Introduction to Transverse Beam Dynamics

Lecture 3: Lattice design

Andrea Latina (CERN)

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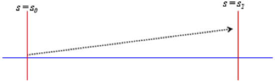
Reminder of the previous lectures

Equation of motion:

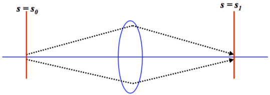
$$x'' + Kx = 0 \qquad \begin{aligned} K &= 1/\rho^2 - k \quad \text{... horiz. plane} \\ K &= k \quad \text{... vert. plane} \end{aligned}$$

Solution of trajectory equations:

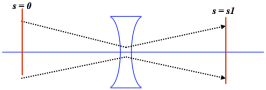
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}$$



$$M_{\mathbf{D}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$



$$M_{\mathbf{QF}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$



$$M_{\mathbf{QD}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

$$M_{\text{total}} = M_{\text{QF}} \cdot M_{\text{D}} \cdot M_{\text{Bend}} \cdot M_{\text{D}} \cdot M_{\text{QD}} \cdot \quad \cdots$$

$$\frac{1}{f} = k\, L_{\text{Q}} \qquad \text{focal length}$$

Beam emittance and phase-space ellipse

General solution of the Hill's equation

$$\begin{cases} x(s) &= \sqrt{\epsilon}\sqrt{\beta(s)}\cos(\psi(s)+\phi) \quad (1) \\ x'(s) &= -\frac{\sqrt{\epsilon}}{\sqrt{\beta(s)}}\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\} \quad (2) \end{cases}$$

To determine the *Twiss parameters* α, β, and γ from Eq. (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\epsilon}\sqrt{\beta(s)}}$$

$$\alpha(s) = -\frac{1}{2}\beta'(s) \qquad \text{beam divergence}$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

that we insert into Eq. (2) and solve for ϵ

$$\epsilon = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$

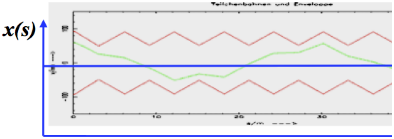
- ϵ is a constant of the motion, independent of s
- parametric representation of an ellipse in the xx' space
- shape and orientation of the ellipse are given by α, β, and γ

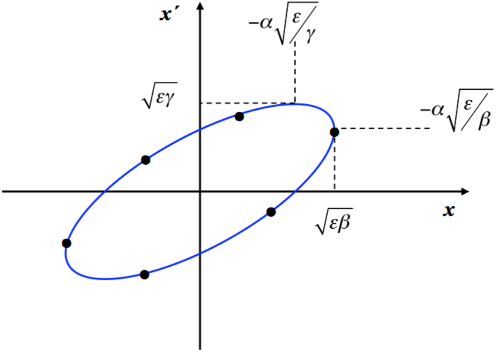
Beam emittance and phase-space ellipse

$$\epsilon = \gamma(s)\, x(s)^2 + 2\alpha(s)\, x(s)\, x'(s) + \beta(s)\, x'(s)^2$$

Liouville: in an ideal storage rings, the area of the ellipse in the phase space is constant

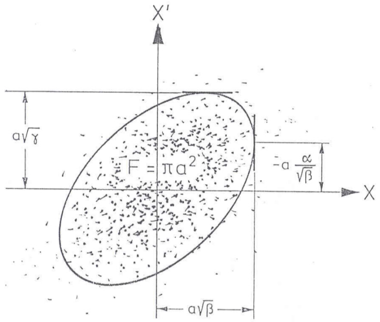
$$A = \pi \cdot \epsilon = \text{const}$$





ϵ beam emittance = with lots of particles, it's the area of the particle ensemble. It is an intrinsic beam parameter that cannot be changed by the focal properties. In short: it's the area covered in transverse x, x' phase-space ... and is constant

Beam emittance



$$A = \pi \cdot \epsilon = \text{const}$$

- A particle beam is reasonably well described by a two dimensional Gaussian distribution in phase space
- The lines of constant phase-space density are then ellipses
- Since the phase-space density decreases only slowly with amplitude, the phase-space area containing all particles might be hard to determine (experimentally as well as theoretically)
- Also, it is not the quantity relevant for most of the applications. Therefore, the emittance is defined as $1/\pi$ times the phase-space area containing a certain fraction of the particles (e.g. 90%).

The transfer matrix M

- Transformation of particle coordinates:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M_{2\times 2} \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

- using matrix notation in terms of the magnet parameter K:

$$M_{\text{foc}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}} \sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

- in Twiss form, i.e. for a periodic system:

$$M_{\text{Twiss}} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} = \cos\mu \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} + \sin\mu \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{J}$$

with $\cos\mu = \frac{1}{2}\text{trace}\,(M)$

- Transport of Twiss parameters:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

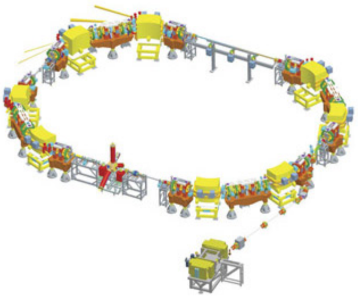
Lattice design in particle accelerators

Or..."how to build a storage ring"

High energy accelerators — are mostly circular machines
we need to juxtapose a number of **dipole** magnets,
to bend the design orbit to a closed ring, then add
quadrupole magnets (FODO cells) to focus the beam
transversely

The geometry of the system is determined by the following equality

$$\text{centrifugal force} = \text{Lorentz force}$$



$$\begin{aligned} \text{Lorentz force} \quad F_L &= evB \\ \text{Centrifugal force} \quad F_{centr} &= \frac{\gamma m v^2}{\rho} \\ \frac{\gamma m v^{\not 2}}{\rho} &= e\not v B \end{aligned}$$

$$\frac{p}{q} = B\rho$$

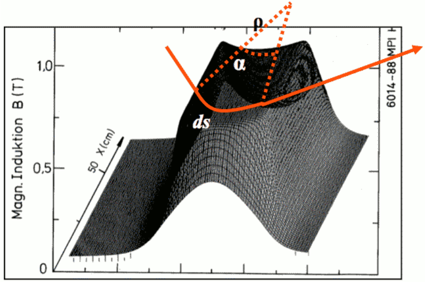
$$B\rho = \text{"beam ridigity"}$$

Lattice design: the magnetic guide

$$B\rho = p/q$$

Circular orbit: the dipole magnets define the geometry

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{Bdl}{B\rho}$$



field map of a storage ring dipole magnet

The angle spanned in one revolution must be 2π, so, for a full circle:

$$\alpha = \frac{\int Bdl}{B\rho} = 2\pi \quad \rightarrow \quad \int Bdl \approx N L_{\mathrm{Bend}} B = 2\pi \frac{p}{q}$$

this defines the integrated dipole field around the machine.

Note that usually $\frac{\Delta B}{B} \approx 10^{-4}$ is required!

7000 GeV proton storage ring

$N = 1232$ ***dipole magnets***

$L_{Bend} = 15 \; \boldsymbol{m}$

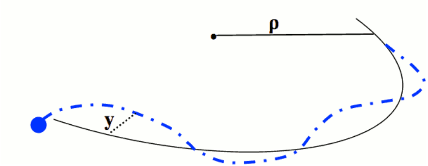
$q = +e$

$$\int B d\boldsymbol{l} \approx N L_{Bend} B = 2\pi p/e$$

$$B \approx \frac{2\pi \cdot 7000 \cdot 10^9 \; \boldsymbol{eV}}{1232 \cdot 15 \; \boldsymbol{m} \cdot 3 \cdot 10^8 \frac{\boldsymbol{m}}{\boldsymbol{s}} e} = 8.3 \; \boldsymbol{T}$$

Focusing forces for single particles

$$x'' + Kx = 0$$



$$K = -k + 1/\rho^2 \quad \text{hor. plane}$$
$$K = k \quad \text{vert. plane}$$

$$\left.\begin{array}{lll} \text{dipole magnet} & \frac{1}{\rho} & = \frac{B}{p/q} \\ \text{quadrupole magnet} & k & = \frac{g}{p/e} \end{array}\right\}$$

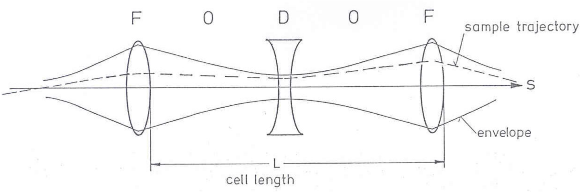
Example: the LHC ring
Bending radius: $\rho = 2.53$ km
Quad gradient: $g = 220$ T/m

$k = 9.4 \cdot 10^{-3}$ m^{-2}
$1/\rho^2 = 1.3 \cdot 10^{-7}$ m^{-2}

For estimates, in large accelerators, the weak focusing term $1/\rho^2$ can in general be neglected

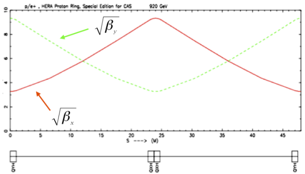
The FODO lattice

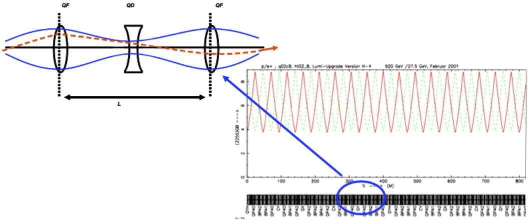
- Most high energy accelerators or storage rings have a periodic sequence of quadrupole magnets of alternating polarity in the arcs



- A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between
- Nota bene: "nothing" here means the elements that can be neglected on first sight: drift, bending magnet, RF structures ... and experiments...

Periodic solution in a FODO Cell





Output of MAD-X

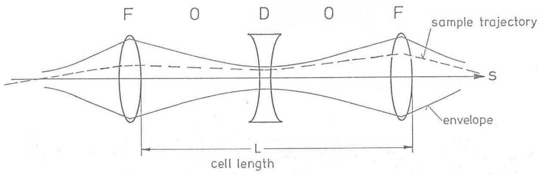
Nr	Type	Length	Strength	β_x	α_x	φ_x	β_z	α_z	φ_z
		m	1/m2	m		$1/2\pi$	m		$1/2\pi$
0	IP	0,000	0,000	11,611	0,000	0,000	5,295	0,000	0,000
1	QFH	0,250	-0,541	11,228	1,514	0,004	5,488	-0,781	0,007
2	QD	3,251	0,541	5,488	-0,781	0,070	11,228	1,514	0,066
3	QFH	6,002	-0,541	11,611	0,000	0,125	5,295	0,000	0,125
4	IP	6,002	0,000	11,611	0,000	0,125	5,295	0,000	0,125

QX= 0,125 QZ= 0,125

$0.125 * 2\pi = 45^0$

The FODO cell

The transfer matrix gives all the information we need.



In thin-lens approximation, we have:

$$M_\text{F} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}; \qquad M_\text{O} = \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix}; \qquad M_\text{D} = \begin{pmatrix} 1 & 0 \\ +\frac{1}{f} & 1 \end{pmatrix}$$

the transformation matrix of the cell is:

$$M_\text{FODO} = M_\text{F} \cdot M_\text{O} \cdot M_\text{D} \cdot M_O$$

(notice that you can also write $M = M_{\text{F}/2} \cdot M_\text{O} \cdot M_\text{D} \cdot M_O \cdot M_{\text{F}/2}$, or other permutations), which corresponds to

$$M_\text{FODO} = \begin{pmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{2L}{f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix}$$

The FODO cell (cont.)

If we compare the previous matrix with the Twiss representation over one period,

$$M_{\text{FODO}} = \begin{pmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{2L}{f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix}$$

$$M_{\text{Twiss}} = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix} = \cos\mu \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathrm{I}} + \sin\mu \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathrm{J}}$$

we can derive interesting properties.

- Phase advance

$$\cos\mu = \frac{1}{2}\text{trace}\,(M) = 1 - \frac{L^2}{8f^2}$$

remembering that $\cos\mu = 1 - 2\sin^2\frac{\mu}{2}$

$$\left|\sin\frac{\mu}{2}\right| = \frac{L}{4f}$$

This equation allows to compute the phase advance per cell from the cell length and the focal length of the quadrupoles.

The FODO cell (cont.)

- Example: compute the focal length in order to have a phase advance of 90° per cell

$$f = \frac{1}{\sqrt{2}} \frac{L}{2}$$

  line an emittance measurement station

- Stability requires that $|\cos \mu| < 1$, that is

$$\frac{L}{4f} < 1 \qquad \rightarrow \qquad \text{stability is for:} \quad f > L/4 \quad (\text{or } L < 4f)$$

- Compute the phase advance per cell from the transfer matrix: From $\cos \mu = \frac{1}{2}\text{trace}\,(M)$

$$\mu = \arccos\left(\frac{1}{2}\text{trace}\;(M)\right)$$

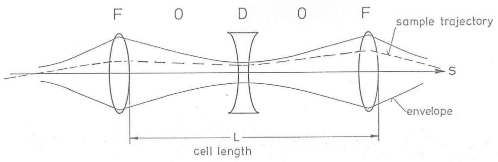
- Computeβ-function and α parameter

$$\beta = \frac{M_{12}}{\sin \mu}$$

$$\alpha = \frac{M_{11} - \cos \mu}{\sin \mu}$$

The FODO cell: useful formulae

For a FODO cell like in figure, with two thin quads separated by length $L/2$



one has:

$$f = \pm \frac{L}{4 \sin \frac{\mu}{2}}$$

$$\beta^{\pm} = \frac{L \left(1 \pm \sin \frac{\mu}{2}\right)}{\sin \mu}$$

$$\alpha^{\pm} = \frac{\mp 1 - \sin \frac{\mu}{2}}{\cos \frac{\mu}{2}}$$

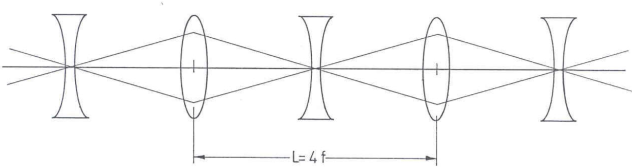
$$D^{\pm} = \frac{L\phi \left(1 \pm \frac{1}{2} \sin \frac{\mu}{2}\right)}{4 \sin^2 \frac{\mu}{2}}$$

ϕ is the total bending angle of the whole cell.

The FODO cell (example 1)

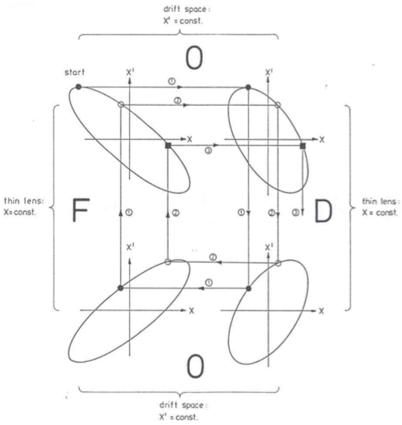
The limiting case $L = 4f$ has a simple interpretation.

- It is well known from optics that an object at a distance $a = 2f$ from a focusing lens has its image at $b = 2f$



- The defocusing lenses have no effect if a point-like object is located exactly on the axis at distance $2f$ from a focusing lens, because they are traversed on the axis
- If however the lens system is moved further apart ($L > 4f$), this is no more true and the divergence of the light or particle beam is increased by every defocusing lens

The FODO cell (example 2)



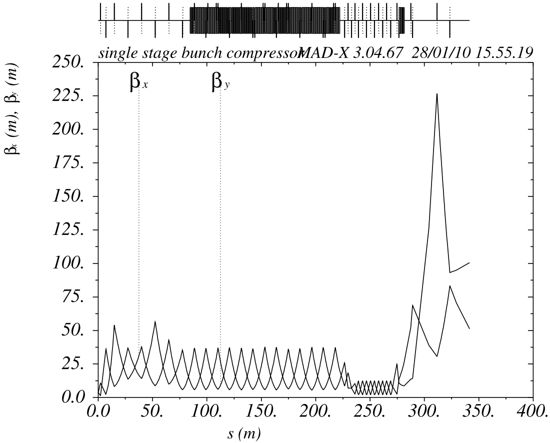
- Phase space dynamics in a simple circular accelerator consisting of one FODO cell with two 180° bending magnets located in the drift spaces (the O's)
- The periodicity of α, β, and γ is reflected by the fact that the phase-space ellipse is transformed into itself after each turn
- An individual particle trajectory, however, which starts, for instance, somewhere on the ellipse at the exit of the focusing quadrupole (small circle), is seen to move on the ellipse from turn to turn as determined by the phase angle μ
- Thus, an individual particle trajectory is not periodic, while the envelope of a whole beam is

Non-periodic beam optics

- In the previous sections the Twiss parameters α, β, γ, and μ have been derived for a periodic, circular accelerator. The condition of periodicity was essential for the definition of the beta function (Hill's equation)
- Often, however, a particle beam moves only **once** along a **beam transfer line**, but one is nonetheless interested in quantities like beam envelopes and beam divergence
- In a circular accelerator α, β, and γ are completely determined by the magnet optics and the condition of periodicity (beam properties are not involved - only the beam emittance is chosen to match the beam size)
- In a transfer lineα, β, and γ are no longer uniquely determined by the transfer matrix, but they also depend on initial conditions which have to be specified in an adequate way

Non-periodic beam optics (example)

Optics of a non-periodic system including non-periodic optics. "Matching" sections connect parts with different periodic conditions.



The matrix

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = M_{3\times 3} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

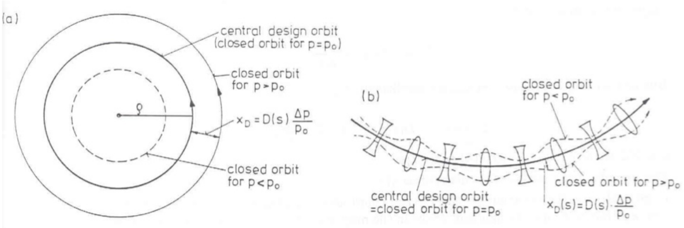
with

$$M_{3\times 3} = \begin{pmatrix} c^2 & -2sc & s^2 \\ -cc' & sc' + s'c & -ss' \\ c'^2 & -2s'c' & s'^2 \end{pmatrix}$$

allows to compute the magnets parameters for the matching sections

Introducing dispersion

- in a circular particle accelerator, a particle with $p = p_0$ and $x = y = x' = y' = 0$ (i.e. zero displacement and zero slope) will move on the design orbit for an arbitrary number of revolutions
- particles with $p = p_0$ but non-zero displacement and/or slope will perform betatron oscillation with a certain tune Q
- particles with momentum $p \neq p_0$ will no longer move on the design orbit



Closed orbit for particles with momentum $p \neq p_0$ in a weakly (a) and strongly (b) focusing circular accelerator.

Solution of the inhomogeneous Hill's equation

A particle with $\Delta p = p - p_0 \neq 0$ satisfies the inhomogeneous Hill equation for the horizontal motion:

$$x''(s) + K(s)\,x(s) = \frac{1}{\rho}\frac{\Delta p}{p_0}$$

the total deviation of the particle from the reference orbit can be written as

$$x(s) = x_\beta(s) + x_D(s)$$

where:

- $x_D(s) = D(s)\frac{\Delta p}{p_0}$ describes the deviation of the closed orbit for off-momentum particles p_0 with a fixed Δp from the reference orbit, where $D(s)$ is the solution of the equation

$$D''(s) + K(s)\,D(s) = \frac{1}{\rho}$$

- $x_\beta(s)$ describes the betatron oscillation around this closed dispersion orbit, solution if the homogeneous equation $x_\beta''(s) + K(s)\,x_\beta(s) = 0$

Dispersion function and orbit

The dispersion function $D(s)$ is the solution of the inhomogeneous Hill's equation:

$$D''(s) + K(s)\,D(s) = \frac{1}{\rho}$$

The Dispersion function $D(s)$:

- is that special orbit that an ideal particle would have for $\Delta p/p = 1$

The orbit $x(s) = x_\beta(s) + x_D(s)$, with $x_D(s) = D(s)\,\frac{\Delta p}{p_0}$, can be rewritten in matrix formalism

$$\begin{cases} x(s) = x_\beta(s) + x_D(s) \\ x(s) = C(s)\,x_0 + S(s)\,x_0' + D(s)\,\dfrac{\Delta p}{p} \end{cases}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}_0$$

Summary

integrated dipole field over a turn	$\int B dl \approx N L_{\text{Bend}} B = 2\pi \frac{p}{q}$
FODO cell	$M_{\text{FODO}} = \begin{pmatrix} 1 + \frac{L}{2f} & L + \frac{L^2}{4f} \\ -\frac{2L}{f^2} & 1 - \frac{L}{2f} - \frac{L^2}{4f^2} \end{pmatrix}$
stability in a FODO cell	$f > L/4$
phase advance in a FODO cell	$\mu = \arccos\left(\frac{1}{2}\text{trace}\left(M\right)\right)$
there exist matching sections	$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = M_{3\times 3} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$
inhomogeneous Hill's equation	$x'' + \left(\frac{1}{\rho^2} - k\right) x = \frac{1}{\rho} \frac{\Delta p}{p_0}$
...and its solution	$x\left(s\right) = x_\beta\left(s\right) + D\left(s\right) \frac{\Delta p}{p}$

Appendix: The stability criterion demonstrated

Recall the matrix for a period L:

$$M = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix} =$$

$$= \cos\mu \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{I}} + \sin\mu \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{\mathbf{J}}$$

We have seen that

$$M^N = (\mathbf{I}\cos\mu + \mathbf{J}\sin\mu)^N = \mathbf{I}\cos N\mu + \mathbf{J}\sin N\mu$$

We want to have that

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M^N \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \qquad \text{remains bounded for } N \to \infty$$

A necessary and sufficient condition for stable motion is that the elements of the matrix M^N remain bounded for $N \to \infty$. To derive a condition for this, we need to consider the eigenvalues of the matrix M

Appendix: The stability condition demonstrated (cont.)

$X = M^N X_0$ is stable if

$$X = M^N (AV_1 + BV_2)$$

where we decomposed the vector X_0 into the eigenvectors on M: V_1 and V_2. If $\lambda_{1,2}$ are the corresponding eigenvalues:

$$X = M^N (AV_1 + BV_2) = A\lambda_1^N V_1 + B\lambda_2^N V_2 \qquad (\star)$$

As $\det(M) = 1$*:* $\quad \det(M) = 1 = \lambda_1 \lambda_2 \rightarrow \lambda_2 = \frac{1}{\lambda_1} \rightarrow \lambda_{1,2} = e^{\pm i\mu}$.
Eq. $(\star)$ is stable if μ is real. If μ is imaginary it gives exponential grow. From the characteristic equation $\det(M - \lambda I) = 0$

$$\det(M - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

we have $(a - \lambda)(d - \lambda)$-bc=0

$$\begin{aligned} &\lambda^2 - (a + d)\lambda + (ad - bc) = 0 \\ &\lambda^2 - \text{trace}(M)\lambda + 1 = 0 \\ &\lambda + 1/\lambda = \text{trace}(M) \\ &e^{i\mu} + e^{-i\mu} = 2\cos\mu = \text{trace}(M) \end{aligned}$$

So the stability is achieved when

$$\mu \in \mathrm{R} \quad \rightarrow \quad |\text{trace}(M)| < 2$$