Introduction to Transverse Beam Dynamics

Lecture 2: Particle trajectories & beams

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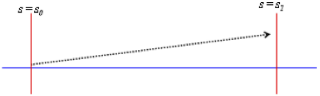
Reminder of lecture 1

Equation of motion:

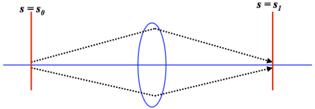
$$x'' + Kx = 0 \qquad \begin{aligned} K &= 1/\rho^2 - k \quad \text{... horiz. plane} \\ K &= k \quad \text{... vert. plane} \end{aligned}$$

Solution of trajectory equations:

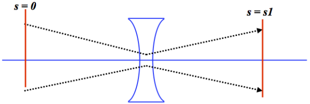
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$



$$M_{\mathbf{D}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$



$$M_{\mathbf{QF}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}} \sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix}$$



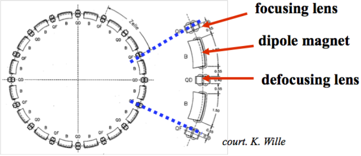
$$M_{\mathbf{QD}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|}L\right) \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) \end{pmatrix}$$

Transformation through a system of lattice elements

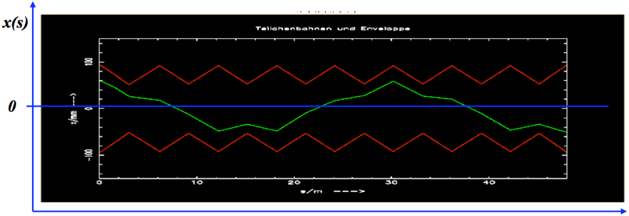
One can compute the solution of a system of elements, by multiplying the matrices of each single element:

$$M_{\text{total}} = M_{\text{QF}} \cdot M_{\text{D}} \cdot M_{\text{Bend}} \cdot M_{\text{D}} \cdot M_{\text{QD}} \cdot \quad \cdots$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = M_{s_1 \to s_2} \cdot M_{s_0 \to s_1} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$



In each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator.



...typical values are:

$$x \approx \text{mm}$$

$$x' \leq \text{mrad}$$

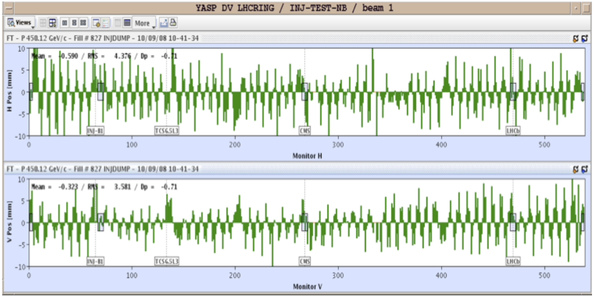
Orbit and Tune

Tune: the number of oscillations per turn.

Example:

64.31

59.32

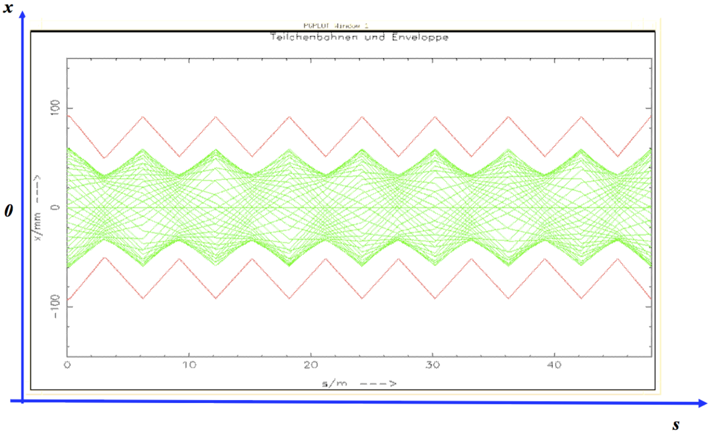


Relevant for beam stability studies is : the non-integer part

Envelope

Question: what will happen, if the particle performs a second turn ?

- ... or a third one or ... 10^{10} turns ...



The Hill's equation

In 19th century George William Hill, one of the greatest master of celestial mechanics of his time, studied the differential equation for "motions with periodic focusing properties": the "Hill's equation"

$$x''(s) + K(s)\,x(s) = 0$$

with:

- a restoring force $\neq$ const
- $K(s)$ depends on the position s
- $K(s+L) = K(s)$ periodic function, where L is the "lattice period"

We expect a solution in the form of a quasi harmonic oscillation: amplitude and phase will depend on the position s in the ring.

The beta function

General solution of Hill's equation:

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)}\cos(\psi(s) + \phi) \qquad (1)$$

ϵ, ϕ =integration constants determined by initial conditions

$\beta(s)$ is a periodic function given by the focusing properties of the lattice $\leftrightarrow$ quadrupoles

$$\beta(s + L) = \beta(s)$$

Inserting Eq. (1) in the equation of motion, we get (Floquet's theorem) the following result

$$\psi(s) = \int_0^s \frac{\mathrm{d}s}{\beta(s)}$$

$\psi(s)$ is the "phase advance" of the oscillation between the points 0 and s in the lattice. For one complete revolution, $\psi(s)$ is the number of oscillations per turn, the "tune"

$$Q = \frac{1}{2\pi}\oint \frac{\mathrm{d}s}{\beta(s)}$$

Beam emittance and phase-space ellipse

General solution of the Hill's equation

$$\begin{cases} x(s) &= \sqrt{\epsilon}\sqrt{\beta(s)}\cos(\psi(s)+\phi) \quad (1) \\ x'(s) &= -\frac{\sqrt{\epsilon}}{\sqrt{\beta(s)}}\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\} \quad (2) \end{cases}$$

From Eq. (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\epsilon}\sqrt{\beta(s)}}$$

$$\alpha(s) = -\frac{1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

Insert into Eq. (2) and solve for ϵ

$$\epsilon = \gamma(s)\,x(s)^2 + 2\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'(s)^2$$

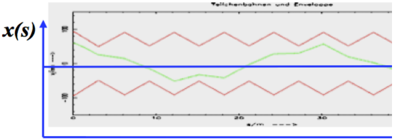
- ϵ is a constant of the motion, independent of s
- parametric representation of an ellipse in the xx' space
- shape and orientation of the ellipse are given by α, β, and γ

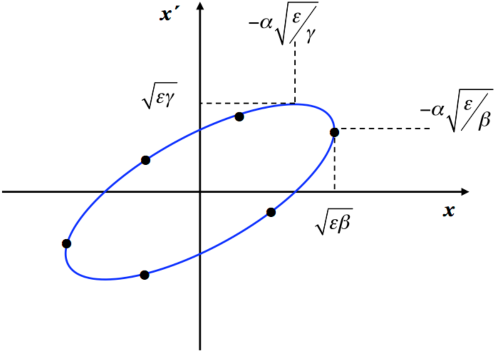
Beam emittance and phase-space ellipse

$$\epsilon = \gamma(s)\,x(s)^2 + 2\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'(s)^2$$

Liouville: in an ideal storage ring, if there is no beam energy change, the area of the ellipse in the phase space $x - x'$ is constant

$$A = \pi \cdot \epsilon = \text{const}$$

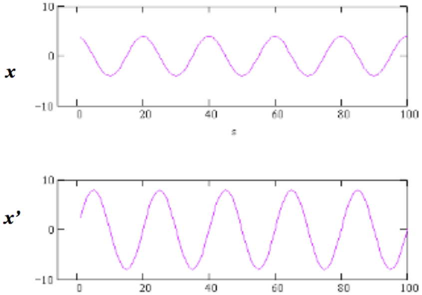


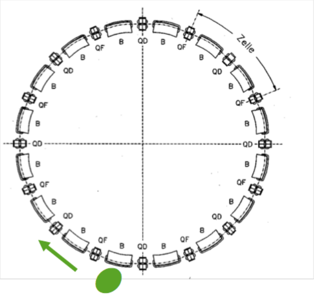


ϵ beam emittance = area of the particle ensemble. It is an intrinsic beam parameter and cannot be changed by the focal properties. In short: the area covered in transverse x, x' phase-space ... and is constant

Particle tracking in a storage ring

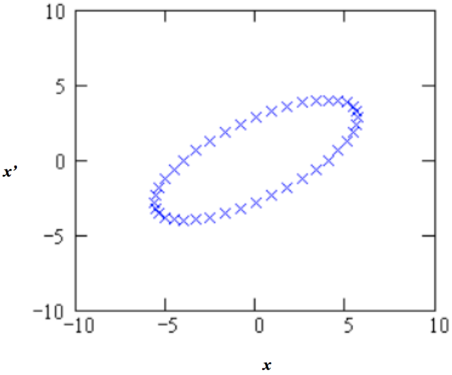
Computation of x and x' for each linear element, according to matrix formalism. We plot x and x' as a function of s

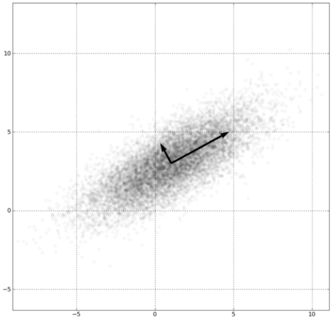




Particle tracking and the beam ellipse

For each turn x, x' at a given position s_1 and plot in the phase-space diagram





Plane: $x - x'$

The emittance and the phase-space ellipse

Given the particle trajectory:

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)}\cos(\psi(s) + \phi)$$

- the max. amplitude is:
$$\hat{x}(s) = \sqrt{\epsilon\beta}$$
- the corresponding angle, in $\hat{x}(s)$, can be found putting $\hat{x}(s) = \sqrt{\epsilon\beta}$ in Eq.
$$\epsilon = \gamma(s)\,x(s)^2 + 2\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'(s)^2$$
and solving for x':
$$\epsilon = \gamma \cdot \epsilon\beta + 2\alpha\sqrt{\epsilon\beta} \cdot x' + \beta x'^2$$
$$\rightarrow \quad x' = -\alpha\sqrt{\frac{\epsilon}{\beta}} \quad \leftarrow$$

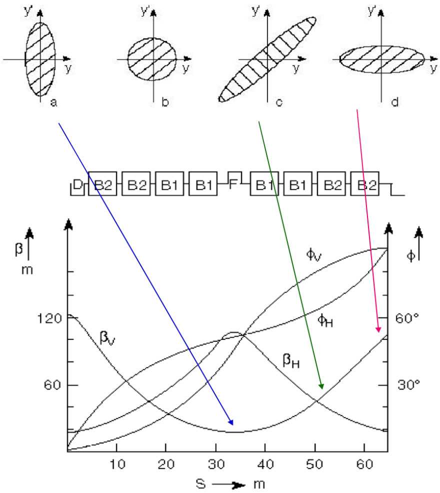
Important remarks:

- A large β-function corresponds to a large beam size and a small beam divergence
- In the middle of a quadrupole, β is maximum, and $\alpha = 0 \Rightarrow x' = 0$

The emittance and the phase-space ellipse

Let's repeat the remarks:

- A large β-function corresponds to a large beam size and a small beam divergence
- In the middle of a quadrupole, β is maximum, and $\alpha = 0 \Rightarrow x' = 0$



The phase-space ellipse

$$\epsilon = \gamma(s)\,x(s)^2 + 2\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'(s)^2$$

$$\alpha(s) = -\frac{1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

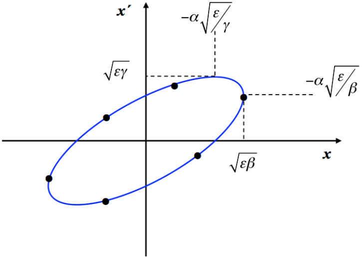
This can be written as

$$\epsilon = \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha \cdot xx' + \beta \cdot x'^2$$

We solve for x', $x'_{1,2} = \frac{-\alpha \cdot x \pm \sqrt{\epsilon\beta - x^2}}{\beta}$ and determine x' max via: $\frac{\mathrm{d}x'}{\mathrm{d}x} = 0$

$$\hat{x}' = \sqrt{\epsilon\gamma}$$

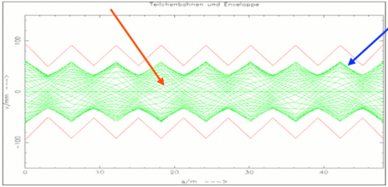
$$\hat{x} = \pm\alpha\sqrt{\frac{\epsilon}{\gamma}}$$



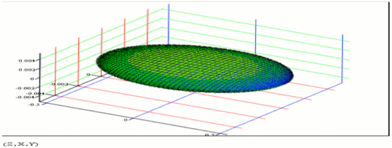
Shape and orientation of the phase-space ellipse depend on the Twiss parameters α, β, and γ.

Emittance of an ensemble of particles

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



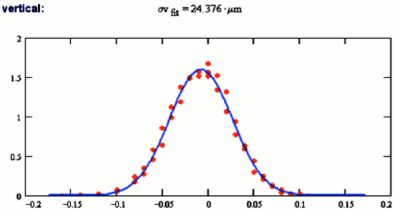
single particle trajectories, $N \approx 10^{11}$ per bunch

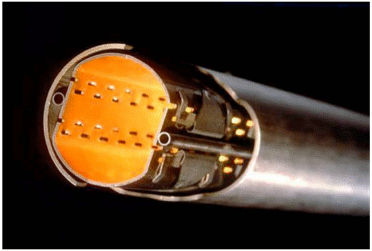


Gauss Particle Distribtion:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

particle at distance 1 σ from centre ↔ 68.3 % of all beam particles

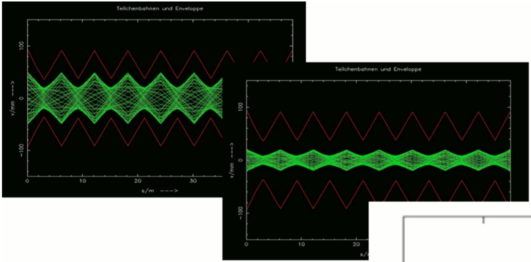




LHC: $\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5*10^{-10}m*180m} = 0.3\ mm$

aperture requirements: $r_0 \geq 10 * \sigma$

Emittance of an ensemble of particles



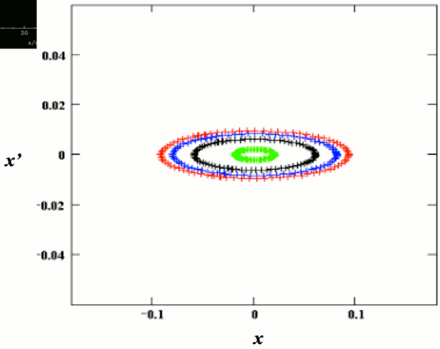
Example: LHC

beam parameters in the arc

$\beta(x) \approx 180\ m$

$\varepsilon \approx 5 * 10^{-10}\ rad \cdot m \qquad (\leftrightarrow 1\sigma)$

$\sigma = \sqrt{\varepsilon\beta} \approx 0.3\ mm$



The transfer matrix M

As we have already seen, a general solution of the Hill's equation is:

$$x(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s) + \phi)$$
$$x'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}}\left[\alpha(s)\cos(\psi(s) + \phi) + \sin(\psi(s) + \phi)\right]$$

Let's remember some trigonometric formulæ:

$$\sin(a \pm b) = \sin a \, \cos b \pm \cos a \, \sin b,$$
$$\cos(a \pm b) = \cos a \, \cos b \mp \sin a \, \sin b, \ \ldots$$

then,

$$x(s) = \sqrt{\epsilon\beta(s)}\left(\cos\psi(s) \, \cos\phi - \sin\psi(s) \, \sin\phi\right)$$
$$x'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}}\left[\alpha(s)\left(\cos\psi(s) \, \cos\phi - \sin\psi(s) \, \sin\phi\right) + \right.$$
$$\left. + \sin\psi(s) \, \cos\phi + \cos\psi(s) \, \sin\phi\right]$$

At the starting point, $s(0) = s_0$, we put $\psi(0) = 0$. Therefore we have

$$\cos\phi = \frac{x_0}{\sqrt{\epsilon\beta_0}}$$
$$\sin\phi = -\frac{1}{\sqrt{\epsilon}}\left(x_0'\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right)$$

If we replace this in the formulæ, we obtain:

$$\underline{x(s)} = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s + \alpha_0\sin\psi_s\right\}\underline{x_0} + \left\{\sqrt{\beta_s\beta_0}\sin\psi_s\right\}\underline{x_0'}$$
$$\underline{x'(s)} = \frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s\right\}\underline{x_0} + \sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s - \alpha_s\sin\psi_s\right\}\underline{x_0'}$$

The linear map follows easily,

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M\begin{pmatrix} x \\ x' \end{pmatrix}_0 \rightarrow \quad M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\psi_s + \alpha_0\sin\psi_s) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos\psi_s - \alpha_s\sin\psi_s) \end{pmatrix}$$

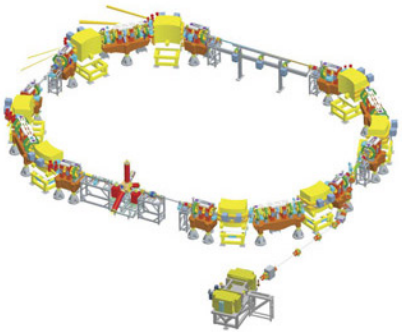
- We can compute the single particle trajectories between two locations in the ring, if we know the α, β, and γ at these positions!

Periodic lattices

The transfer matrix for a particle trajectory

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$

simplifies considerably if we consider one complete turn...



$$M = \begin{pmatrix} \cos\psi_L + \alpha_s \sin\psi_L & \beta_s \sin\psi_L \\ -\gamma_s \sin\psi_L & \cos\psi_L - \alpha_s \sin\psi_L \end{pmatrix}$$

where ψ_L is the phase advance per period

$$\psi_L = \int_s^{s+L} \frac{ds}{\beta(s)}$$

Remember: the tune is the phase advance in units of 2π:

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)} = \frac{\psi_L}{2\pi}$$

Stability criterion

We have seen that if our particles perform one complete turn, the transport map is

$$M = \begin{pmatrix} \cos\psi_L + \alpha_s \sin\psi_L & \beta_s \sin\psi_L \\ -\gamma_s \sin\psi_L & \cos\psi_L - \alpha_s \sin\psi_L \end{pmatrix} =$$

$$= \cos\psi_L \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{I}} + \sin\psi_L \underbrace{\begin{pmatrix} \alpha_s & \beta_s \\ -\gamma_s & -\alpha_s \end{pmatrix}}_{\mathbf{J}}$$

What happens if we consider N turns?

$$M^N = (\mathbf{I}\cos\psi_L + \mathbf{J}\sin\psi_L)^N = \mathbf{I}\cos N\psi_L + \mathbf{J}\sin N\psi_L$$

$\Rightarrow$ The motion for N turns remains bounded if the elements of M^N remain bounded:

$$\psi_L \in \mathbf{R} \qquad |\cos\psi_L| < 1 \qquad |\mathrm{trace}\,(M)| < 2$$

Stability criterion ... the demonstration

Matrix for 1 turn:

$$M = \cos\psi_L \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{I}} + \sin\psi_L \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

Matrix for 2 turns:

$$\begin{aligned} M^2 &= (\mathbf{I}\cos\psi_1 + \mathbf{J}\sin\psi_1)(\mathbf{I}\cos\psi_2 + \mathbf{J}\sin\psi_2) \\ &= \mathbf{I}^2\cos\psi_1\cos\psi_2 + \mathbf{IJ}\cos\psi_1\sin\psi_2 + \mathbf{JI}\sin\psi_1\cos\psi_2 + \mathbf{J}^2\sin\psi_1\sin\psi_2 \end{aligned}$$

now

$$\mathbf{I^2} = \mathbf{I}$$

$$\mathbf{IJ} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \mathbf{J}$$

$$\mathbf{JI} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \mathbf{J}$$

$$\mathbf{J^2} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 - \gamma\beta & \alpha\beta - \beta\alpha \\ -\gamma\alpha + \alpha\gamma & \alpha^2 - \gamma\beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}$$

which brings us to

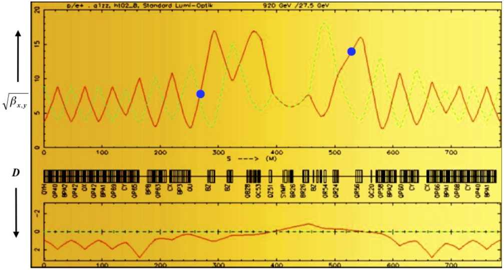
$$M^2 = \mathbf{I}\cos(\psi_1 + \psi_2) + \mathbf{J}\sin(\psi_1 + \psi_2)$$

$$M^2 = \mathbf{I}\cos(2\psi) + \mathbf{J}\sin(2\psi)$$

The transformation for α, β, and γ

Consider two positions in the storage ring: s_0, s

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \quad \text{with} \quad \begin{array}{l} M = M_{\mathbf{QF}} \cdot M_{\mathbf{D}} \cdot M_{\mathbf{Bend}} \cdot M_{\mathbf{D}} \cdot M_{\mathbf{QD}} \cdot \quad \cdots \\ M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix} \end{array}$$



Since the Liouville theorem holds, $\epsilon = \text{const}$:

$$\epsilon = \beta x'^2 + 2\alpha x x' + \gamma x^2$$

$$\epsilon = \beta_0 x_0'^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0^2$$

We express x_0 and x_0' as a function of x and x':

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} = M^{-1} \begin{pmatrix} x \\ x' \end{pmatrix}_{s} \quad \Rightarrow \quad \begin{aligned} x_0 &= S'x - Sx' \\ x_0' &= -C'x + Cx' \end{aligned}$$

Inserting into ϵ we obtain:

$$\begin{aligned} \epsilon &= \beta x'^2 + 2\alpha x x' + \gamma x^2 \\ \epsilon &= \beta_0 \left(-C'x + Cx'\right)^2 + 2\alpha_0 \left(S'x - Sx'\right)\left(-C'x + Cx'\right) + \gamma_0 \left(S'x - Sx'\right)^2 \end{aligned}$$

We need to sort by x and x':

$$\begin{aligned} \beta(s) &= C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0 \\ \alpha(s) &= -CC'\beta_0 + \left(SC' + S'C\right)\alpha_0 - SS'\gamma_0 \\ \gamma(s) &= C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0 \end{aligned}$$

The transformation for α, β, and γ

In matrix notation:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

1. this expression is important, and useful
2. given the twiss parameters α, β, γ at any point in the lattice we can transform them and compute their values at any other point in the ring
3. the transfer matrix is given by the focusing properties of the lattice elements, the elements of M are just those that we used to compute single particle trajectories
4. go back to point 1.

Summary

equation of motion: $x''(s) + K(s)\,x(s) = 0, \quad K = 1/\rho - k$

general solution of the Hill's equation: $x(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s) + \phi)$

phase advance & tune: $\psi_{12} = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}, \quad Q = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}$

emittance: $\epsilon = \gamma(s)\,x(s)^2 + 2\alpha(s)\,x(s)\,x'(s) + \beta(s)\,x'(s)^2$

transfer matrix $s_1 \to s_2$: $M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}(\cos\psi_s + \alpha_0 \sin\psi_s) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}(\cos\psi_s - \alpha_s\sin\psi_s) \end{pmatrix}$

matrix for 1 turn: $M = \begin{pmatrix} \cos\psi_L + \alpha_s\sin\psi_L & \beta_s\sin\psi_L \\ -\gamma_s\sin\psi_L & \cos\psi_L - \alpha_s\sin\psi_L \end{pmatrix}$

stability criterion: $|\mathrm{trace}(M)| < 2$

Bibliography

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