Non-linear imperfections

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References

- O. Bruning, Non-linear dynamics, JUAS courses, 2006.
- O. Bruning, Non-linear imperfections, CERN Accelerator School Intermediate level, courses, 2009.
- M. Tabor, Chaos and Integrability in Nonlinear Dynamics, An Introduction, Willey, 1989.
- H. Wiedemann, Particle accelerator physics, 3rd edition, Springer 2007

Summary

- Oscillators and resonance condition
- Field imperfections and normalized field errors
- Perturbation treatment for a sextupole
- Poincaré section
- Chaotic motion
- Octupole effect and fringe fields
- Singe-particle diffusion
  - Dynamic aperture
  - Frequency maps

Harmonic oscillator including damping

- Damped harmonic oscillator:

$$\frac{d^2u(t)}{dt^2} + \frac{\omega_0}{Q}\frac{du(t)}{dt} + \omega_0^2 u(t) = 0$$

  - $Q = \frac{1}{2\zeta}$ is the ratio between the stored and lost energy per cycle with ζ the damping ratio
  - ω_0 is the eigen-frequency of the harmonic oscillator

- A general solution can be found by the ansatz

$$u(t) = u_0 e^{\lambda t}$$

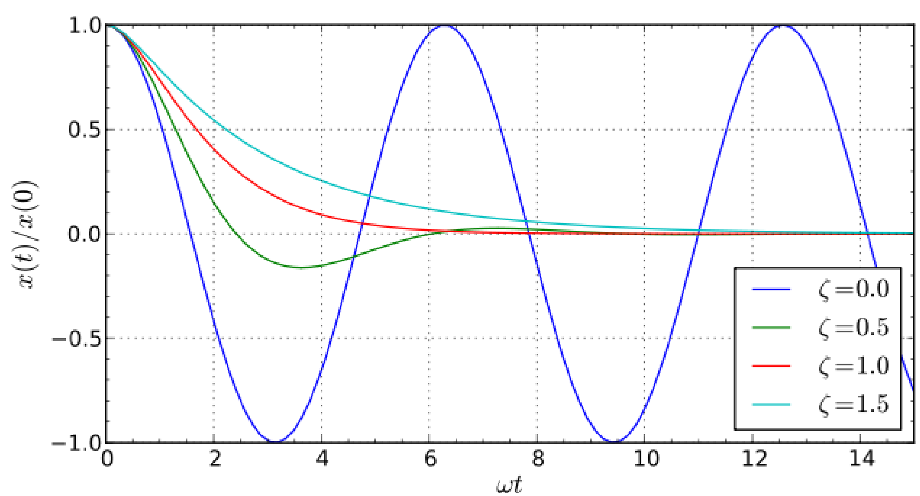
leading to an auxiliary 2nd order equation

$\lambda^2 + \frac{\omega_0}{Q}\lambda + \omega_0^2 = 0$ with solutions

$$\lambda_\pm = -\frac{\omega_0}{2Q}(-1 \pm \sqrt{1-4Q^2}) = -\omega_0\zeta(-1 \pm \sqrt{1-\frac{1}{\zeta^2}})$$

Harmonic oscillator including damping II

- Three cases can be distinguished
  - *Overdamping* (λ real, i.e. $\zeta > 1$ or $Q < 1/2$): The system exponentially decays to equilibrium (slower for larger damping ratio values)
  - *Critical damping* (ζ = 1): The system returns to equilibrium as quickly as possible without oscillating.
  - *Underdamping* (λ complex, i.e. $\zeta < 1$ or $Q > 1/2$): The system oscillates with the amplitude gradually decreasing to zero, with a slightly different frequency than



$$\omega_d = \omega_0 \sqrt{1 - \zeta^2}$$

Damped oscillator with periodic driving

- Consider periodic force pumping energy into the system

$$\frac{d^2u(t)}{dt^2} + \frac{\omega_0}{Q}\frac{du(t)}{dt} + \omega_0^2 u(t) = \frac{F}{m}\cos(\omega t)$$

- General solution is a combination of a transient and a steady state term

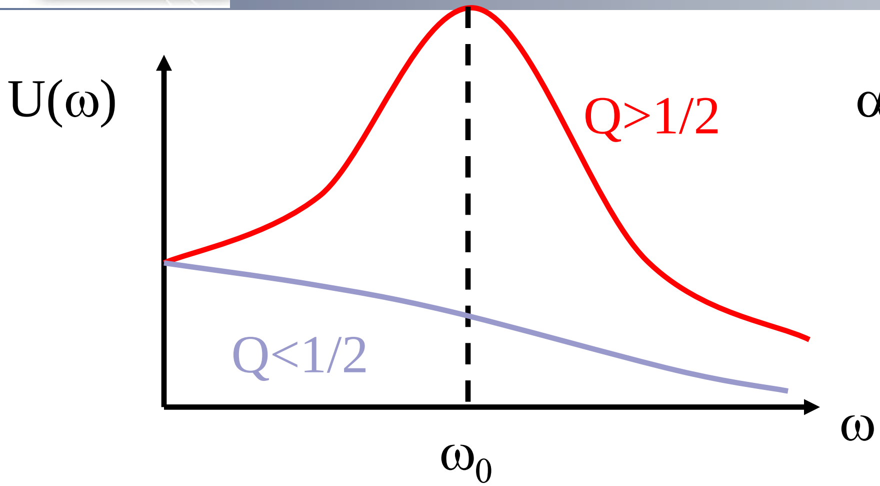
$$u(t) = u_t(t) + u_s(t)$$

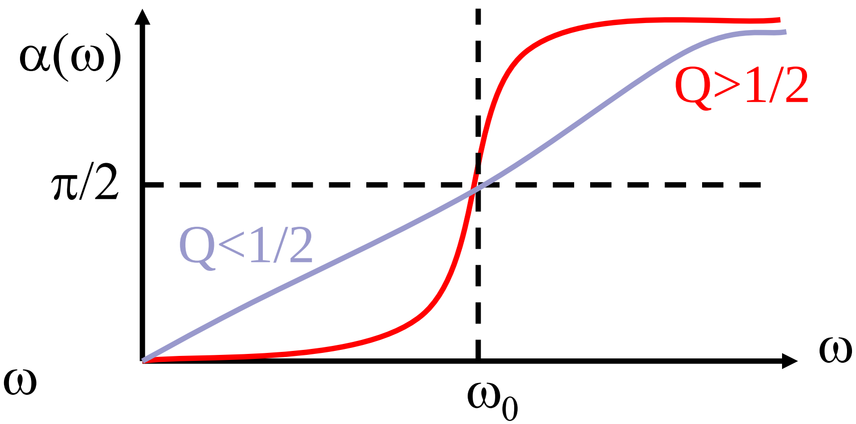
- The transient solution corresponds to the one of the homogeneous system (damped oscillator) and "dies" out after some time leaving only the steady state one

$$u_s(t) = U(\omega)\cos(\omega t + \phi(\omega))$$

  - ω the frequency of the driven oscillation
  - Amplitude $U(\omega)$ can become large for certain frequencies

Resonance effect



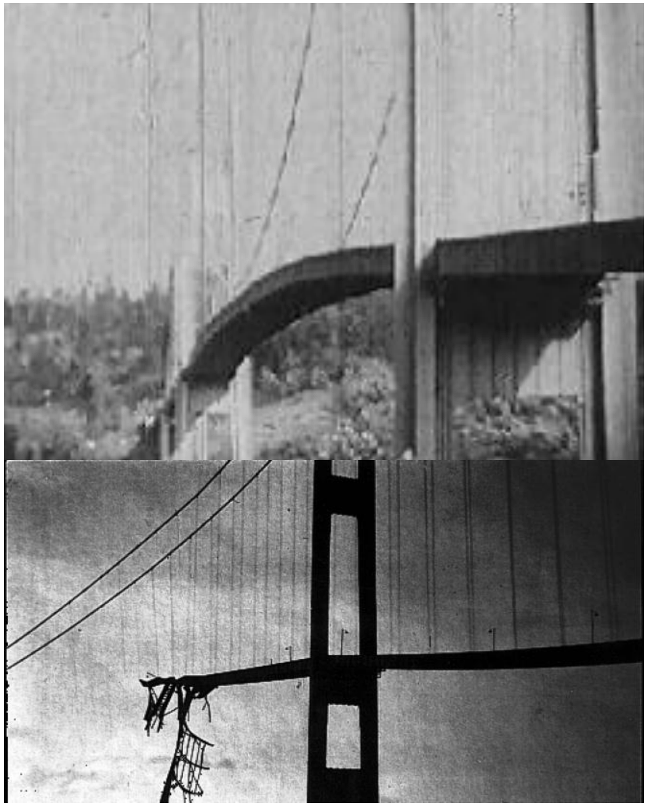


$$U(\omega) = \frac{U(0)}{\sqrt{(1-(\frac{\omega}{\omega_0})^2)^2 + (\frac{\omega}{Q\omega_0})^2}}$$

- Without or with weak damping a resonance condition occurs for $\omega = \omega_0$
- Infamous example:

Tacoma Narrow bridge 1940

excitation by strong wind on the eigenfrequencies



Accelerator performance parameters

$$L = \frac{N_b^2 k_b \gamma}{4\pi \epsilon_n \beta^*}$$

- Colliders
  - Luminosity (i.e. rate of particle production)
    - N_b bunch population
    - k_b number of bunches
    - γ relativistic reduced energy
    - ε_n normalized emittance
    - β^* "betatron" amplitude function at collision point

$$\bar{P} = \bar{I} E = f_N N e E$$

- High intensity accelerators
  - Average beam power
    - $\bar{I}$ mean current intensity
    - E energy
    - f_N repetition rate
    - N number of particles/pulse

$$B = \frac{N_p}{4\pi^2 \epsilon_x \epsilon_y}$$

- Synchrotron light sources
  - Brightness (photon density in phase space)
    - N_p number of photons
    - $\varepsilon_{x,,y}$ transverse emittances
- Performance issues due to non-linear effects
  - Reduced dynamic aperture, lifetime and availability, beam loss (radio-activation, magnet quench

Normalized coordinates

- Recall that $u(s) = \sqrt{\epsilon\beta(s)}\cos(\psi(s)+\psi_0)$

$$u'(s) = -\sqrt{\frac{\epsilon}{\beta(s)}}\left(\sin(\psi(s)+\psi_0) + \alpha(s)\cos(\psi(s)+\psi_0)\right)$$

- Introduce new variables

$$\mathcal{U} = \frac{u}{\sqrt{\beta}}\,, \quad \mathcal{U}' = \frac{d\mathcal{U}}{d\phi} = \frac{\alpha}{\sqrt{\beta}}u + \sqrt{\beta}u'\,, \quad \phi = \frac{\psi}{\nu} = \frac{1}{\nu}\int\frac{ds}{\beta(s)}$$

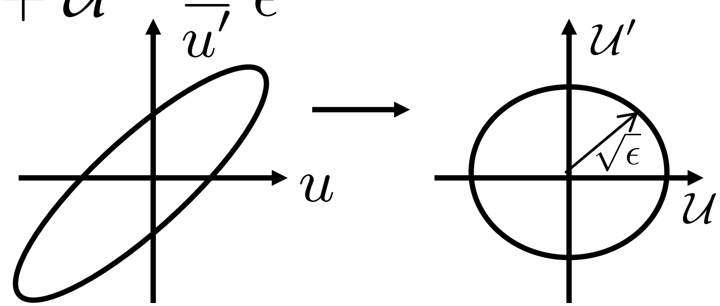
- In matrix form $\begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{\beta}} & 0\\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta}\end{pmatrix}\begin{pmatrix}u\\ u'\end{pmatrix}$

- Hill's equation becomes $\frac{d^2\mathcal{U}}{d\phi^2} + \nu^2\mathcal{U} = 0$

- System becomes harmonic oscillator with frequency

$$\begin{pmatrix}\mathcal{U}\\ \mathcal{U}'\end{pmatrix} = \sqrt{\epsilon}\begin{pmatrix}\cos(\nu\phi)\\ -\sin(\nu\phi)\end{pmatrix} \qquad \mathcal{U}^2 + \mathcal{U}'^2 = \epsilon$$

- **Floquet transformation** transforms phase space in circles



Perturbation in Hill's equations

- Hill's equations in normalized coordinates with harmonic perturbation, using $\mathcal{U} = \mathcal{U}_x$ or $\mathcal{U}_y$ and $\phi = \phi_x$ or ϕ_y

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu^2\mathcal{U} = \nu^2\beta^{3/2}F(\mathcal{U}_x(\phi_x),\mathcal{U}_y(\phi_y))$$

where the *F* is the Lorentz force from perturbing fields

  - **Linear magnet imperfections**: deviation from the design dipole and quadrupole fields due to powering and alignment errors
  - **Time varying fields**: feedback systems (damper) and wake fields due to collective effects (wall currents)
  - **Non-linear magnets**: sextupole magnets for chromaticity correction and octupole magnets for Landau damping
  - **Beam-beam interactions**: strongly non-linear field
  - **Space charge effects**: very important for high intensity beams
  - **non-linear magnetic field imperfections**: particularly difficult to control for super conducting magnets where the field quality is entirely determined by the coil winding accuracy

- From Gauss law of magnetostatics, a vector potential exist

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \exists \mathbf{A} : \quad \mathbf{B} = \nabla \times \mathbf{A}$$

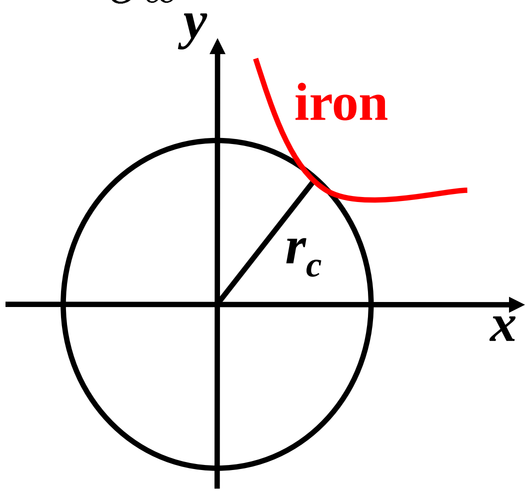
- Assuming a 2D field in x and y, the vector potential has only one component A_s. The Ampere's law in vacuum (inside the beam pipe)

$$\nabla \times \mathbf{B} = 0 \quad \rightarrow \quad \exists V : \quad \mathbf{B} = -\nabla V$$

- Using the previous equations, the relations between field components and potentials are

$$B_x = -\frac{\partial V}{\partial x} = \frac{\partial A_s}{\partial y}, \quad B_y = -\frac{\partial V}{\partial y} = -\frac{\partial A_s}{\partial x}$$

i.e. Riemann conditions of an analytic function



There exist a complex potential of $z = x+iy$ with a power series expansion convergent in a circle with radius $|z| = r_c$ (distance from iron yoke)

$$\mathcal{A}(x+iy) = A_s(x,y) + iV(x,y) = \sum_{n=1}^{\infty} \kappa_n z^n = \sum_{n=1}^{\infty} (\lambda_n + i\mu_n)(x+iy)^n$$

Multipole expansion II

- From the complex potential we can derive the fields

$$B_y + iB_x = -\frac{\partial}{\partial x}(A_s(x,y) + iV(x,y)) = -\sum_{n=1}^{\infty} n(\lambda_n + i\mu_n)(x+iy)^{n-1}$$

- Setting $b_n = -n\lambda_n \;,\quad a_n = n\mu_n$

$$B_y + iB_x = \sum_{n=1}^{\infty}(b_n - ia_n)(x+iy)^{n-1}$$

- Define normalized coefficients

$$b'_n = \frac{b_n}{10^{-4}B_0} r_0^{n-1} \;,\; a'_n = \frac{a_n}{10^{-4}B_0} r_0^{n-1}$$

on a reference radius r_0, 10^{-4} of the main field to get

$$B_y + iB_x = 10^{-4} B_0 \sum_{n=1}^{\infty}(b'_n - ia'_n)\left(\frac{x+iy}{r_0}\right)^{n-1}$$

- **Note**: $n' = n - 1$ is the US convention

Perturbation by single dipole

- Hill's equations in normalized coordinates with single dipole perturbation:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{3/2}b_1(\phi) = \overline{b_1}(\phi)$$

- The dipole perturbation is periodic, so it can be expanded in a Fourier series

$$\overline{b_1}(\phi) = \sum_{m=-\infty}^{\infty} \overline{b_{1m}} e^{im\phi}$$

- Note that a periodic kick introduces infinite number of integer driving frequencies
- The resonance condition occurs when $\nu_0 = m$

i.e. **integer tunes** should be avoided (remember orbit distortion due to single dipole kick)

Perturbation by single quadrupole

- Consider single quadrupole kick in one normalized plane:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^2 b_2(\phi)\mathcal{U} = \overline{b_2}(\phi)\mathcal{U}$$

- The quadrupole perturbation is periodic, so it can be expanded in a Fourier series $\overline{b_2}(\phi) = \sum_{m=-\infty}^{\infty} \overline{b_{2m}} e^{im\phi}$

- As the perturbation is small insert on the right hand side the unperturbed solution $\mathcal{U} \approx \mathcal{U}_0 = W_1 e^{i\nu_0\phi} + W_{-1}e^{-i\nu_0\phi}$ and the equation of motion can be written as

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \sum_{q=-1}^{1}\sum_{m=-\infty}^{\infty} W_q \overline{b_{2m}} e^{i(m+q\nu_0)\phi} \quad \text{with} \quad W_0 = 0$$

- The resonance conditions are $m - \nu_0 = \nu_0 \rightarrow \nu_0 = \frac{m}{2}$ i.e. **integer** and **half-integer tunes** should be avoided

- The condition $m + \nu_0 = \nu_0 \rightarrow m = 0$ corresponds to a non-vanishing average value $\overline{b_{20}}$, which can be absorbed in the left-hand side providing a **tune-shift**: $\nu^2 = \nu_0^2 - b_{20}$ or $\delta\nu \approx -\frac{b_{20}}{2\nu_0}$

Perturbation by single multi-pole

- For a generalized multi-pole perturbation, Hill's equation is:

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{\frac{n}{2}+1}b_n(\phi)\mathcal{U}^{n-1} = \overline{b_n}(\phi)\mathcal{U}^{n-1}$$

- As before, the multipole coefficient can be expanded in Fourier series

$$\overline{b_n}(\phi) = \sum_{m=-\infty}^{\infty} \overline{b_{nm}} e^{im\phi}$$

- As before, we insert the unperturbed solution on the right side and $\mathcal{U}^{n-1} \approx \mathcal{U}_0^{n-1} = \sum_{q=-n+1}^{n-1} W_q e^{iq\nu_0\phi}$ the equation of motion can be written as

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \sum_{q=-n+1}^{n-1}\sum_{m=-\infty}^{\infty} W_q\overline{b_{nm}}e^{i(m+q\nu_0)\phi}$$

with $W_{n-2} = W_{n-4} = \cdots = W_{-n+2} = 0$

- The resonance conditions are $\boxed{m + q\nu_0 = \nu_0}$ with $q = -n+1, -n+3, \ldots, n-1$
- If $q=1$ does not correspond to a vanishing coefficient (even multi-poles), there is an (amplitude dependent, for n>2) frequency shift

Single Sextupole Perturbation

- Consider a localized sextupole perturbation in the horizontal plane

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \nu_0^2\beta^{\frac{5}{2}}b_3(\phi)\mathcal{U}^2 = \overline{b_3}(\phi)\mathcal{U}^2$$

- After replacing the perturbation by its Fourier transform and inserting the unperturbed solution to the right hand side

$$\frac{d^2\mathcal{U}}{d\phi^2} + \nu_0^2\mathcal{U} = \sum_{q=-2}^{2}\sum_{m=-\infty}^{\infty} W_q\overline{b_{3m}}e^{i(m+q\nu_0)\phi} \quad \text{with} \quad W_1 = W_{-1} = 0$$

- Resonance conditions:

3rd integer $\rightarrow 3\nu_0 = m \text{ for } q = -2$

integer $\rightarrow \nu_0 = m \text{ for } q = 0, 2$

- Note that there is not a tune-spread associated. This is only true for small perturbations (first order perturbation treatment)
- No exact solution
- Need numerical tools to integrate equations of motion

General resonance conditions

- Equations of motion including any multi-pole error

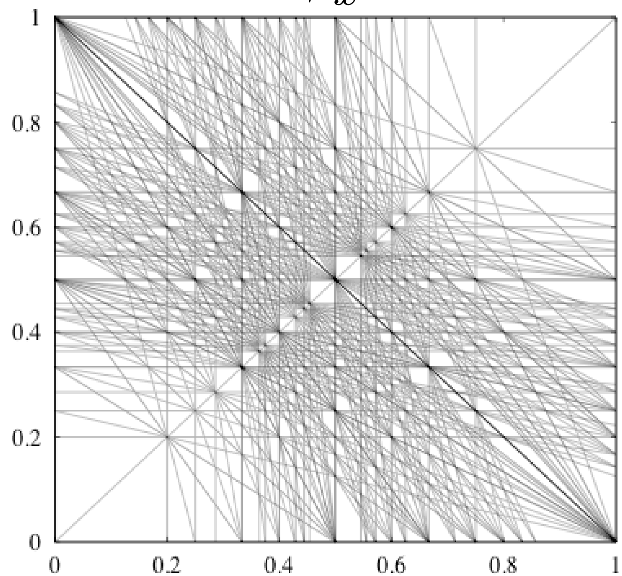
$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2 \mathcal{U}_x = \overline{b_{n,r}}(\phi_x)\mathcal{U}_x^{n-1}\mathcal{U}_y^{r-1}$$

- Expanding perturbation coefficient in Fourier series and inserting the solution of the unperturbed system gives the following series:

$$\overline{b_{nr}}(\phi_x) = \sum_{m=-\infty}^{\infty} \overline{b_{nrm}} e^{im\phi_x} \quad \mathcal{U}_x^{n-1} \approx \mathcal{U}_{0x}^{n-1} = \sum_{q_x=-n+1}^{n-1} W_{q_x}^x e^{iq_x\nu_{0x}\phi_x} \quad \mathcal{U}_y^{r-1} \approx \mathcal{U}_{0y}^{r-1} = \sum_{q_y=-r+1}^{r-1} W_{q_y}^y e^{iq_y\nu_{0y}\phi_x}$$

- The equation of motion becomes

$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2 \mathcal{U}_x = \sum_{m,q_x,q_y} \overline{b_{nrm}} W_{q_x}^x W_{q_y}^y e^{i(m+q_x\nu_{0x}+q_y\nu_{0y})\phi_x}$$



Resonance conditions

$$m + q_x\nu_{0x} + q_y\nu_{0y} = \nu_{0x}$$

or

$$m + q'_x\nu_{0x} + q_y\nu_{0y} = 0$$

with the resonance order $|q_x| + |q_y| + 1$

There are resonance lines everywhere !!!

Example: Linear Coupling

- For a localized skew quadrupole we have

$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2\mathcal{U}_x = \overline{b_{1,2}}(\phi_x)\mathcal{U}_y$$

- Expanding perturbation coefficient in Fourier series and inserting the solution of the unperturbed system gives the following equation:

$$\frac{d^2\mathcal{U}_x}{d\phi_x^2} + \nu_{0x}^2\mathcal{U}_x = \sum_{m=-\infty}^{\infty}\sum_{q_y=-1}^{q_y=1}\overline{b_{12m}}W_{q_y}^y e^{i(m+q_y\nu_{0y})\phi_x} \text{ with } W_0^y = 0$$

- The coupling resonance are found for $q_y = \pm 1$

Linear sum resonance

$$m = \nu_{0x} + \nu_{0y}$$

Linear difference resonance

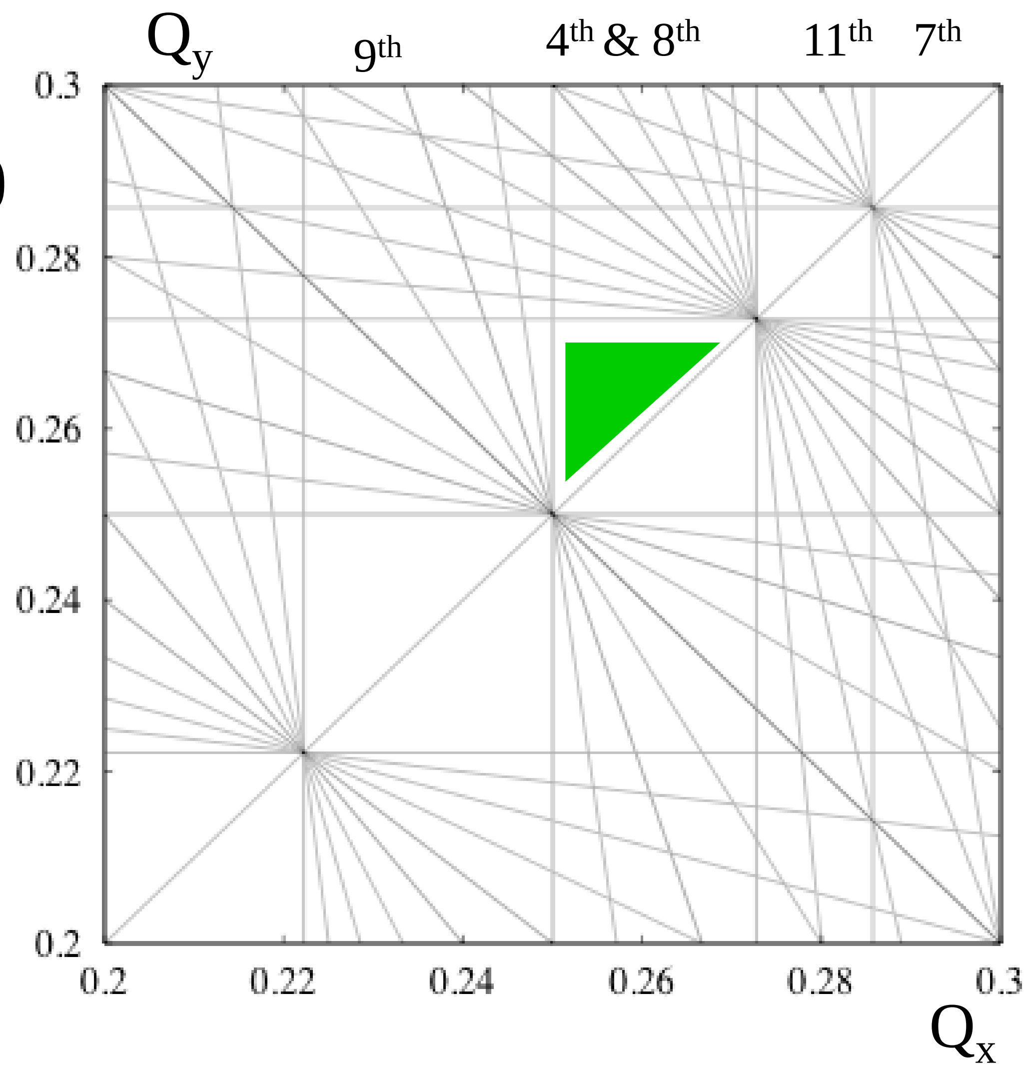
$$m = \nu_{0x} - \nu_{0y}$$

Choice of the working point

- Regions with few resonances:

$$m + q_x \nu_{0x} + q_y \nu_{0y} = 0$$

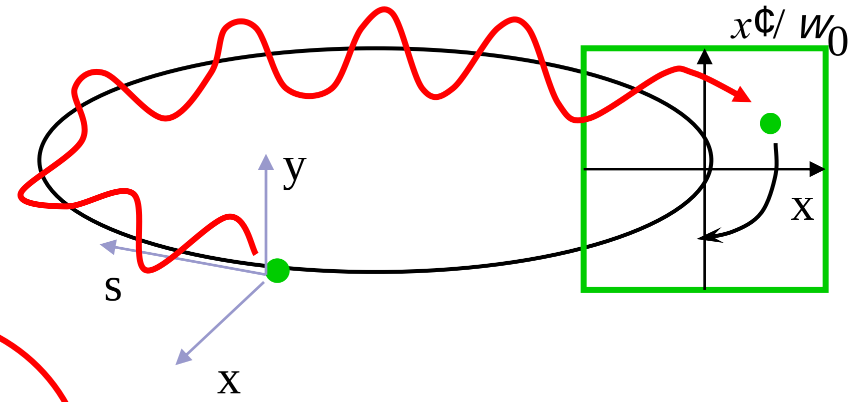
- Avoid low order resonances
- < 12^{th} order for a proton beam without damping
- < 3^{rd} ⇔ 5^{th} order for electron beams with damping
- Close to coupling resonances: regions without low order resonances but relatively small!

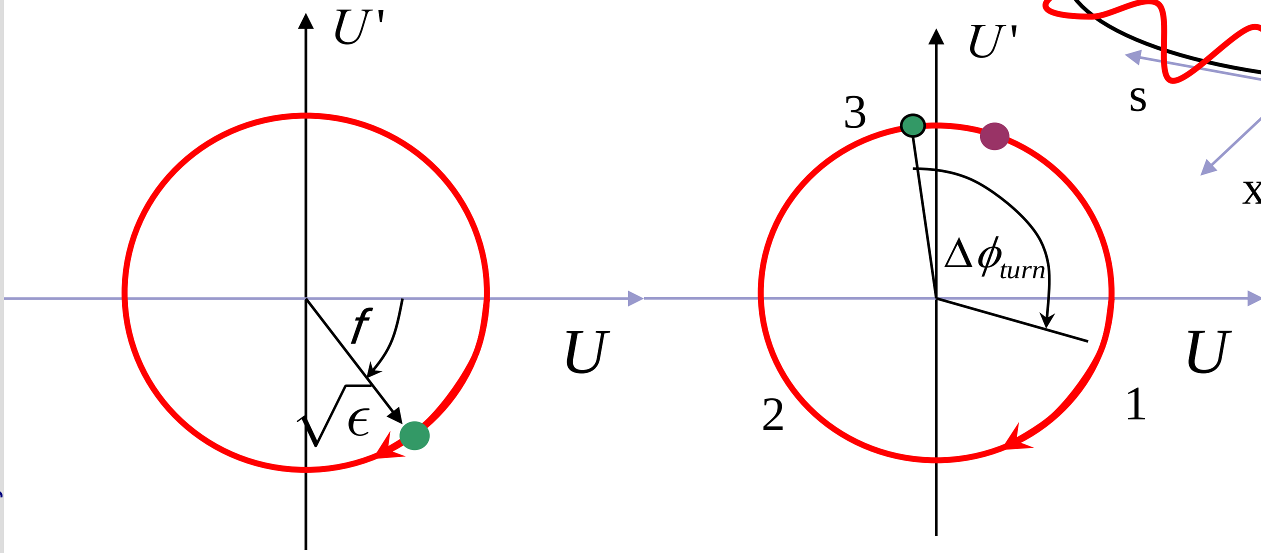


Poincaré Section

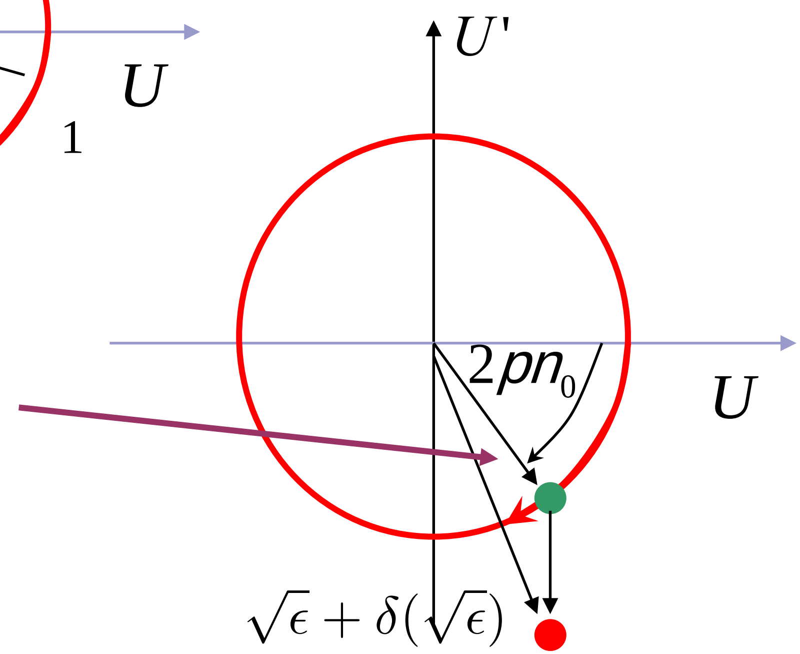
- Record the particle coordinates at one location (BPM)
- Unperturbed motion lies on a circle in normalized coordinates (simple rotation)

Poincaré Section:





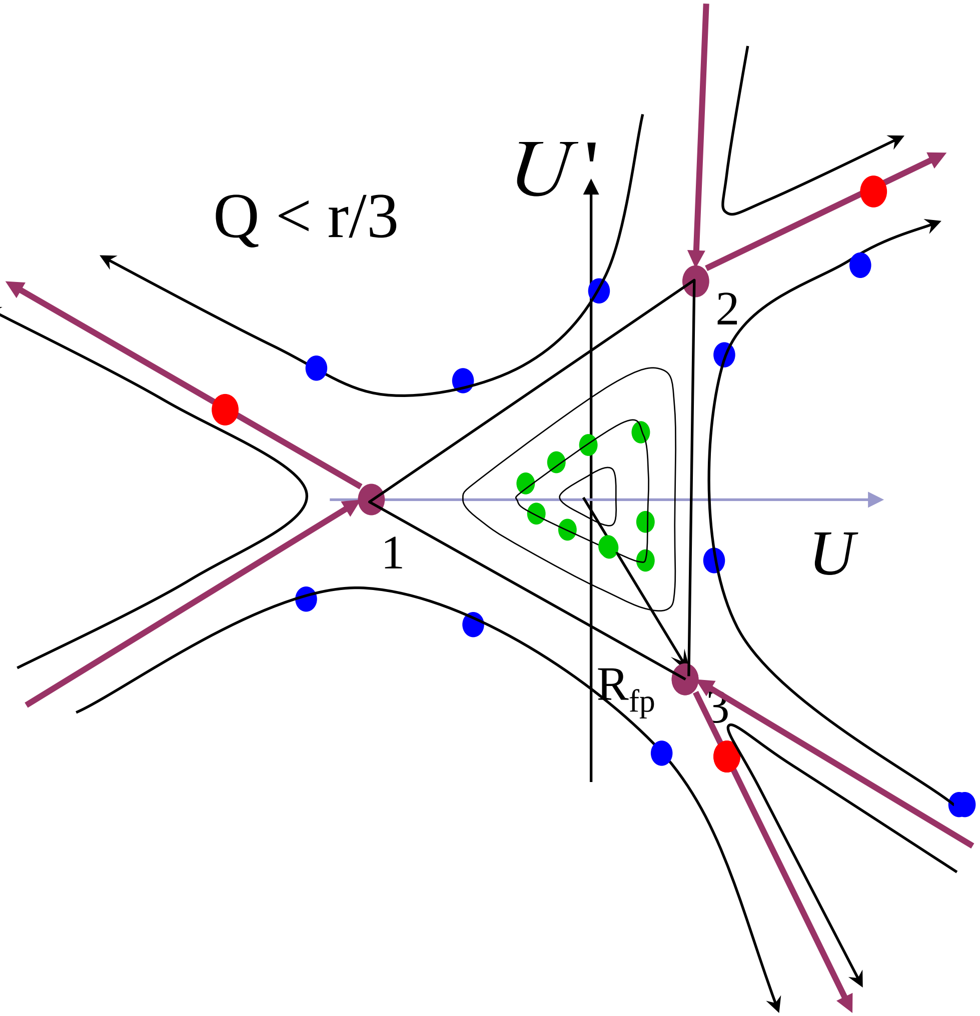
- Resonance condition corresponds to a periodic orbit or in fixed points in phase space
- For a sextupole $\delta\mathcal{U}' = \overline{b_3}\mathcal{U}^2$



- The particle does not lie on a circle!

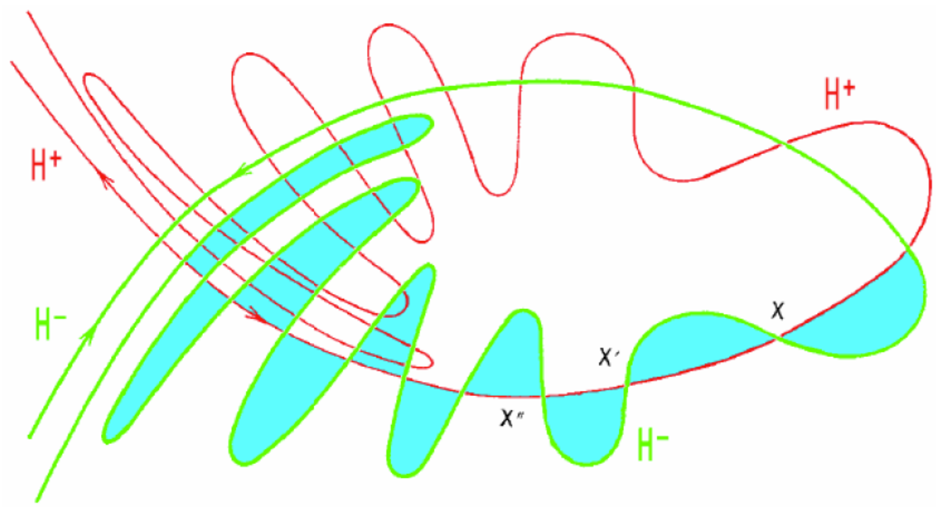
Topology of a sextupole resonance

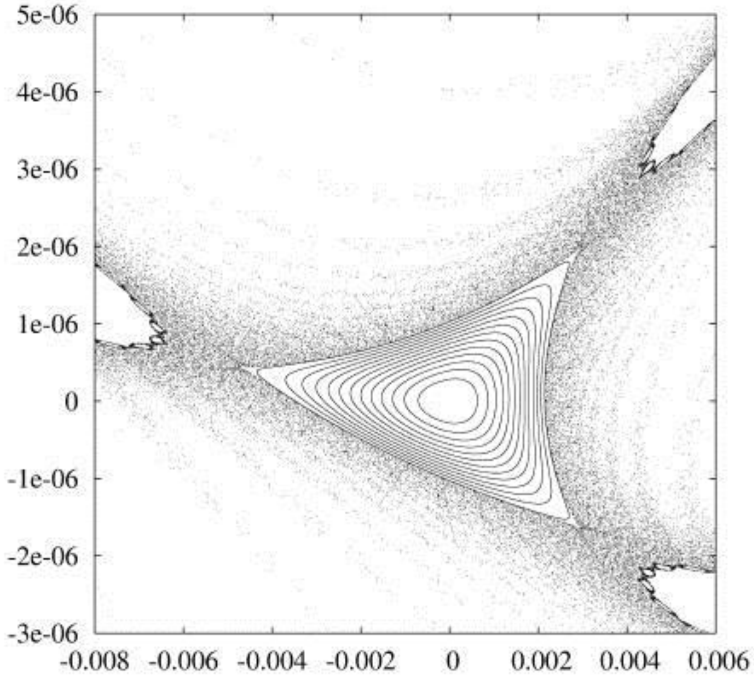
- Small amplitude, regular motion (circles)
- Larger amplitude deformation of phase space towards a triangular shape
- Separatrix: curve passing through unstable (hyperbolic) fixed points (and going to infinity)
- Its location (width) depends on distance to the resonance of the unperturbed tune
- Exactly on the resonance, sepratrix collapses to a single unstable fixed point (bifurcation)
- Stable fixed points should exist but they are found in much larger amplitudes



Path to chaos

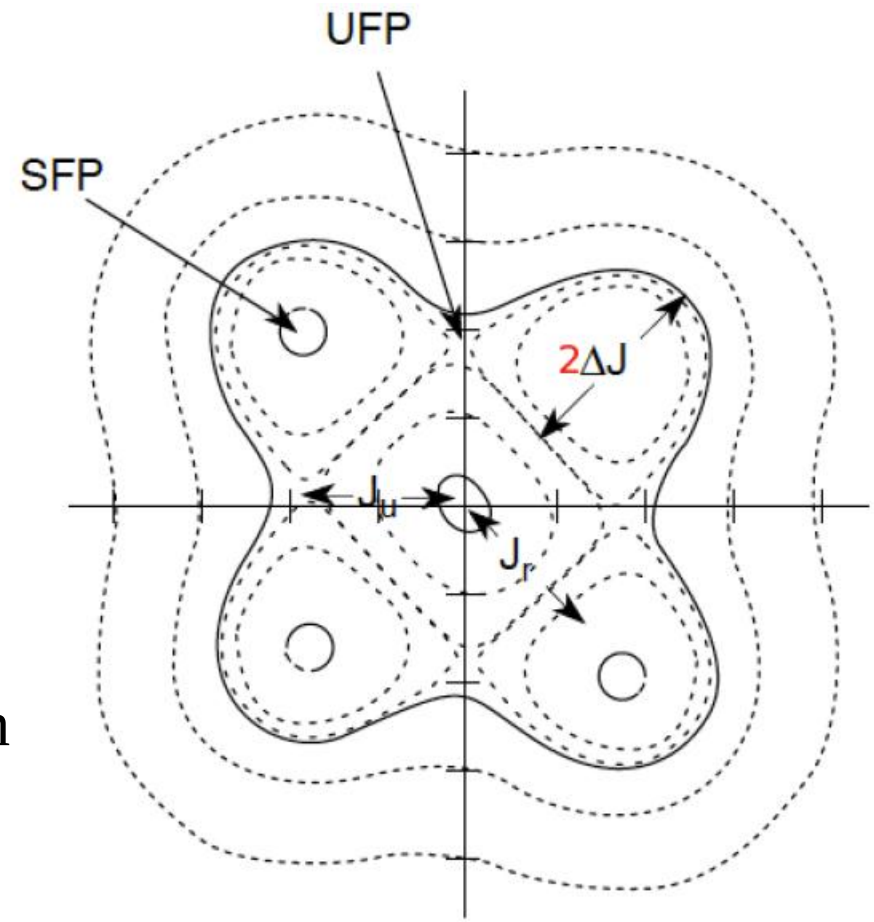
- When perturbation becomes higher, motion around the separatrix becomes chaotic (producing tongues or splitting of the separatrix)
- Unstable fixed points are indeed the source of chaos when a perturbation is added





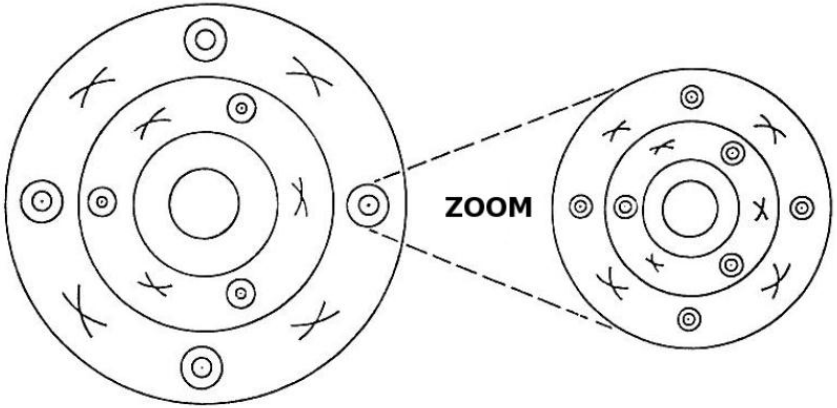
Topology of an octupole resonance

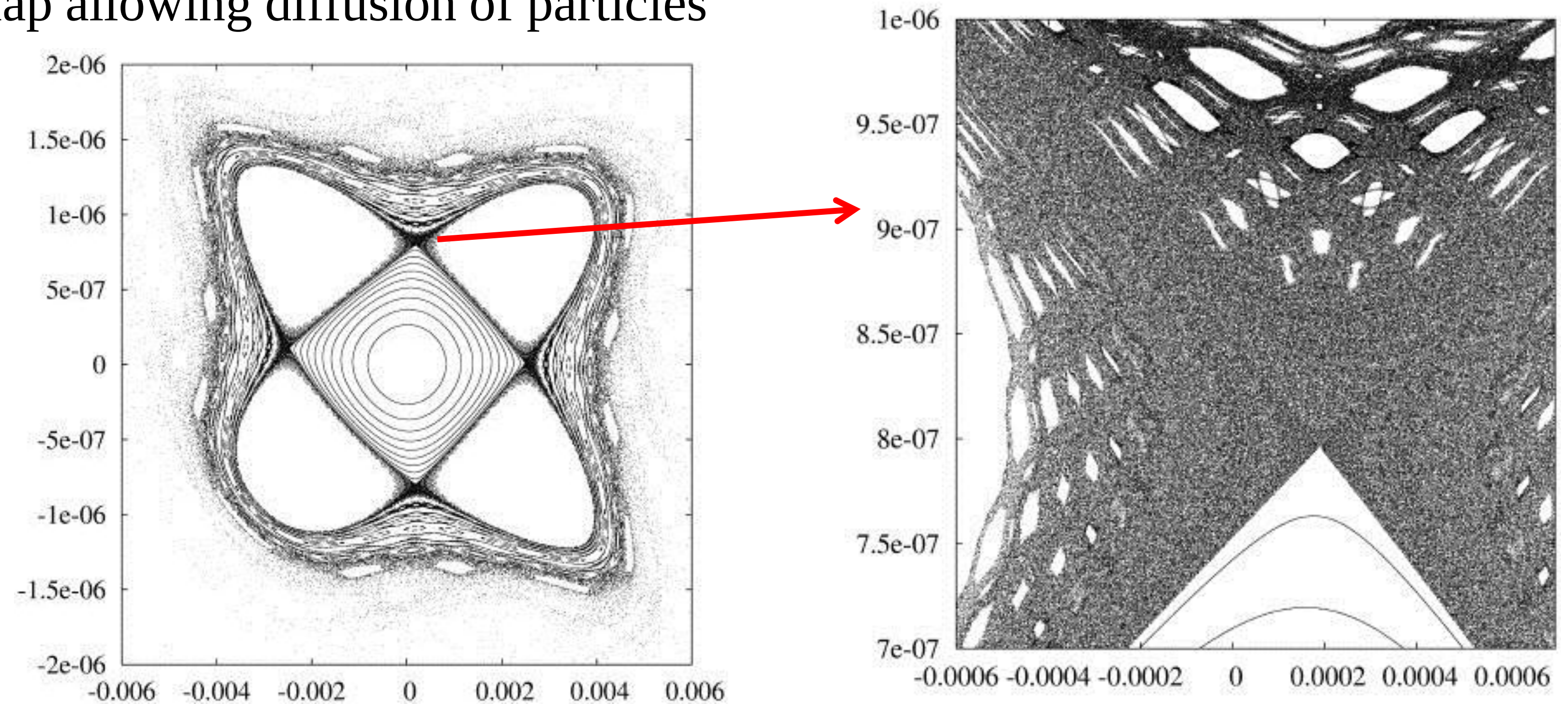
- Regular motion near the center, with curves getting more deformed towards a rectangular shape
- The separatrix passes through 4 unstable fixed points, but motion seems well contained
- Four stable fixed points exist and they are surrounded by stable motion (islands of stability)



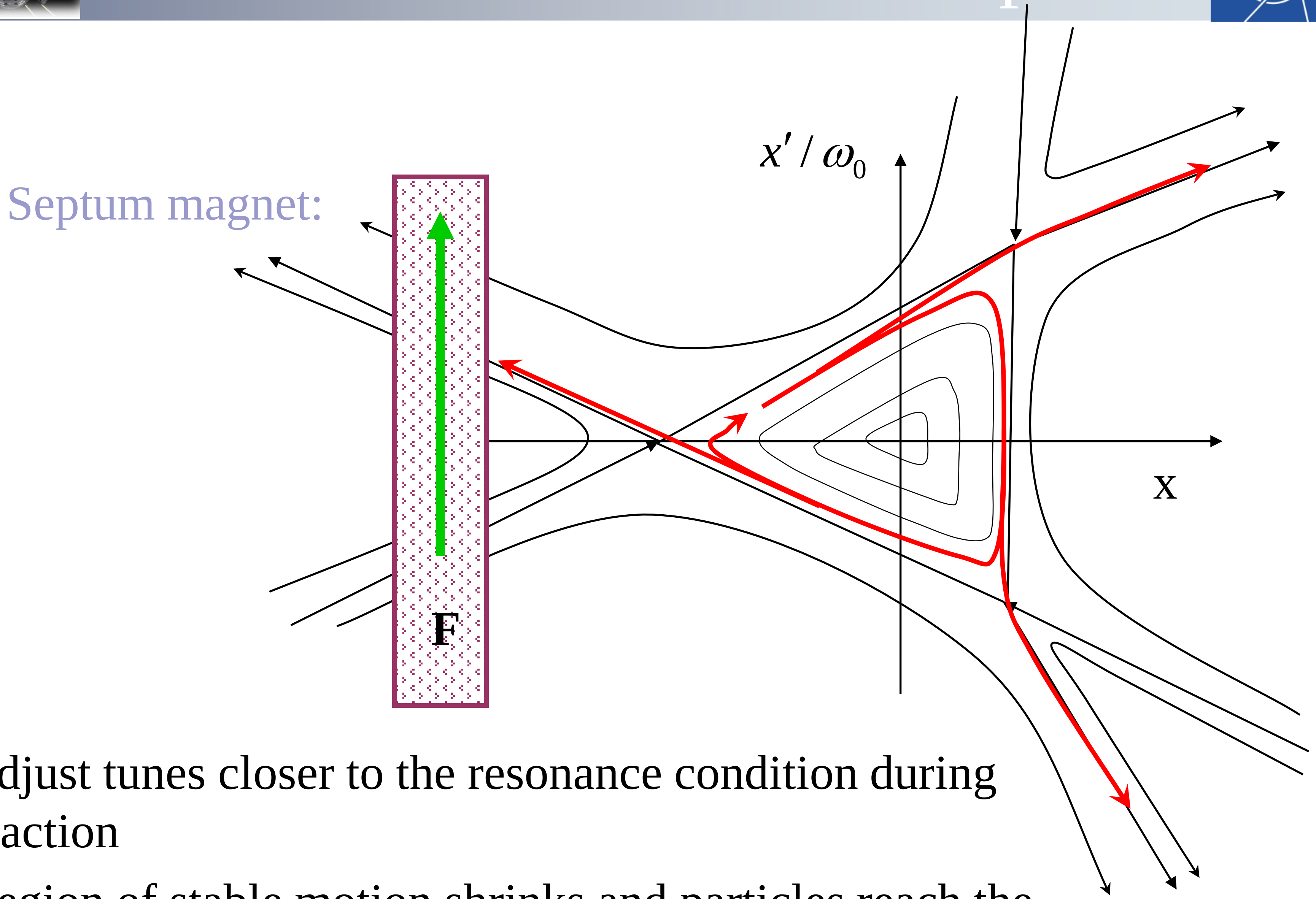
Chaotic motion

- Poincare-Birkhoff theorem states that under perturbation of a resonance only an even number of fixed points survives (half stable and the other half unstable)
- Themselves get destroyed when perturbation gets higher, etc. (self-similar fixed points)
- Resonance islands grow and resonances can overlap allowing diffusion of particles





Slow Extraction With Sextupoles



- Adjust tunes closer to the resonance condition during extraction
- Region of stable motion shrinks and particles reach the septum diffusing through the separatrix

9 first order terms:

- 2 chromaticities ξ_x, ξ_y
- 2 off-momentum resonances $2Q_x, 2Q_y \rightarrow d\beta/d\delta \rightarrow \xi^{(2)} = \partial^2 Q/\partial\delta^2$
- 2 terms $\rightarrow$ integer resonances Q_x
- 1 term $\rightarrow 3^{rd}$ integer resonances $3Q_x$
- 2 terms $\rightarrow$ coupling resonances $Q_x \pm 2Q_y$

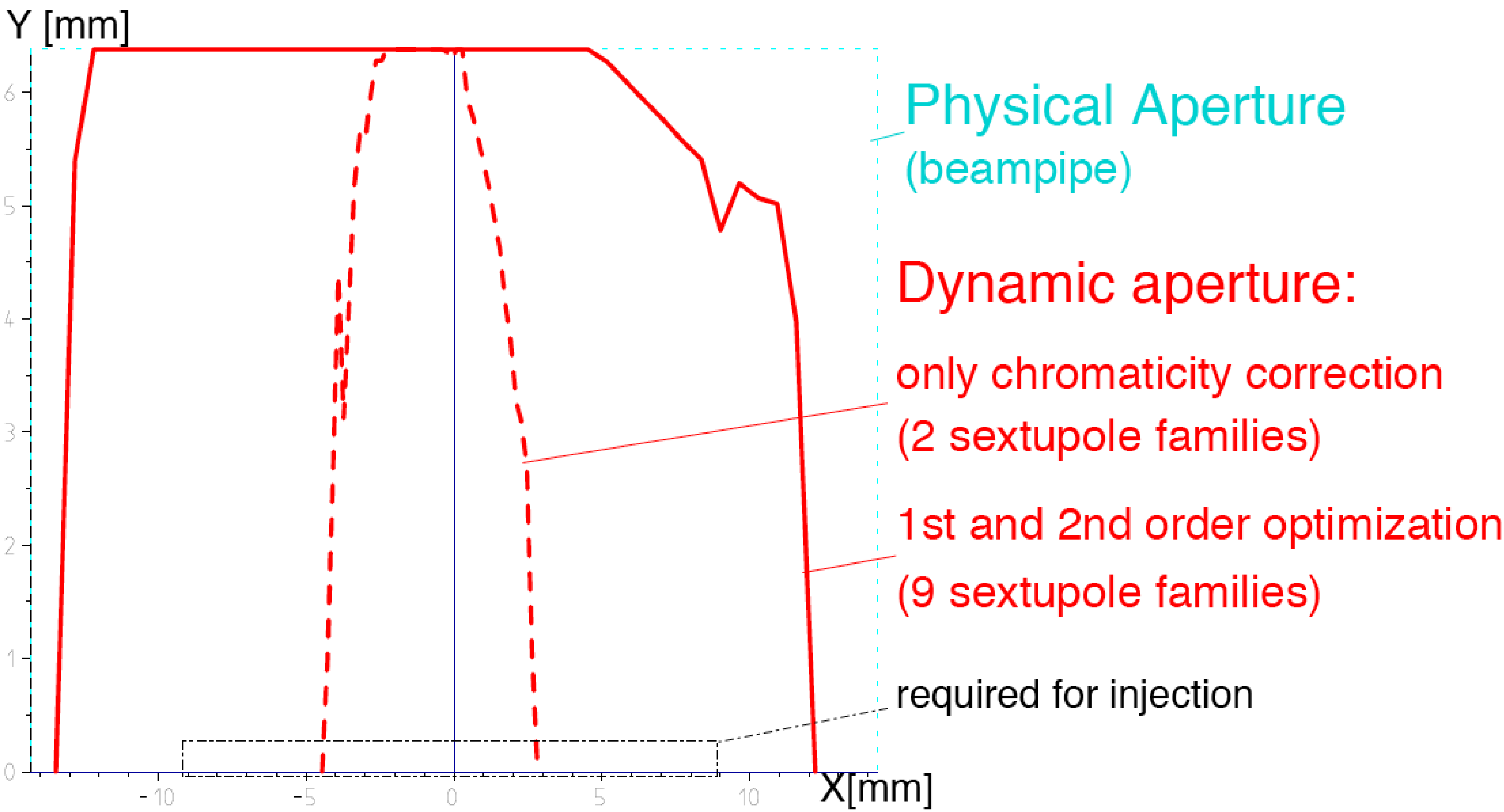
13 second order terms:

- 3 tune shifts with amplitude: $\partial Q_x/\partial J_x, \quad \partial Q_x/\partial J_y = \partial Q_y/\partial J_x, \quad \partial Q_y/\partial J_y$
- 8 terms $\rightarrow$ octupole like resonances: $4Q_x, 2Q_x \pm 2Q_y, 4Q_y, 2Q_x, 2Q_y$
- 2 second order chromaticities: $\partial^2 Q_x/\partial\delta^2$ and $\partial^2 Q_y/\partial\delta^2$

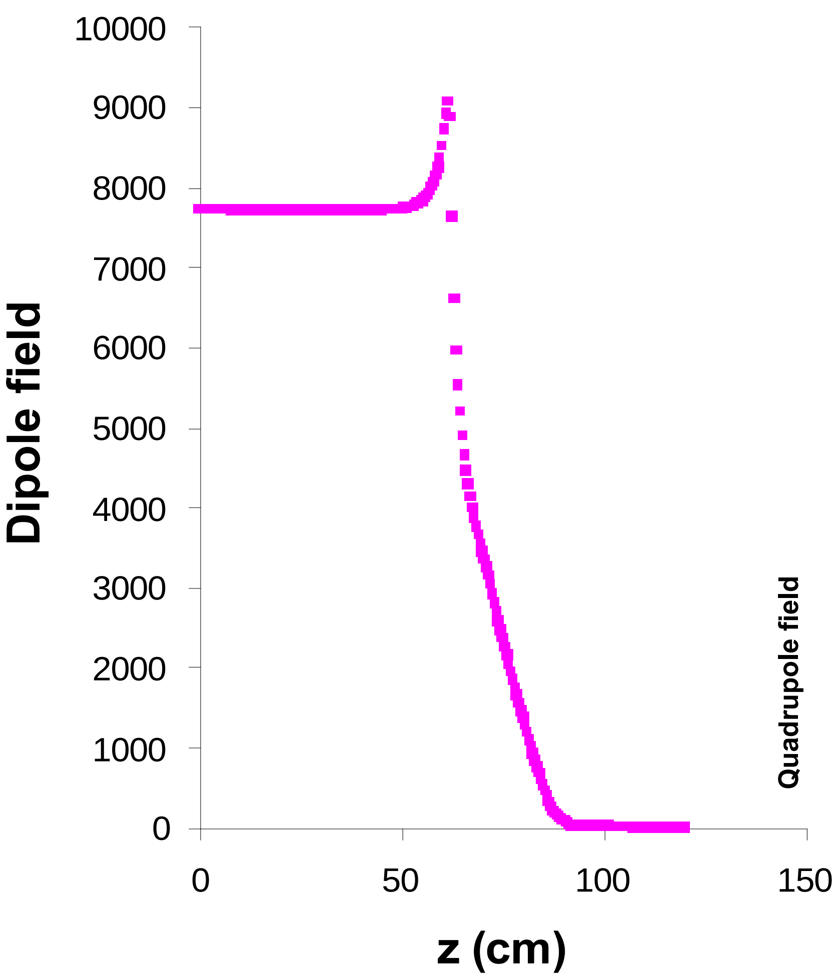
- Enough sextupole families are needed to control all these terms

Optimization of Dynamic aperture

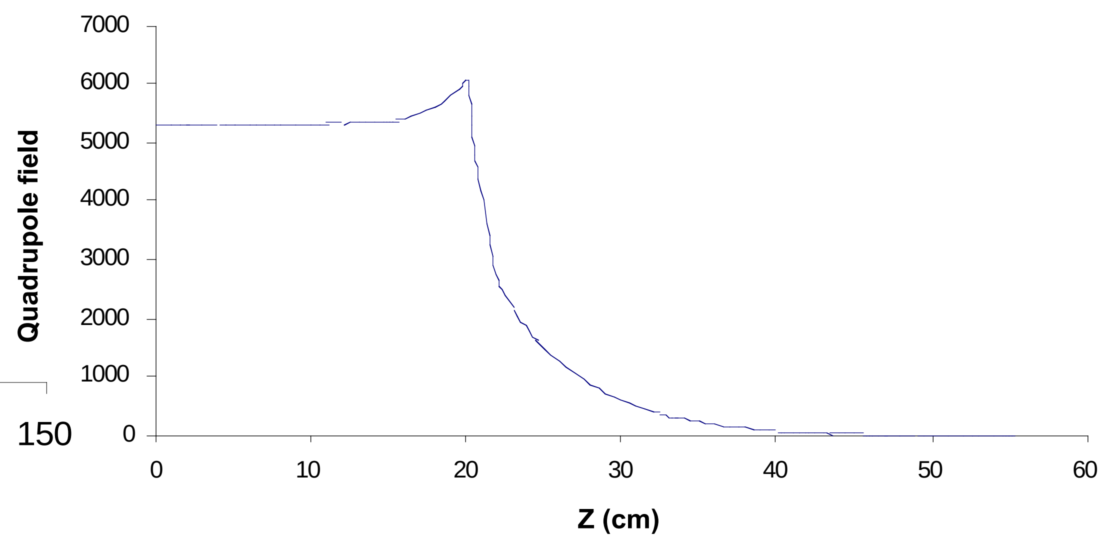
- Keep chromaticity sextupole strength low
- Try an interleaved sextupole scheme (-*I* transformer) to cancel first order third resonance effect
- Choose working point far from systematic resonances
- Iterate between linear and non-linear lattice



Magnet fringe fields



- Up to now we considered only transverse fields
- Magnet fringe field is the longitudinal dependence of the field at the magnet edges
- Important when magnet aspect ratios and/or emittances are big



General field expansion for a quadrupole magnet:

$$B_x = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n} y^{2m+1}}{(2n)!(2m+1)!} \binom{m}{l} b^{[2l]}_{2n+2m+1-2l}$$

$$B_y = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m}}{(2n+1)!(2m)!} \binom{m}{l} b^{[2l]}_{2n+2m+1-2l} \quad .$$

$$B_z = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m+1}}{(2n+1)!(2m+1)!} \binom{m}{l} b^{[2l+1]}_{2n+2m+1-2l}$$

and to leading order

$$\begin{aligned} B_x &= y\left[b_1 - \frac{1}{12}(3x^2 + y^2) b_1^{[2]}\right] + O(5) \\ B_y &= x\left[b_1 - \frac{1}{12}(3y^2 + x^2) b_1^{[2]}\right] + O(5) \\ B_z &= xy b_1^{[1]} + O(4) \end{aligned}$$

The quadrupole fringe to leading order has an octupole-like effect

First order tune spread for an octupole:

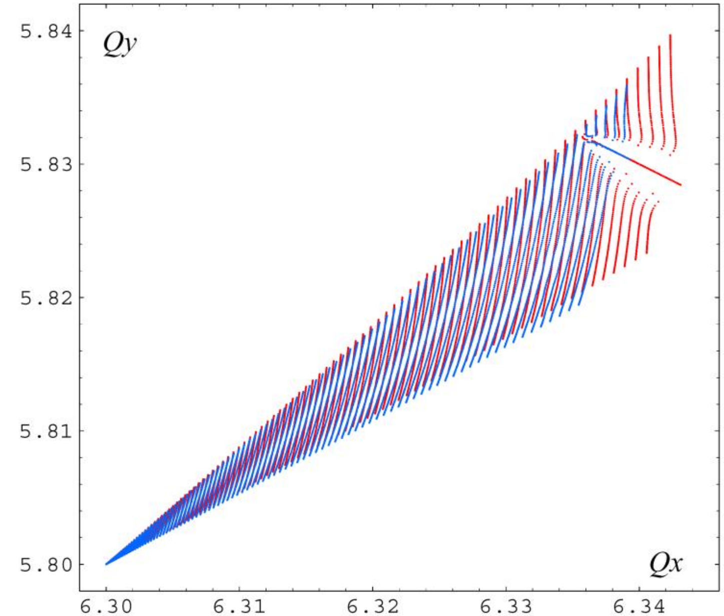
$$\begin{pmatrix}\delta\nu_x\\ \delta\nu_y\end{pmatrix} = \begin{pmatrix}a_{hh} & a_{hv}\\ a_{hv} & a_{vv}\end{pmatrix}\begin{pmatrix}2J_x\\ 2J_y\end{pmatrix},$$

where the normalized anharmonicities are

$$a_{hh} = \frac{-1}{16\pi B\rho}\sum_i \pm Q_i\beta_{xi}\alpha_{xi},$$

$$a_{hv} = \frac{1}{16\pi B\rho}\sum_i \pm Q_i(\beta_{xi}\alpha_{yi} - \beta_{yi}\alpha_{xi}),$$

$$a_{vv} = \frac{1}{16\pi B\rho}\sum_i \pm Q_i\beta_{yi}\alpha_{yi}.$$



Tune footprint for the SNS based on hard-edge (red) and realistic (blue) quadrupole fringe-field

Quasi-periodic approximation through **NAFF** algorithm

$$f'_j(t) = \sum_{k=1}^{N} a_{j,k} e^{i\omega_{j,k} t}$$

of a complex phase space function $f_j(t) = q_j(t) + ip_j(t)$

defined over $t = \tau$,

for each degree of freedom $j = 1, \ldots, n$ with $\omega_{j,k} = \boldsymbol{k}_j \cdot \boldsymbol{\omega}$

and $a_{j,k} = A_{j,k} e^{i\phi_{j,k}}$

Advantages of NAFF:

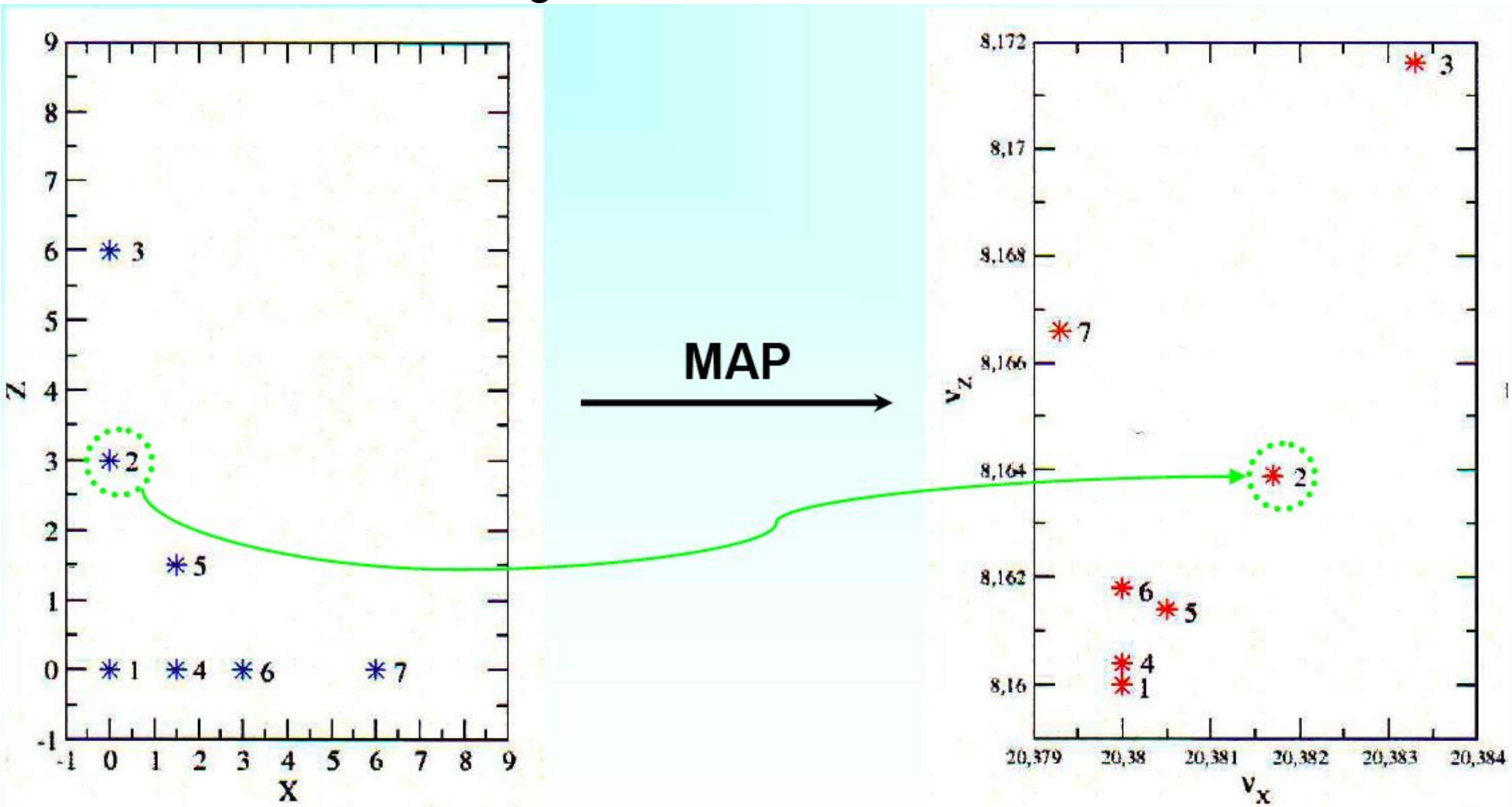
a) Very accurate representation of the "signal" $f_j(t)$ (if quasi-periodic) and thus of the amplitudes

b) Determination of frequency vector $\boldsymbol{\omega} = 2\pi\boldsymbol{\nu} = 2\pi(\nu_1, \nu_2, \ldots, \nu_n)$

with high precision $\frac{1}{\tau^4}$ for Hanning Filter

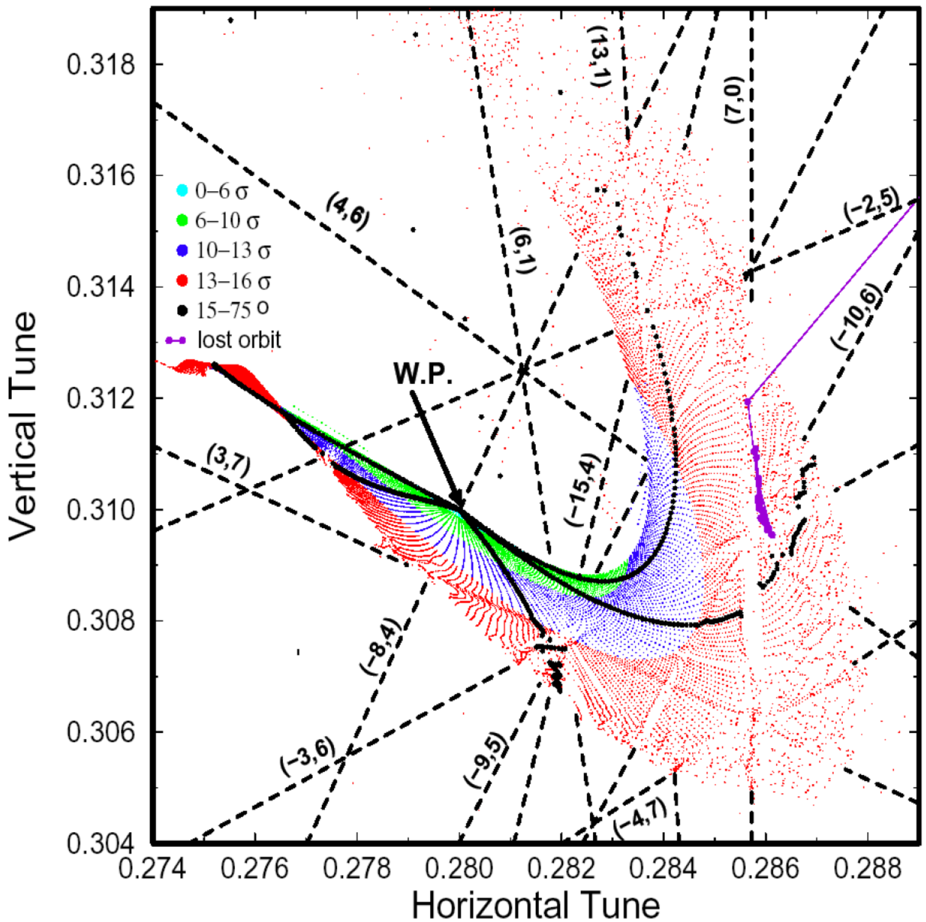
Building the frequency map

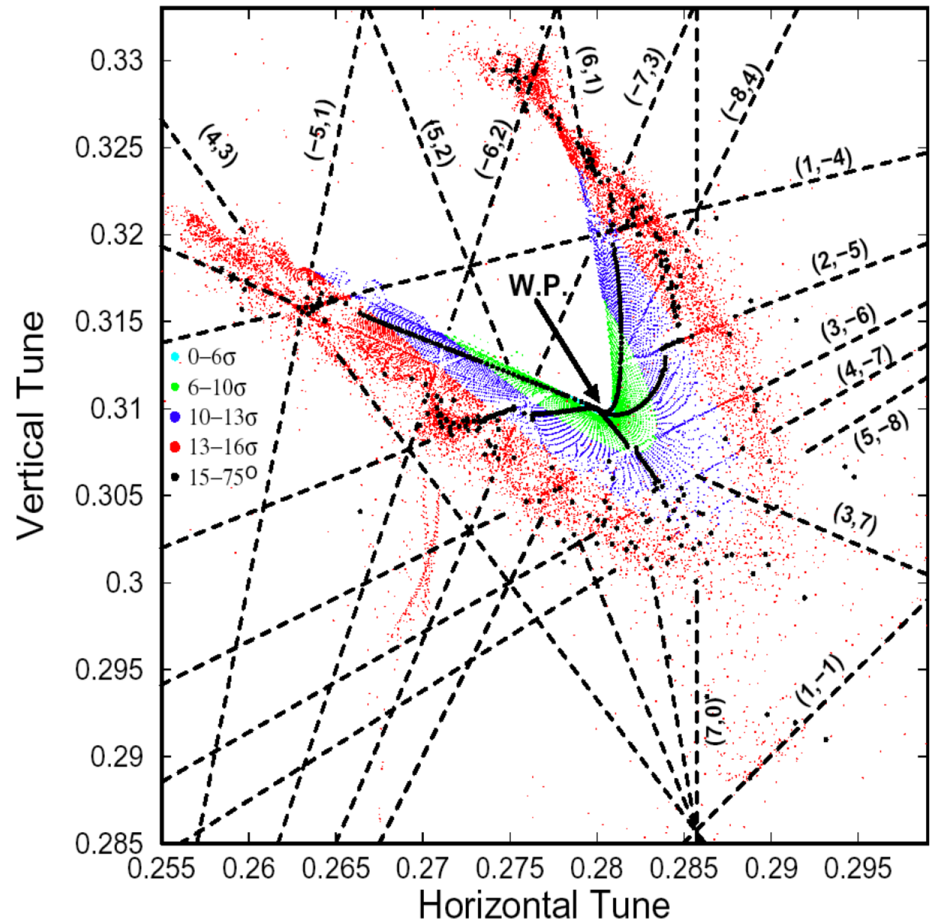
- Choose coordinates (x_i, y_i) with p_x and p_y=0
- Numerically integrate the phase trajectories through the lattice for sufficient number of turns
- Compute through NAFF Q_x and Q_y after sufficient number of turns
- Plot them in the tune diagram



Frequency maps for the LHC

Y. Papaphilippou, PAC1999





- Frequency maps for the target error table (left) and an increased random skew octupole error in the super-conducting dipoles (right)

Diffusion Maps

J. Laskar, PhysicaD, 1993

- Calculate frequencies for two equal and successive time spans and compute frequency diffusion vector:

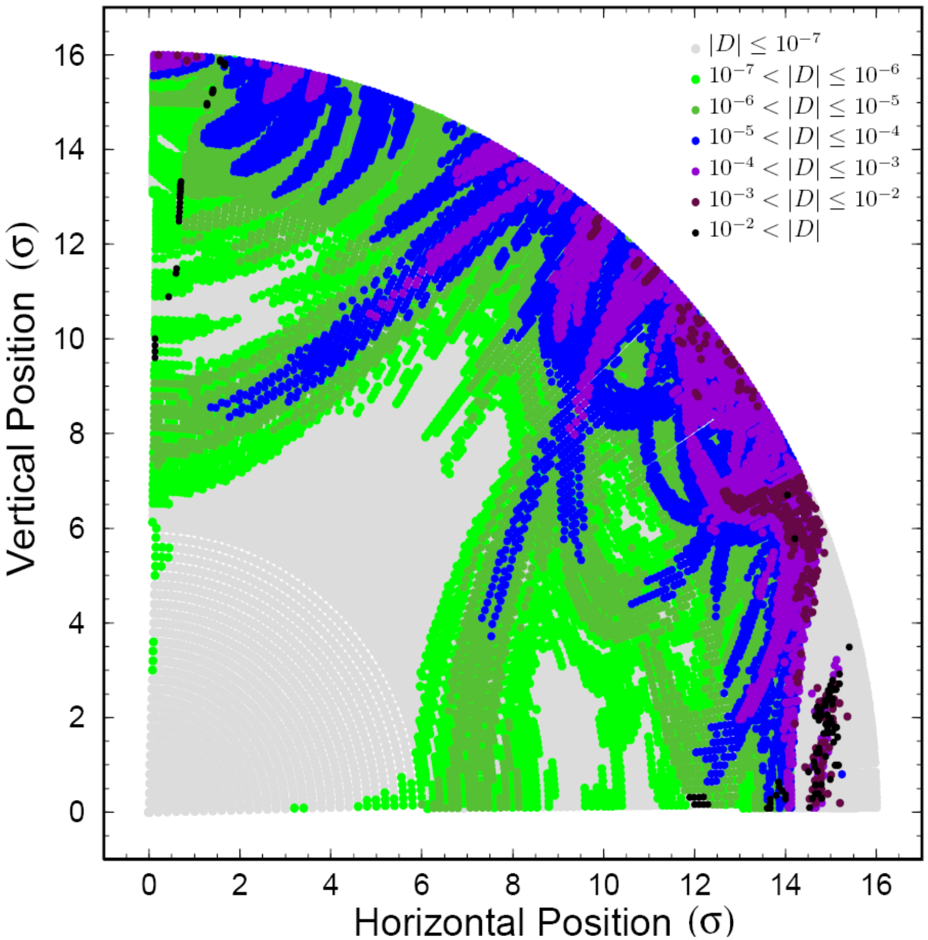
$$\boldsymbol{D}\big|_{t=\tau} = \boldsymbol{\nu}\big|_{t\in(0,\tau/2]} - \boldsymbol{\nu}\big|_{t\in(\tau/2,\tau]}$$

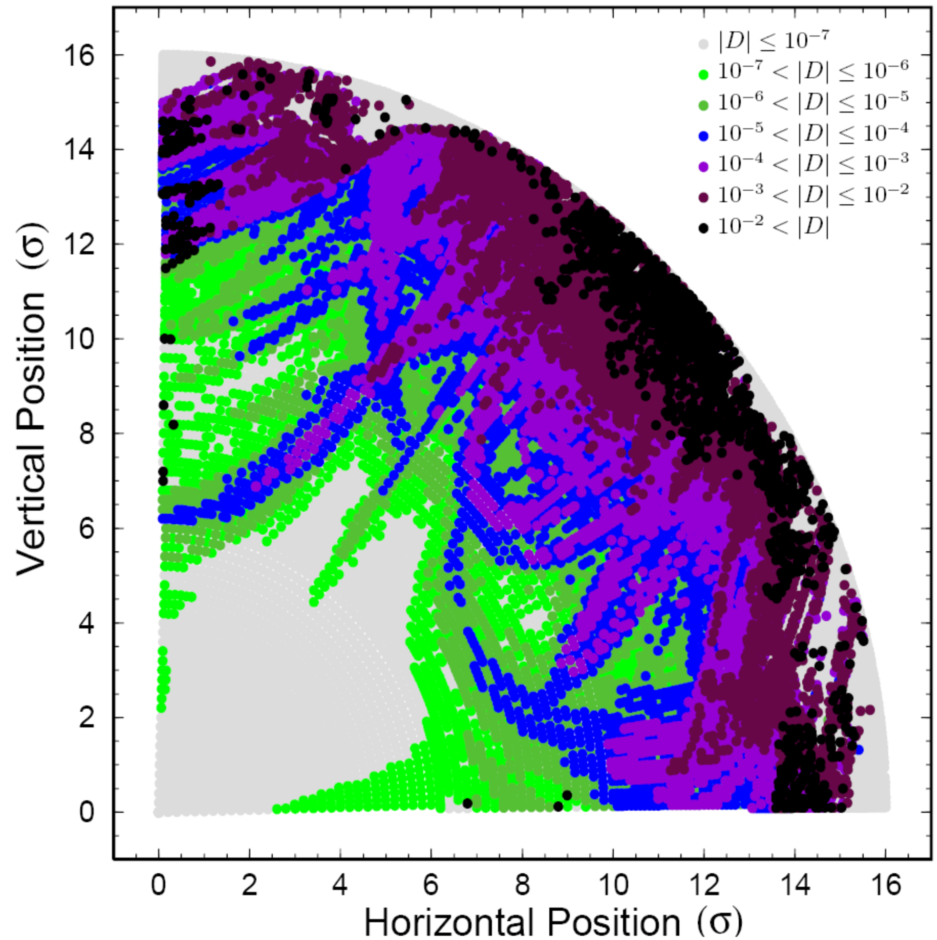
- Plot the initial condition space color-coded with the norm of the diffusion vector
- Compute a diffusion quality factor by averaging all diffusion coefficients normalized with the initial conditions radius

$$D_{QF} = \left\langle \frac{|\boldsymbol{D}|}{(I_{x0}^2 + I_{y0}^2)^{1/2}} \right\rangle_R$$

Diffusion maps for the LHC

Y. Papaphilippou, PAC1999

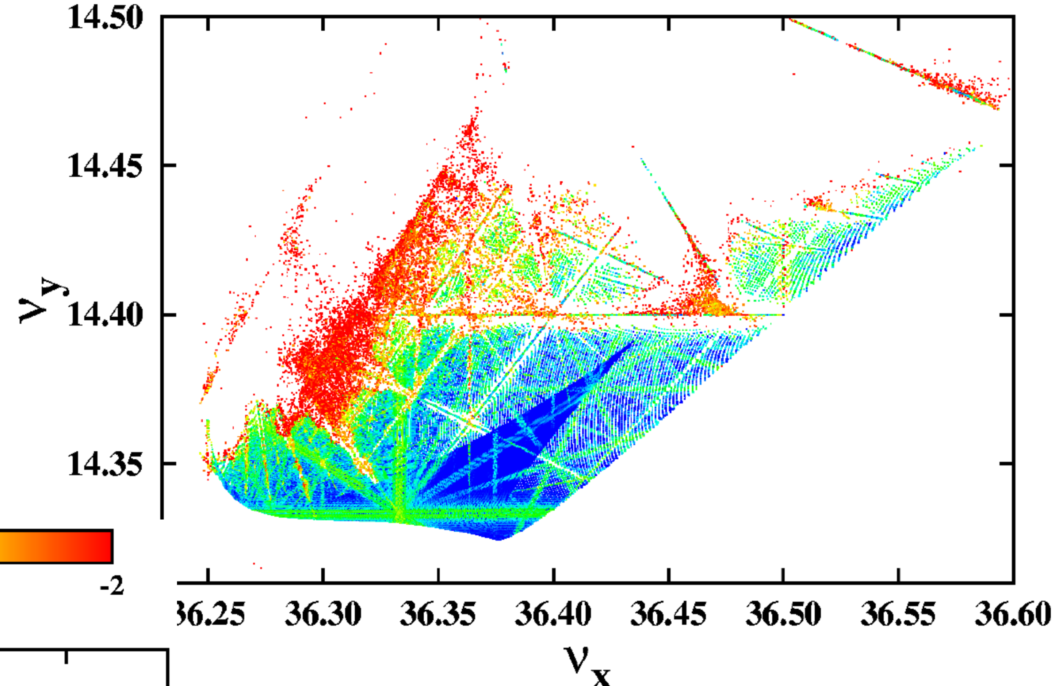


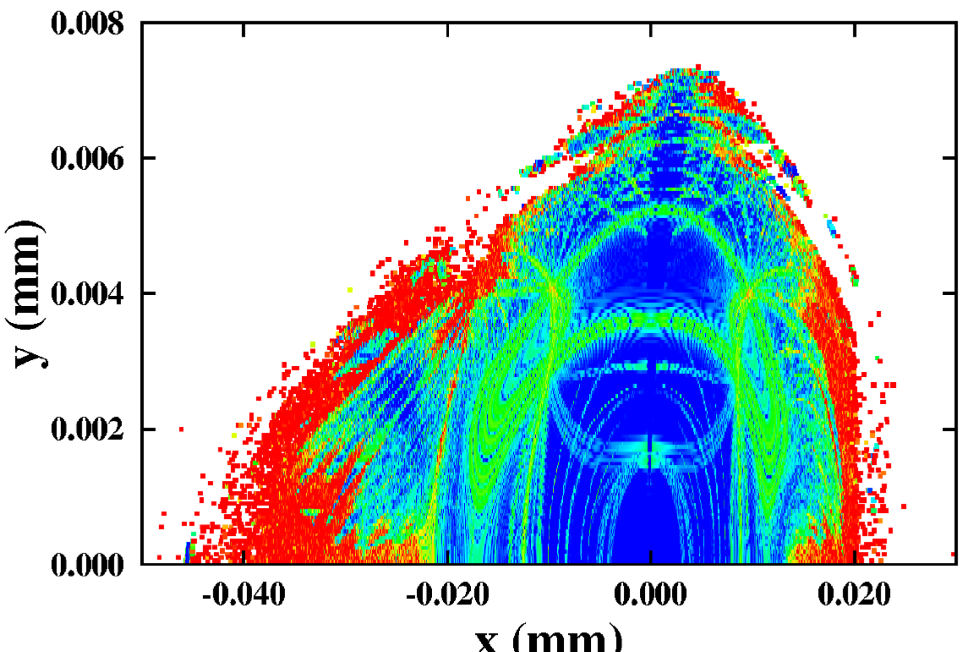


Diffusion maps for the target error table (left) and an increased random skew octupole error in the super-conducting dipoles (right)

Frequency Map for the ESRF

- All dynamics represented in these two plots
- Regular motion represented by blue colors (close to zero amplitude particles or working point)





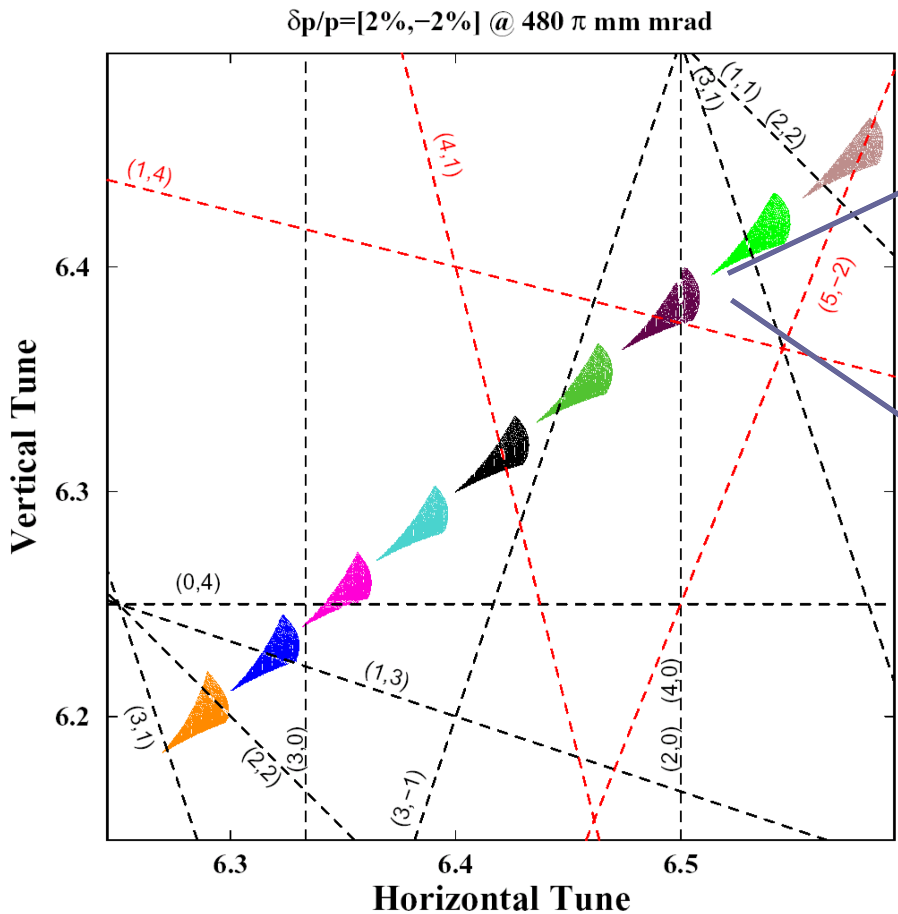
- Resonances appear as distorted lines in frequency space (or curves in initial condition space
- Chaotic motion is represented by red scattered particles and defines dynamic aperture of the machine

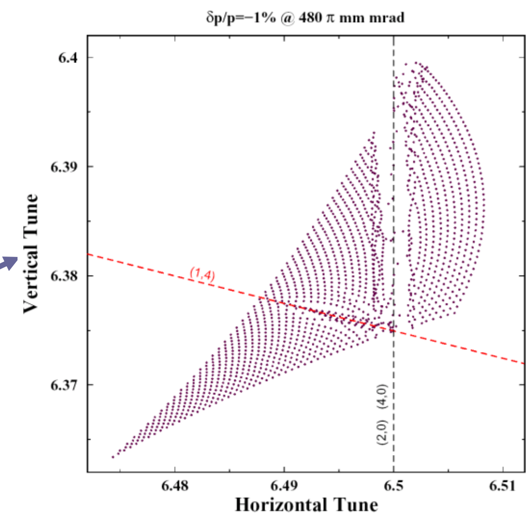
Example for the SNS ring: Working point (6.4,6.3)

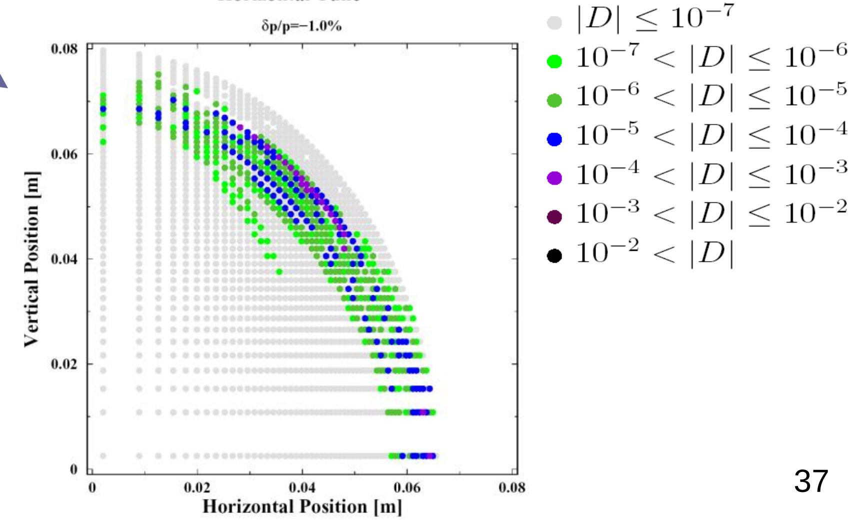
- Integrate a large number of particles
- Calculate the tune with refined Fourier analysis
- Plot points to tune space

$$\mathcal{F}_\tau \; : \; \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (I_x, I_y)|_{p_x,p_y=0}, & \longrightarrow & (\nu_x, \nu_y) \end{array}$$

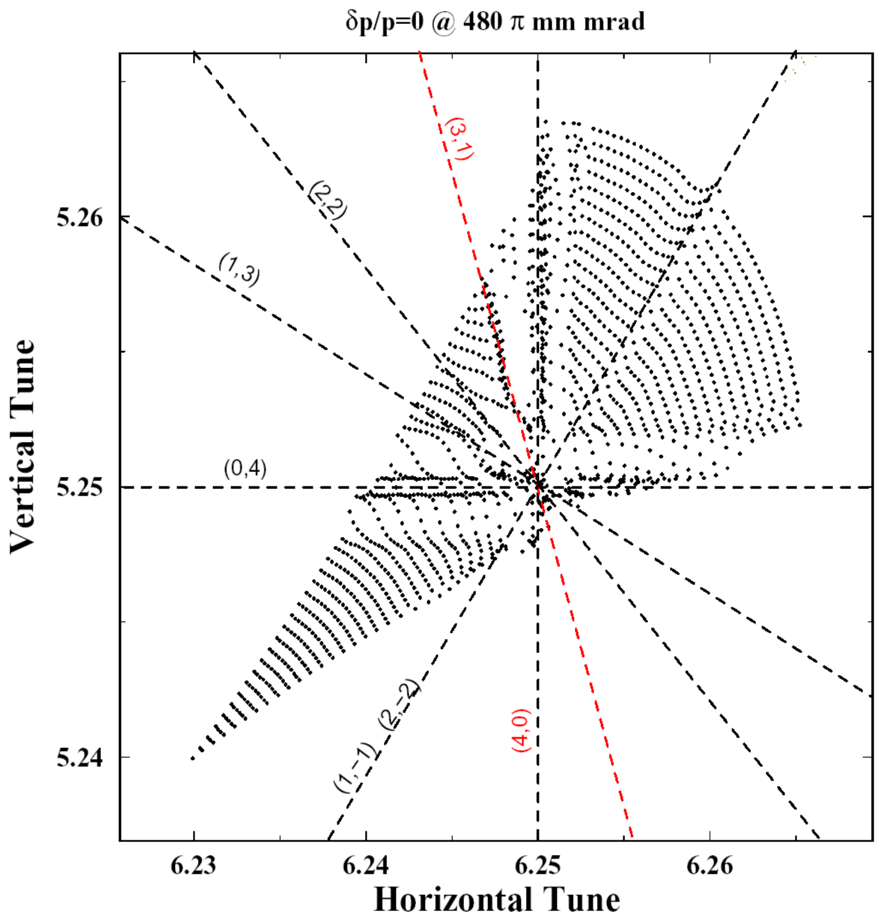
SNS Working Point $(Q_x,Q_y)=(6.4,6.3)$

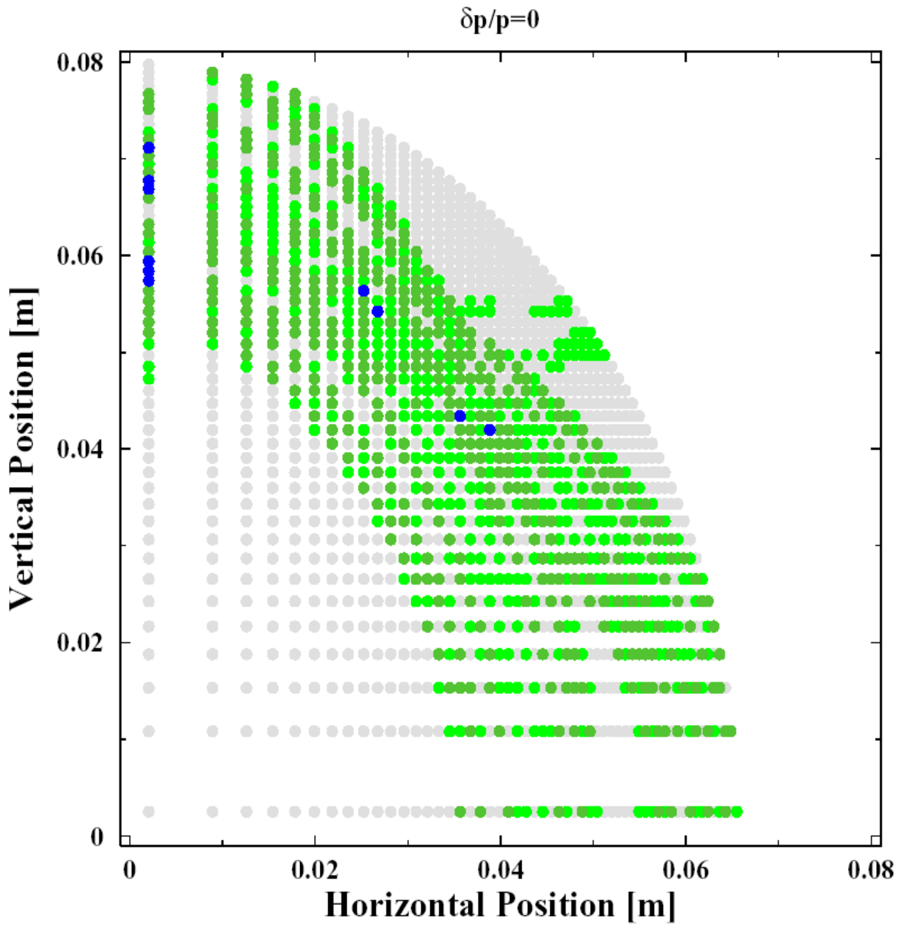






SNS Working point (6.23,5.24)

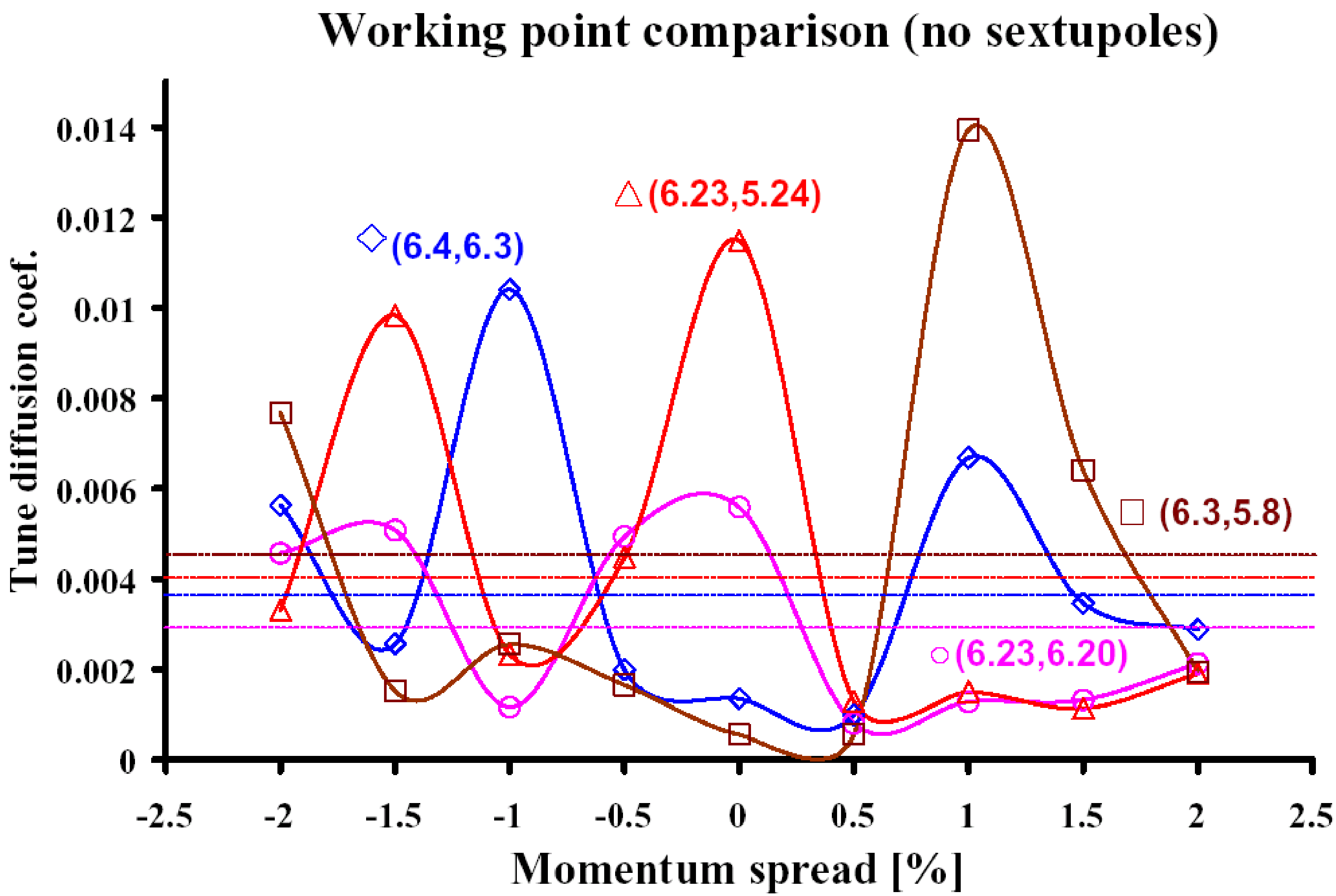




Working Point Comparison

Tune Diffusion quality factor

$$D_{QF} = \left\langle \frac{|\boldsymbol{D}|}{(I_{x0}^2 + I_{y0}^2)^{1/2}} \right\rangle_R$$



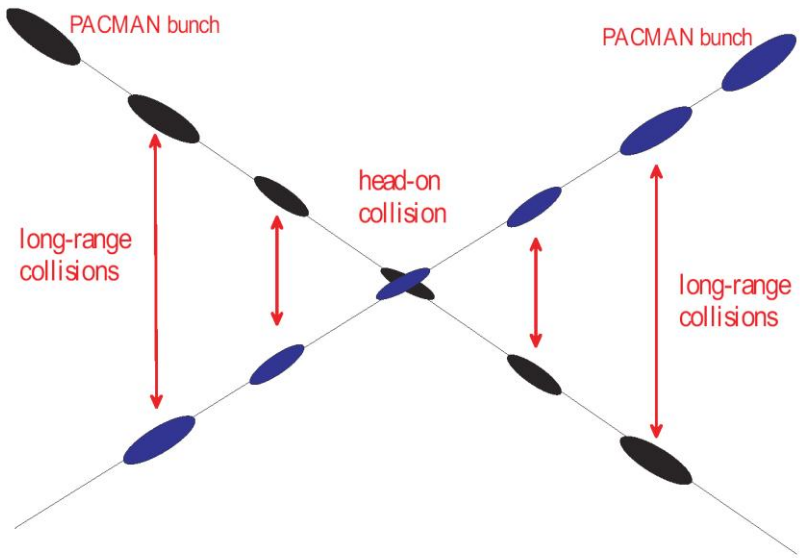
Variable	Symbol	Value
Beam energy	E	7 TeV
Particle species	$\cdots$	protons
Full crossing angle	θ_c	300 μrad
rms beam divergence	σ'_x	31.7 μrad
rms beam size	σ_x	15.9 μm
Normalized transv. rms emittance	$\gamma\varepsilon$	3.75 μm
IP beta function	β^*	0.5 m
Bunch charge	N_b	$(1\times 10^{11}-2\times 10^{12})$
Betatron tune	Q_0	0.31

- Long range beam-beam interaction represented by a 4D kick-map

$$\Delta x = -\, n_{par}\frac{2r_pN_b}{\gamma}\left[\frac{x'+\theta_c}{\theta_t^2}\left(1-e^{-\frac{\theta_t^2}{2\theta_{x,y}^2}}\right) - \frac{1}{\theta_c}\left(1-e^{-\frac{\theta_c^2}{2\theta_{x,y}^2}}\right)\right]$$

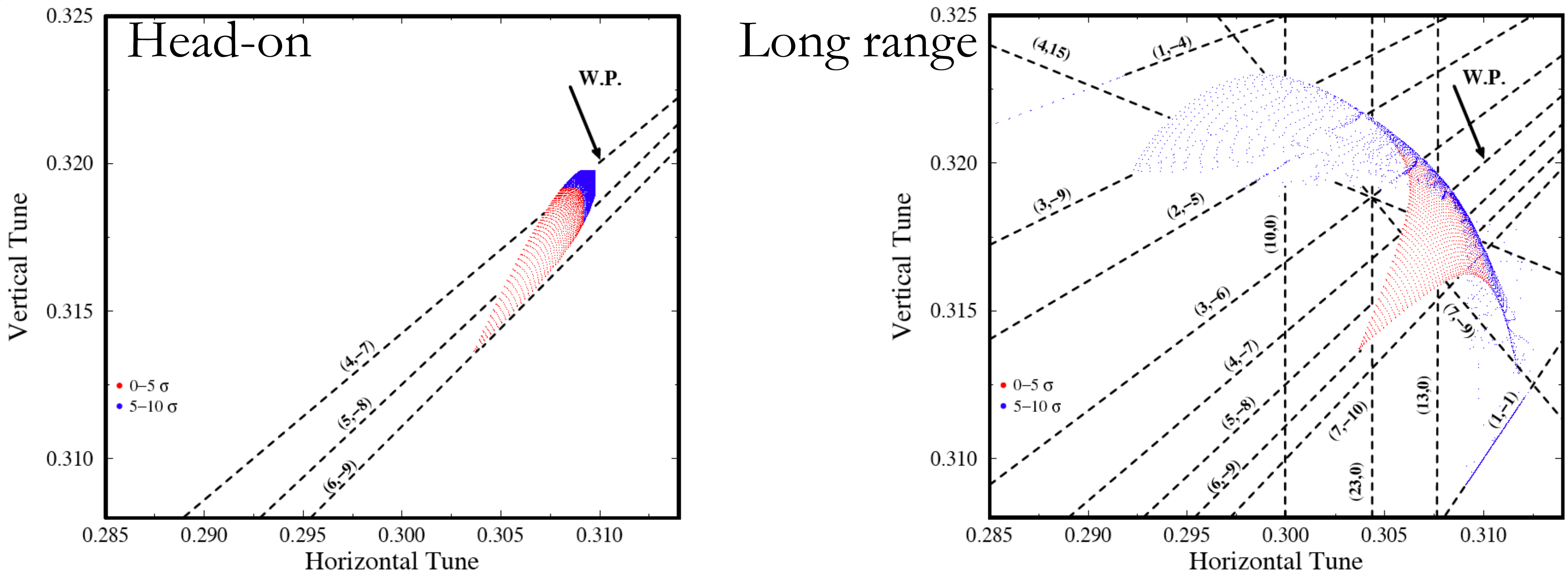
$$\Delta y = -\, n_{par}\frac{2r_pN_b}{\gamma}\,\frac{y'}{\theta_t^2}\left(1-e^{-\frac{\theta_t^2}{2\theta_{x,y}^2}}\right)$$

with $$\theta_t \equiv \left((x'+\theta_c)^2 + {y'}^2\right)^{1/2}$$



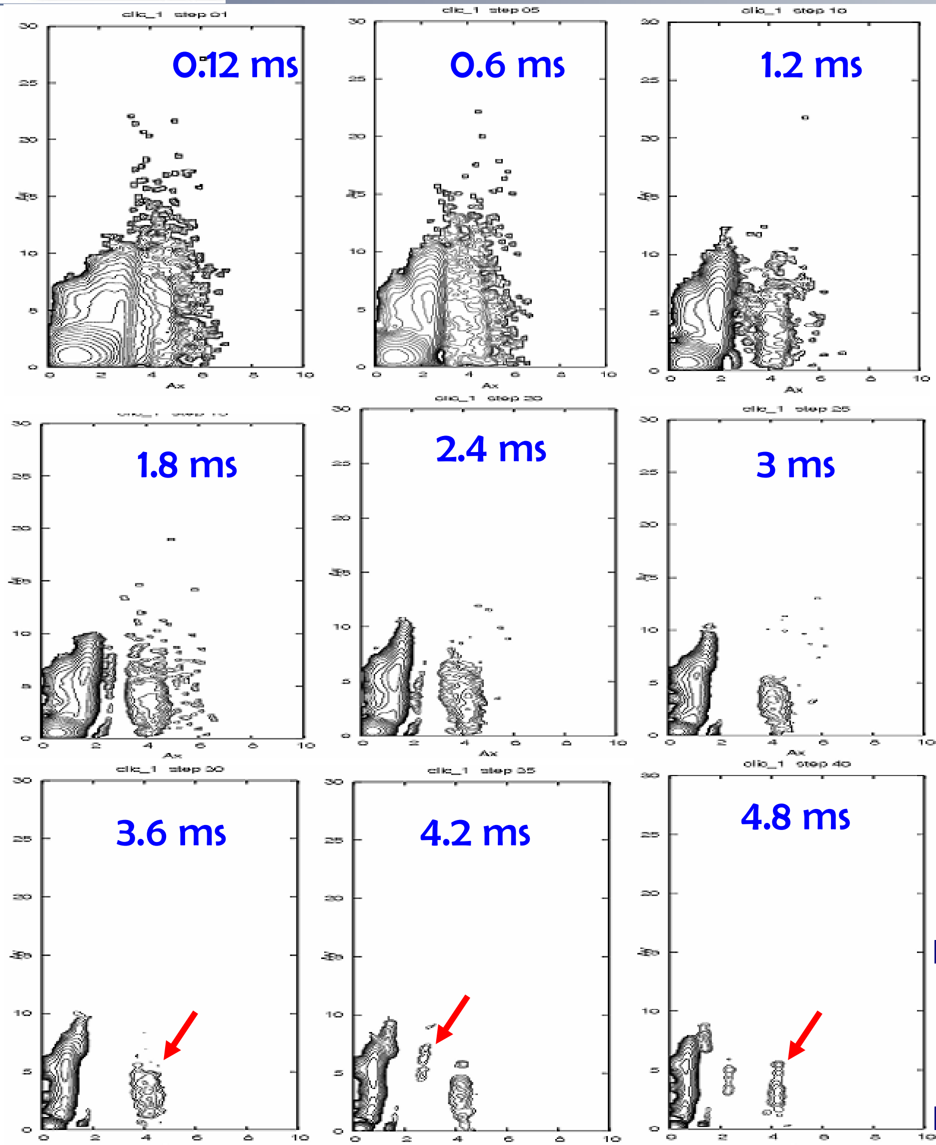
Head-on vs Long range interaction

YP and F. Zimmermann, PRSTAB 1999, 2002

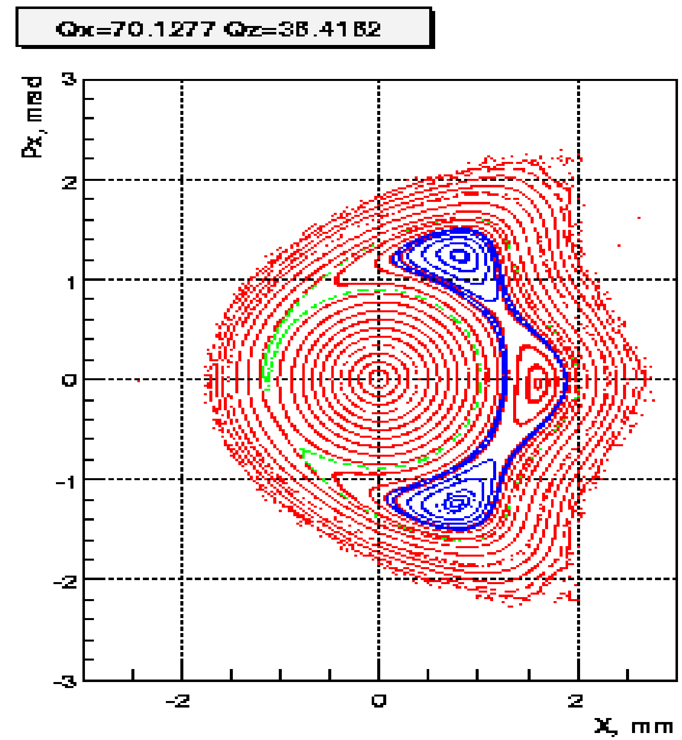


- Proved dominant effect of long range beam-beam effect
- Dynamic Aperture (around 6σ) located at the folding of the map (indefinite torsion)
- Dynamics dominated by the 1/r part of the force, reproduced by electrical wire, which was proposed for correcting the effect
- Experimental verification in SPS and installation to the LHC IPs

CLIC Damping ring dynamics



E. Levichev et al. PAC2009

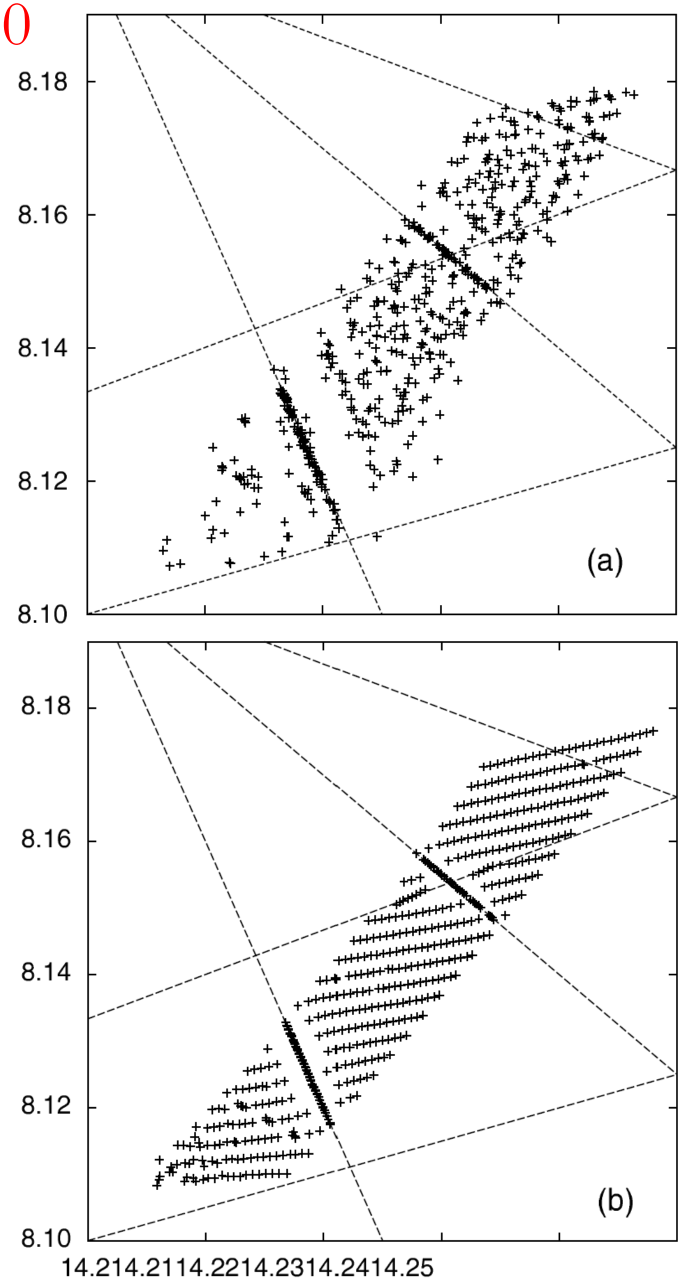


- Including radiation damping and excitation shows that 0.7% of the particles are lost during the damping
- Certain particles seem to damp away from the beam core, on resonance islands

Experimental frequency maps

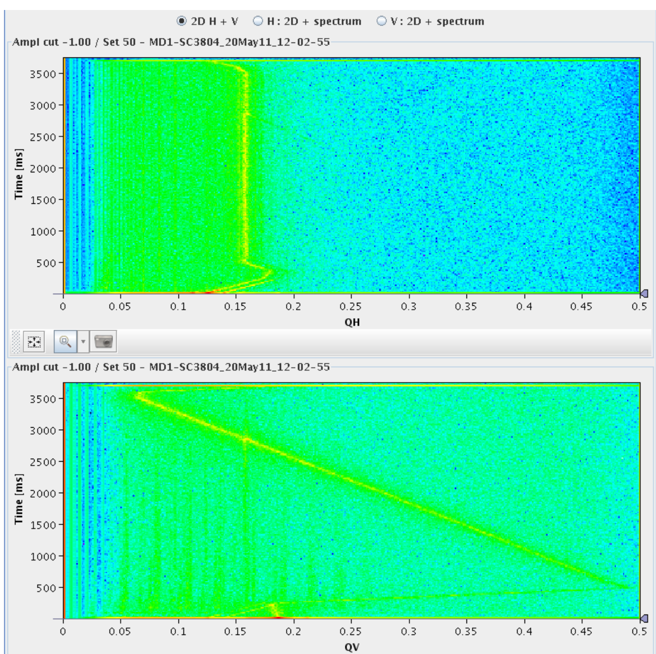
D. Robin et al. PRL 2000

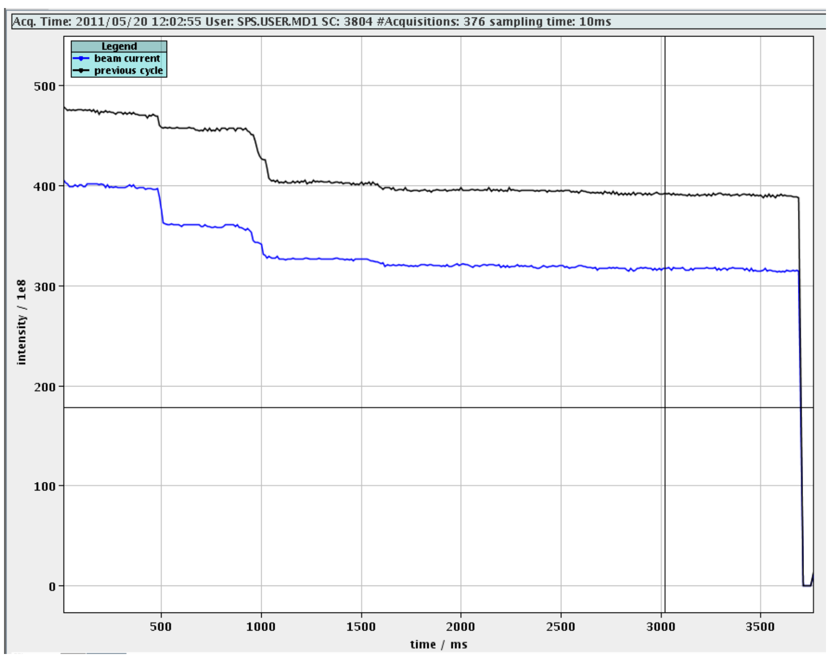
- Frequency analysis of turn-by-turn data of beam oscillations produced by a fast kicker magnet and recorded on a Beam Position Monitors
- Reproduction of the non-linear model of the Advanced Light Source storage ring and working point optimization for increasing beam lifetime



Experimental Methods – Tune scans

- Study the resonance behavior around different working points in SPS
- Strength of individual resonance lines can be identified from the beam loss rate, i.e. the derivative of the beam intensity at the moment of crossing the resonance
- Vertical tune is scanned from about 0.45 down to 0.05 during a period of 3s along the flat bottom
- Low intensity 4-5e10 p/b single bunches with small emittance injected
- Horizontal tune is constant during the same period
- Tunes are continuously monitored using tune monitor (tune post-processed with NAFF) and the beam intensity is recorded with a beam current transformer





- Resonances in low γ_t optics
  - Normal sextupole Qx+2Qy is the strongest
  - Skew sextupole 2Qx+Qy quite strong
  - Normal sextupole Qx-2Qy, skew sextupole at 3Qy and 2Qx+2Qy fourth order visible
- Resonances in the nominal optics
  - Normal sextupole resonance Qx+2Qy is the strongest
  - Coupling resonance (diagonal, either Qx-Qy or some higher order of this), Qx-2Qy normal sextupole
  - Skew sextupole resonance 2Qx+Qy weak compared to Q20 case
  - Stop-band width of the vertical integer is stronger (predicted by simulations)