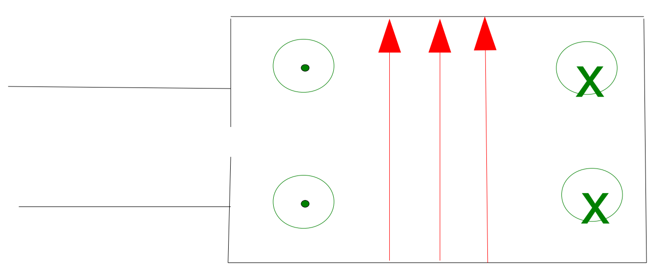
Loop / magnetic coupling



Probe / electric coupling



Electromagnetic coupling



Tut-Ex 7

Give the E- and H-field of a z-polarized plane wave which propagates in x-direction.
What is the radiated power density?

Tut-Ex 8

Derive the longitudinal vector potential for TM-waves in a rectangular waveguide.
What is the equation for the separation constants?

Tut-Ex 9

Give the longitudinal wavelength and phase and group velocity of a TE_{10}-mode in a rectangular waveguide?

Tut-Ex 10

What is the lowest mode in a circular waveguide?
Show the field pattern.
In which frequency range is mono-mode operation possible?

Tut-Ex 11

Calculate the accelerating voltage, shunt impedance and R-upon-Q of a TM_{010}-mode pill-box cavity.