Special Relativity: Tutorial

Exercise 1

An object has a velocity of 30km/s in system S' moving also with 30km/s. What is its velocity in the lab system? Repeat the calculation with 270 000km/s.

$$u_x' = v = 30\mathrm{km}/s = 10^{-4}c$$

$$u_x = \frac{u_x' + v}{1 + u_x' v/c^2} = \frac{2\cdot 10^{-4}c}{1+10^{-8}} \simeq 60\left(1-10^{-8}\right) km/s$$

$$u_x' = v = 270\,000\mathrm{km}/s = 0.9\mathrm{c}$$

$$u_x = 1.8\mathrm{c}/\left(1+0.81\right) = 0.995\mathrm{c}$$

Exercise 2

Prove that the scalar product of any two 4-vectors is Lorentz invariant.

$$A^\mu = \left(A^0, A^1, A^2, A^3\right), \quad B^\mu = \left(B^0, B^1, B^2, B^3\right)$$

$$A^{\mu} B_{\mu} = A^{0} B_{0} - A^{1} B_{1} - A^{2} B_{2} - A^{3} B_{3}$$

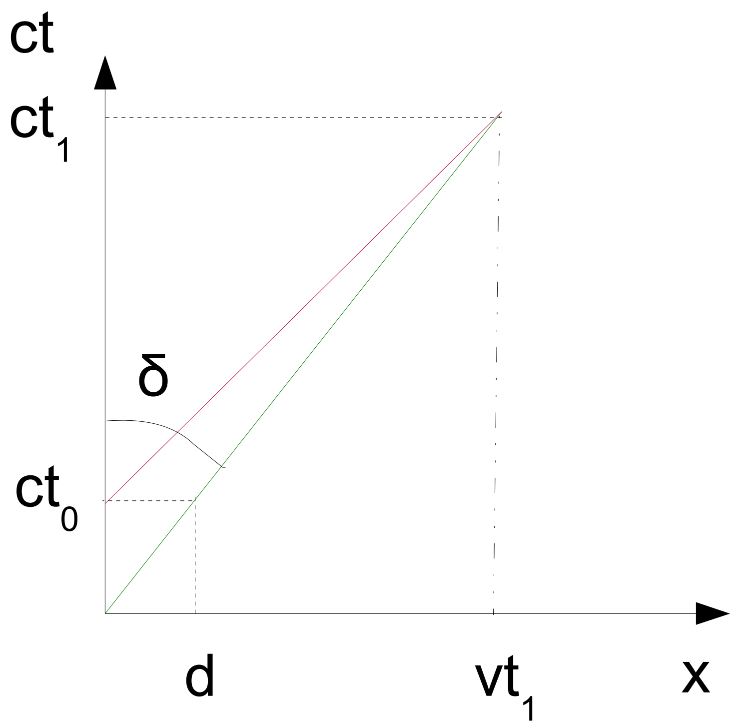
L-T gives

$$A'^{\mu} = \left(\gamma\left(A^{0} - \beta A^{1}\right), \gamma\left(-\beta A^{0} + A^{1}\right), A^{2}, A^{3}\right)$$
$$B'^{\mu} = \left(\gamma\left(B^{0} - \beta B^{1}\right), \gamma\left(-\beta B^{0} + B^{1}\right), B^{2}, B^{3}\right)$$

$$\rightarrow \qquad A'^{\mu} B'_{\mu} = A^{\mu} B_{\mu}$$

Exercise 3

A space craft travels away from earth with ß=0.8. At a distance d= 2.16•10^{8} km a radio signal from earth is transmitted to the space craft.
How long does the signal need to reach the space craft in the system of the earth?



$$x = vt \quad \rightarrow \quad \tan\delta = \beta = \frac{d}{ct_0}$$

$$ct = ct_0 + x \quad \rightarrow \quad ct_1 = ct_0 + vt_1$$

$$t_1 = \frac{t_0}{1-\beta} = 5\mathrm{t}_0$$

$$t_1 - t_0 = 4\mathrm{t}_0 = 3600\mathrm{s} = 1\mathrm{h}$$

Exercise 4

Which mass has to be lifted by 1m in order to provide the energy corresponding to 1mg of mass?

$$E_0 = m_0 c^2 = 10^{-6}\, kg\, 3^2 \cdot 10^{16}\, m^2/s^2 = 9 \cdot 10^{10}\, kg\, m^2/s^2$$

$$x\, 0.981\, m/s^2\, 1\mathrm{m} = E_0 \quad \rightarrow \quad x \simeq 10^{11}\, kg = 10^8\, t$$

Exercise 5

A charge q is at rest. At t=0 an electric field E_x is turned on. Calculate the velocity as a function of time.

$$f_x = \frac{dp_x}{dt} = qE_x \quad \rightarrow \quad p_x = qE_x t = \frac{m_0 u_x}{\sqrt{1-(u_x/c)^2}}$$

$$u_x = \frac{qE_x t/m_0}{\sqrt{1+(qE_x t/m_0 c)^2}}$$

Exercise 6

Consider the charge of exercise 5 in its instantaneous rest frame S' where it experiences a constant acceleration $\alpha = q E_x / m_0$.

It is $$\vec{u}' = (0,0,0),\ \vec{a}' = (\alpha, 0, 0)$$

with the proper acceleration α.

In S $$\vec{a} = (\alpha/\gamma^3, 0, 0)$$

and the relativ velocity between S and S' is the particle velocity

$$v = u_x$$

Integration

$$a_x = \frac{du_x}{dt} = [1 - (\frac{u_x}{c})^2]^{3/2} \alpha \quad \rightarrow \quad u_x = \frac{\alpha t}{\sqrt{1 + (\alpha t/c)^2}}$$

Integration

$$u_x = \frac{dx}{dt} = \frac{\alpha t}{\sqrt{1+(\alpha t/c)^2}} \quad \rightarrow \quad x = \frac{c^2}{\alpha}\left[\sqrt{1+(\alpha t/c)^2} - 1\right]$$

$$\left(x + \frac{c^2}{\alpha}\right)^2 - (ct)^2 = 1$$ Hyperbola

Non-relativistic case (a=α)

$$du = \alpha\, dt \quad \rightarrow \quad u = \alpha t$$

$$dx = u\, dt \quad \rightarrow \quad x = \frac{1}{2}\alpha t^2$$ Parabola

Exercise 7

Particles a and b with u_a and u_b are in frame S. Use the velocity 4-vector to derive u_b' in S', the rest frame of a.

in S: $$U_a^{\mu} U_{b\mu} = \gamma_a \gamma_b (c^2 - \vec{u}_a \cdot \vec{u}_b) \qquad (i)$$

in S': $$U'^{\mu}_a U'_{b\mu} = \gamma_b' c^2 \qquad (ii)$$

equating (i) and (ii) and devision by c^2

$$\gamma_b' = \gamma_a \gamma_b \left(1 - \frac{\vec{u}_a \cdot \vec{u}_b}{c^2}\right) \qquad (iii)$$

inverse of (iii) and squared

$$\frac{1}{\gamma'^2_b} = 1 - \left(\frac{u'_b}{c}\right)^2 = \frac{(1-(u_a/c)^2)(1-(u_b/c)^2)}{(1-(\vec{u}_a \cdot \vec{u}_b)/c^2)^2}$$

$$u'^2_b = \frac{(u_a - u_b)^2 - u_a^2 u_b^2 / c^2 + (u_a \cdot u_b)^2 / c^2}{(1 - (u_a \cdot u_b)/c^2)^2}$$

and using $(\vec{u}_a \times \vec{u}_b)^2 = u_a^2 u_b^2 - (\vec{u}_a \cdot \vec{u}_b)^2$

$$u'^2_b = \frac{(\vec{u}_a - \vec{u}_b)^2 - ((\vec{u}_a \times \vec{u}_b)/c)^2}{(1 - (\vec{u}_a \cdot \vec{u}_b)/c^2)^2}$$

non-relativistic case: $u'_b = |\vec{u}_a - \vec{u}_b|$

Exercise 8

A particle moves in S with velocity $\vec{u}$ and experiences a force $\vec{f}$. What is the force in S'?.

It is $F^{\mu} = \gamma_u \left(\frac{1}{c} \vec{f} \cdot \vec{u}, f_x, f_y, f_z \right)$

L-Transformation of F

$$F'^1 = \gamma_v \left(F^1 - \beta_v F^0 \right) \quad \rightarrow \quad \gamma_{u'} f'_x = \gamma_v \left(\gamma_u f_x - \beta_v \frac{\gamma_u}{c} \vec{f} \cdot \vec{u} \right)$$

$$F'^2 = F^2 \quad \rightarrow \quad \gamma_{u'} f'_y = \gamma_u f_y$$

$$F'^3 = F^3 \quad \rightarrow \quad \gamma_{u'} f'_z = \gamma_u f_z$$

If the particle moves with u_x in S it is in S'

$$u'_x=\frac{u_x-v}{1-(u_x v)/c^2} \quad \rightarrow \quad \gamma_{u'}=\gamma_u\gamma_v\left(1-\frac{u_x v}{c^2}\right)$$

and we obtain

$$f'_x=\frac{f_x-(v/c^2)\vec{f}\cdot\vec{u}}{1-(u_x v)/c^2}$$

$$f'_y=\frac{f_y}{\gamma_v(1-(u_x v)/c^2)}$$

$$f'_z=\frac{f_z}{\gamma_v(1-(u_x v)/c^2)}$$

$\xrightarrow{u_x=0}$

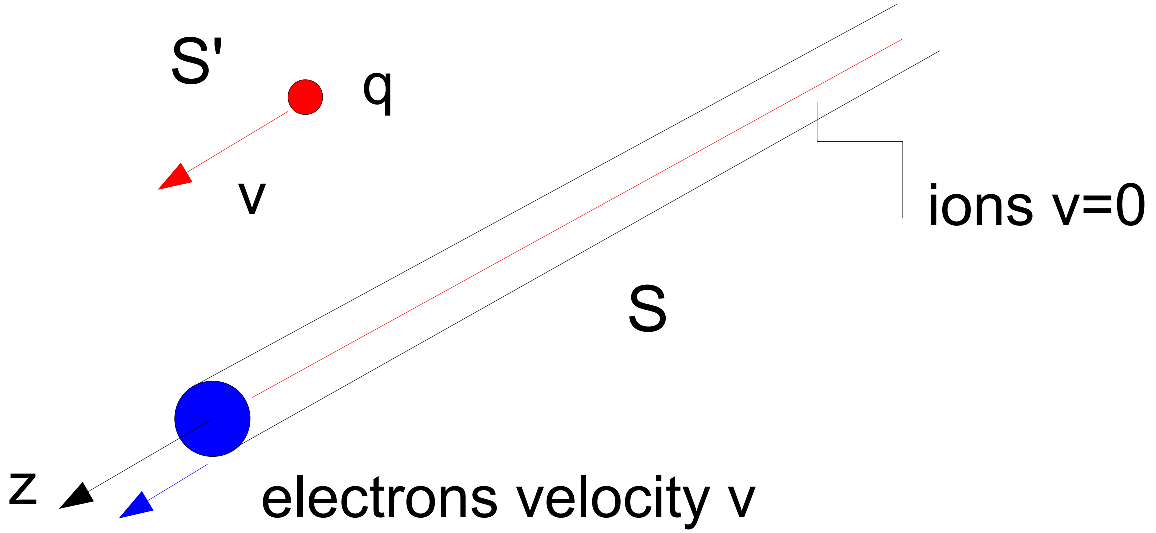
$$f'_x=f_x$$

$$f'_y=\frac{1}{\gamma_v}f_y$$

$$f'_z=\frac{1}{\gamma_v}f_z$$

Exercise 9

A point charge q moves parallel to a current carrying wire. By transforming the e.-m. fields calculate the force in its rest frame.



in S: $\vec{E}=(0,0,0),$ $\vec{B}=(0,B_\varphi,0)$

$\vec{f}_C=(0,0,0),$ $\vec{f}_L=(-qvB_\varphi,0,0)$

Transformation of fields

$$\vec{E}'_{\parallel}=\vec{E}_{\parallel}, \qquad \vec{B}'_{\parallel}=\vec{B}_{\parallel}$$

$$\vec{E}'_{\perp}=\gamma(\vec{E}_{\perp}+\vec{v}\times\vec{B}), \qquad \vec{B}'_{\perp}=\gamma(\vec{B}_{\perp}-\frac{1}{c^2}\vec{v}\times\vec{E})$$

In S':

$$\vec{E}'=(-\gamma v B_{\varphi},0,0), \qquad \vec{B}'=(0,\gamma B_{\varphi},0)$$

$$\vec{f}'_C=q\vec{E}'=(-q\gamma v B_{\varphi},0,0), \quad \vec{f}'_L=(0,0,0)$$

$$\vec{f}'_C=\gamma\vec{f}_L$$