Transverse Beam Dynamics

JUAS tutorial 1 (solutions)

15 January 2013

1 Exercise: local radius, rigidity

We wish to design an electron ring with a radius of R=200 m. Let us assume that only 50% of the circumference is occupied by bending magnets:

- What will be the local radius of bend ρ in these magnets if they all have the same strength?

$$2\pi\rho = 50\% \cdot 2\pi R \longrightarrow \rho = 100 \text{ m}$$

- If the momentum of the electrons is 12 GeV/c, calculate the rigidity $B\rho$ and the field in the dipoles.

Using the rigidity definition:

$$B\rho = 3.3356 \cdot p[\text{GeV}/c] = 40.03 \text{ T·m}$$

and therefore $B = 0.4$ T.

2 Exercise: particle momentum, geometry of a storage ring and thin lenses

The LHC storage ring at CERN will collide proton beams with a maximum momentum of $p = 7$ TeV/c per beam. The main parameters of this machine are:

Circumference	$C_0 = 26658.9$ m	
Particle momentum	$p = 7$ TeV/c	
Main dipoles	$B = 8.392$ T	$l_B = 14.2$ m
Main quadrupoles	$G = 235$ T/m	$l_q = 5.5$ m

- Calculate the magnetic rigidity of the design beam, the bending radius of the main dipole magnets in the arc and determine the number of dipoles that is needed in the machine.

The beam rigidity is obtained in the usual way by the golden rule:

$$B\rho = \frac{p}{e} = \frac{1}{0.299792} \cdot p[\mathrm{GeV}/c] = 3.3356 \cdot p[\mathrm{GeV}/c] = 3.3356 \cdot 7000 \ \mathrm{Tm} = 23349 \ \mathrm{T{\cdot}m}$$

and knowing the magnetic dipole field we get

$$\rho = \frac{3.3356 \cdot 7000\mathrm{Tm}}{8.392\mathrm{T}} = 2782 \ \mathrm{m}$$

The bending angle for one LHC dipole magnet:

$$\theta = \frac{l_B}{\rho} = \frac{14.2\mathrm{m}}{2782\mathrm{m}} = 5.104 \ \mathrm{mrad}$$

and as we want to have a closed storage ring we require an overall bending angle of 2π:

$$N = \frac{2\pi}{\theta} = 1231 \ \text{Magnets}$$

- Calculate the k-strength of the quadrupole magnets and compare its focal length to the length of the magnet. Can this magnet be treated as a thin lens?

We can use the beam rigidity (or the particle momentum) to calculate the normalized quadrupole strength:

$$k = \frac{G}{B\rho} = \frac{G}{p/e} = 0.299792 \cdot \frac{G}{p[\mathrm{GeV/c}]} = 0.299792 \cdot \frac{235\mathrm{T/m}}{7000\mathrm{GeV/c}} = 0.01 \ \mathrm{m}^{-2}$$

an the focal length:

$$f = \frac{1}{k \cdot l_q} = 18.2 \ \mathrm{m} > l_q$$

The focal length of this magnet is still quite bigger than the magnetic length l_q. So it is valid to treat that quadrupole in thin lens approximation.

- What does the matrix for the quadrupoles look like?

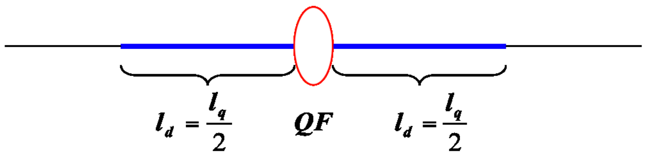
The matrix of a focusing quadrupole is given by

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{|k|}l_q) & \frac{1}{\sqrt{|k|}}\sin(\sqrt{|k|}l_q) \\ -\sqrt{|k|}\sin(\sqrt{|k|}l_q) & \cos(\sqrt{|k|}l_q) \end{pmatrix} = \begin{pmatrix} 0.8525 & 5.22 \\ -0.0522 & 0.8525 \end{pmatrix}$$

In thin lens approximation we replace the matrix above by the expression

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{|f|} & 1 \end{pmatrix} \text{ with the focal length } f = \frac{1}{|kl_q|} = 18.2 \ \mathrm{m}$$

The thin lens description has to be completed by the matrix of a drift space of half the quadrupole length in front and after the thin lens quadrupole. The appropriate description is therefore



So we write

$$M_{thinlens} = \begin{pmatrix} 1 & \frac{l_q}{2} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{l_q}{2} \\ 0 & 1 \end{pmatrix}$$

Multiplying out we get

$$M_{thinlens} = \begin{pmatrix} 1 + \frac{l_q}{2} k l_q & \frac{l_q}{2}(2 + k l_q \frac{l_q}{2}) \\ k l_q & 1 + k l_q \end{pmatrix}$$

With the parameters in the example we get finally

$$M_{thinlens} = \begin{pmatrix} 0.848 & 5.084 \\ -0.055 & 0.848 \end{pmatrix}$$

which is still quite close to the result of the exact calculation above.

3 Exercise: FODO lattice

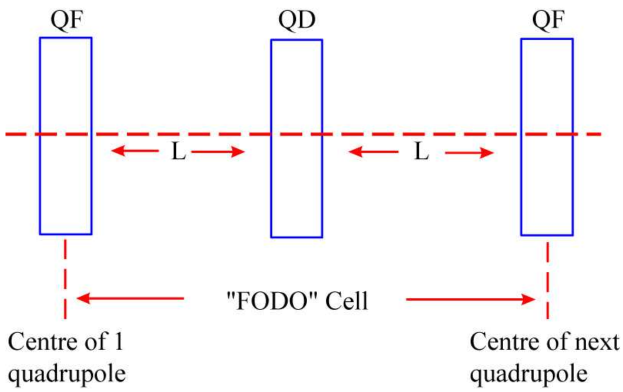
A quadrupole doublet consists of two lenses of focal length f_1 and f_2 separated by a drift length L. Assume that the lenses are thin and show that the transport matrix of this system is

$$M = \begin{pmatrix} 1 - L/f_1 & L \\ -1/f^* & 1 - L/f_2 \end{pmatrix} \text{ where } \frac{1}{f^*} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2}$$

We obtain this matrix from a simple matrix multiplication:

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix}$$

A FODO cell can be considered as the simplest block of the magnetic structure of modern accelerators and storage rings. It consists of a magnet structure of focusing (F) and defocusing (D) quadrupole lenses in alternating order (see schematic below). Its transfer matrix can be calculated using the matrix of the quadrupole doublet (above) with $f_1 = +2f$ and $f_2 = -2f$ followed (and multiplied) by another quadrupole doublet matrix with $f_1 = -2f$ and $f_2 = +2f$.



Show that the transfer matrix of a FODO system in thin lens approximation is as follows:

$$M_{FODO} = \begin{pmatrix} 1-\frac{L^2}{2f^2} & 2L+\frac{L^2}{f} \\ -\frac{L}{2f^2}(1-\frac{L}{2f}) & 1-\frac{L^2}{2f^2} \end{pmatrix}$$

Taking into account the previous result:

$$M_{FODO} = \begin{pmatrix} 1+\frac{L}{2f} & L \\ -\frac{L}{4f^2} & 1-\frac{L}{2f} \end{pmatrix} \begin{pmatrix} 1-\frac{L}{2f} & L \\ -\frac{L}{4f^2} & 1+\frac{L}{2f} \end{pmatrix} = \begin{pmatrix} 1-\frac{L^2}{2f^2} & 2L+\frac{L^2}{f} \\ -\frac{L}{2f^2}(1-\frac{L}{2f}) & 1-\frac{L^2}{2f^2} \end{pmatrix}$$

4 Exercise: Hill equation

Solve the Hill´s equation:

$$y'' + k(s)y = 0$$

by substituting:

$y = A\sqrt{\beta(s)}\cos[\phi(s)+\phi_0]$ with $\phi' = \frac{1}{\beta(s)}$, and where A and ϕ_0 are constants,

demonstrating that a necessary condition is:

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + k(s)\beta^2 = 1$$

The first and second derivative of y:

$$\begin{aligned} y' &= \frac{A}{\sqrt{\beta(s)}}(\frac{\beta'}{2}\cos[\phi(s)+\phi_0] - \sin[\phi(s)+\phi_0]) \\ y'' &= \frac{A}{\sqrt{\beta(s)}}((\frac{\beta'^2}{4} + \frac{\beta''}{2} - \frac{1}{\beta})\cos[\phi(s)+\phi_0] - (\frac{\beta'}{2} + \frac{\beta'}{2\beta})\sin[\phi(s)+\phi_0]) \end{aligned}$$

Substituting in the Hill´s equation

$$\frac{A}{\sqrt{\beta(s)}}((\frac{\beta'^2}{4} + \frac{\beta''}{2} + \beta k(s) - \frac{1}{\beta})\cos[\phi(s)+\phi_0] - (\frac{\beta'}{2} + \frac{\beta'}{2\beta})\sin[\phi(s)+\phi_0]) = 0$$

Since the phase $\phi(s)$ has a different value at every point around the orbit and the amplitude $A \neq 0$, the previous equation can only be satisfied if

$$\begin{array}{rcl} \frac{\beta\beta'^2}{4} + \frac{\beta\beta''}{2} + \beta^2 k(s) - 1 & = & 0 \\ \beta\beta' + \beta' & = & 0 \end{array}$$

and therefore

$$\tfrac{1}{2}\beta\beta'' - \tfrac{1}{4}\beta'^2 + k(s)\beta^2 = 1$$