TRANSVERSE MOTION & ELECTROSTATIC ELEMENTS

JUAS

Joint Universities Accelerator School

Lecture 3

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Introduction

- **There is a fundamental difference between electric and magnetic elements. When traversing a magnetic field the ion's energy is rigorously constant, whereas in an electric field the ion can exchange energy with the field.**

- **This means that both the mass and velocity of the ion can vary as the ion traverses an electrostatic element. Since the velocity affects the transit time the kick is affected.**

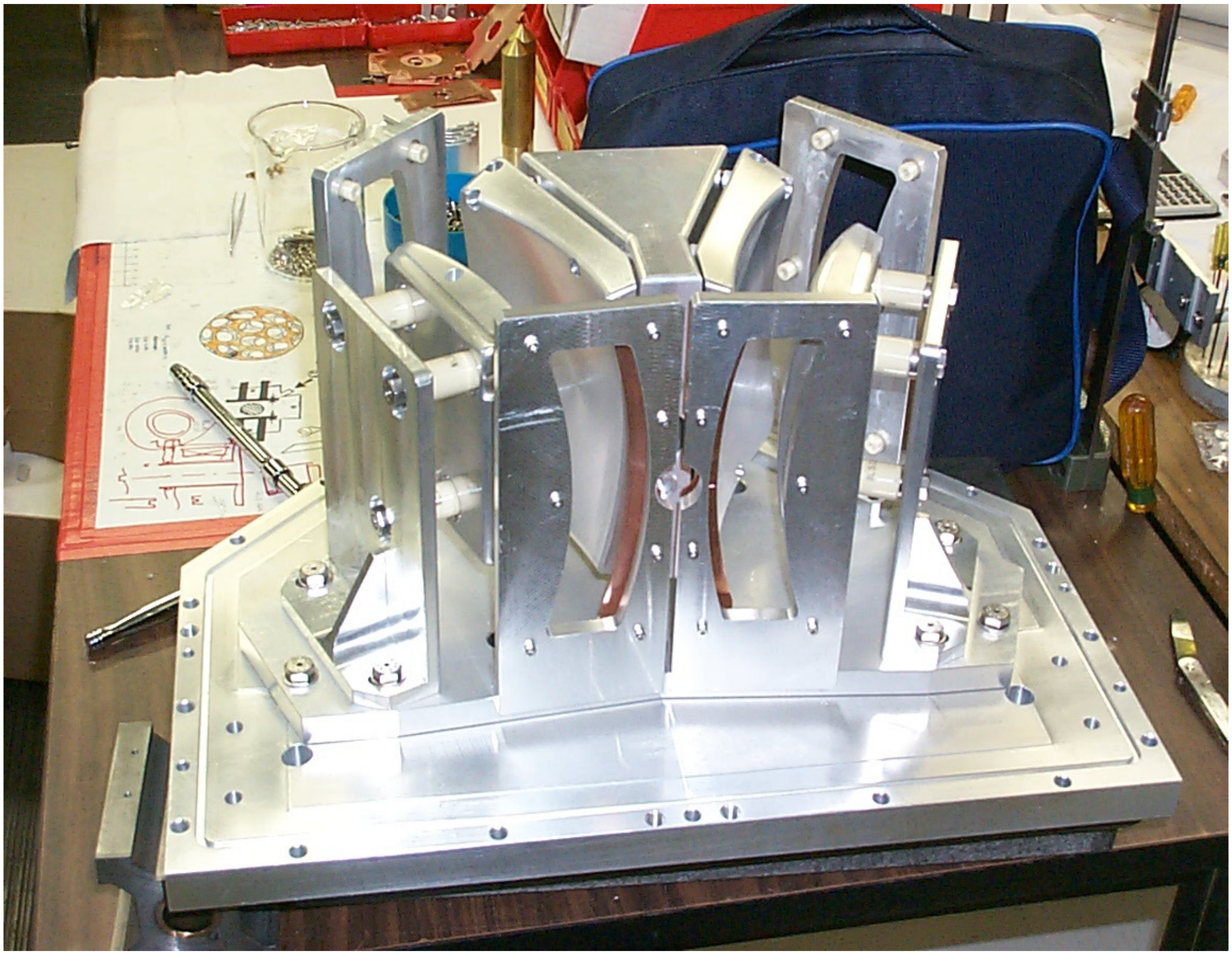
- **In most of the literature, this is elegantly handled by using Lagrangian mechanics, but for consistency we will follow an analysis exactly parallel to the analysis for magnetic elements. This shows more clearly where and when the differences occur and what approximations are being made.**

- **The following will treat the transverse motion and does not apply to elements that are designed to accelerate longitudinally.**

'Small-angle' approximation

- **It will be assumed that the angular deviations and transverse excursions in straight elements, i.e. quadrupoles are always small, so that,**
  - **the transverse electric field is always perpendicular to the particle motion and does not affect the longitudinal velocity and hence the transit time and kick remain unchanged,**
  - **the transverse excursions are small so that the transverse energy change is negligible.**
- **Thus, electrostatic quadrupoles behave in essentially the same way as magnetic quadrupoles.**
- **This leaves the bends, which can have large angles and exhibit new effects.**
- **It is further assumed that the elements are housed in earthed enclosures, so that there can be no net energy difference between the incoming and outgoing ions.**

A complicated electrostatic bend



- **Three-way electrostatic bend: left, right and straight through. Electrodes are spherical (concentric) giving focusing in both planes.**

- **In general electrodes are:**
  - **Concentric cylinders (*cylindrical bend*), or**
  - **Concentric spheres (*spherical bend*), or**
  - **Concentric toroids (*toroidal bend*).**

Equivalent of cyclotron motion

- **The force, *F*, acting on a charged particle in an electric field is,**

$$\boxed{\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}) = q\boldsymbol{E} \qquad (1)}$$

- **The three components of this force in a cylindrical system (ρ, Θ, *y*) are well known and are written as,**

$$F_\rho = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\rho}) - m\rho\dot{\theta}^2 = qE_\rho$$

$$F_\theta = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}t}\left(m\rho^2\dot{\theta}\right) = qE_\theta \qquad (2)$$

$$F_y = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{y}) = qE_y$$

FOR REFERENCE

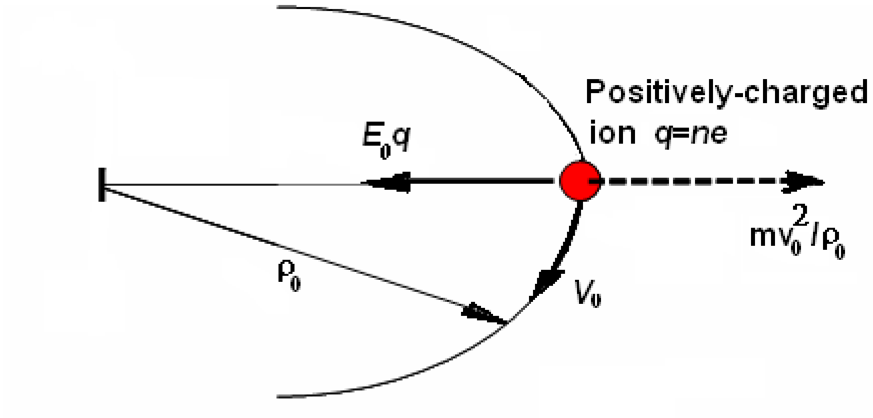
DO NOT MEMORIZE

- **The equivalent of cyclotron motion is obtained by launching an ion perpendicular to a radial field, so that $E_\rho = E_0$ (constant), $E_\Theta = E_y = 0$, $\rho = \rho_0$ (constant) and $y = y_0$ (constant), to give,**

$$\boxed{E_0 q = -\frac{m v_0^2}{\rho_0} \qquad (3)}$$

where $q = ne$ is the ion's charge and m its relativistic mass.

Electrical rigidity



- Alternatively, the equivalent of cyclotron motion is found simply by equating the expressions for the centripetal force,

$$E_0 q = -\frac{mv_0^2}{\rho_0}$$

- This gives the *electric rigidity*, the equivalent of the *magnetic rigidity*. Re-writing (3) as an 'engineering' formula gives,

$$\boxed{\left|E_0\rho_0\right| [\mathrm{kV}] = \frac{(\gamma+1)}{\gamma}\frac{A}{n}\left|\overline{T}\right| [\mathrm{keV}] \qquad (4)}$$

- where n is the charge number of the ion so that $q = ne$, A is the atomic mass number and T is the average kinetic energy per nucleon. This relation defines the central orbit in an electrostatic bend just as the magnetic rigidity defines the central orbit in a dipole.

Note on derivation of (4)

$$E_0 q = -\frac{mv_0^2}{\rho_0}$$

$$E_0 \rho_0 = -\frac{mv_0^2}{q}$$

$$\left|E_0 \rho_0\right| = \frac{mc^2 \beta^2}{q} = \frac{mc^2}{q}\left(1 - \frac{1}{\gamma^2}\right) \qquad \text{(A)}$$

$$mc^2 = m_0 c^2 + T$$

$$T = m_0 c^2 (\gamma - 1) \quad \text{so that} \quad m_0 c^2 = \frac{T}{(\gamma - 1)}$$

$$mc^2 = \frac{T}{(\gamma - 1)} + T = T\frac{\gamma}{(\gamma - 1)} \qquad \text{(B)}$$

Substitute (B) into (A)

$$\left|E_0 \rho_0\right| = \frac{1}{q}\left(1 - \frac{1}{\gamma^2}\right) T\frac{\gamma}{(\gamma - 1)} = \frac{T}{e}\frac{(\gamma + 1)(\gamma - 1)}{\gamma^2}\frac{\gamma}{(\gamma - 1)}$$

$$\left|E_0 \rho_0\right| = \frac{T}{e}\frac{(\gamma + 1)}{\gamma}$$

Electric rigidity continued

- The electric rigidity leads to a second 'engineering' formula for the bending angle,

$$\boxed{\alpha = \frac{\int E_{\perp}[\mathrm{kV/m}]\,\mathrm{d}s[\mathrm{m}]}{|E_0\rho_0|\,[\mathrm{kV}]} \qquad (5)}$$

Note, as with the magnetic rigidity, these formulæ avoid applying sign conventions.

- Compare the magnetic and electrostatic derivations. There is an additional factor *v* in the electrostatic case. This occurs because the magnetic force contains the velocity whereas the electrostatic force does not. This difference is the root of the complications in this lecture.

- As in the magnetic case the bending angle is found by,

$$\alpha = \frac{\ell}{\rho} \quad \Rightarrow \quad \alpha = \frac{\ell}{\rho}\frac{E\rho}{E\rho} \quad \Rightarrow \quad \alpha = \frac{\int E\mathrm{d}s}{|E\rho|}$$

Transverse motion in an electrostatic bend

- **Consider ions that enter with a momentum deviation, Δp that contains the mass and velocity deviations Δm and Δv. Since the bend is in a screened enclosure, the ion must leave with the same momentum deviation.**
- **Inside the device, the ion can exchange energy with the electric field and suffer variable mass and velocity deviations denoted by δm and δv.**
- **Re-writing the radial equation from (2) with the deviations in evidence,**

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((m+\Delta m+\delta m)\frac{\mathrm{d}}{\mathrm{d}t}\rho\right)-(m+\Delta m+\delta m)\frac{(v_0+\Delta v+\delta v)^2}{\rho}=qE_\rho$$

- **This only differs from the magnetic case in that the force term is changed and the mass and velocity deviations are separated into constant and variable parts.**
- **As always, we look for approximations and first we neglect the effect of the variable mass deviation δm inside the differential, so that,**

$$(m+\Delta m+\delta m)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\rho-(m+\Delta m+\delta m)\frac{(v_0+\Delta v+\delta v)^2}{\rho}=qE_\rho$$

Transverse motion in an electrostatic bend continued

- **As usual we transform the independent variable from time, *t*, to distance, *s*, and introduce the local coordinate *x* for the excursion,**

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \left(v_0 + \Delta v + \delta v\right)\frac{\mathrm{d}}{\mathrm{d}s}$$

$$\rho = \rho_0 + x$$

which gives,

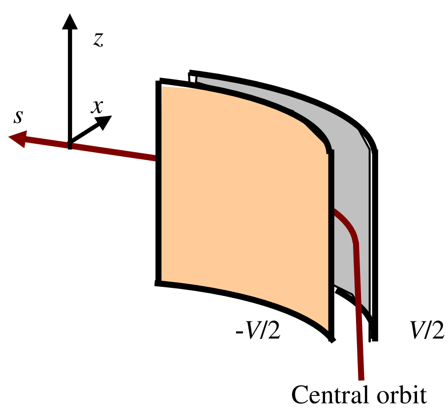
$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{1}{(\rho_0 + x)} = qE_\rho \frac{1}{(m + \Delta m + \delta m)(v_0 + \Delta v + \delta v)^2}$$

- **Expanding the RHS and truncating to 1st order gives,**

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{1}{(\rho_0 + x)} = \frac{qE_\rho}{m v_0^2}\left(1 - \underbrace{\left[\frac{\Delta m}{m} + 2\frac{\Delta v}{v_0}\right]}_{\text{Constant increments defined by incoming beam.}} - \underbrace{\left[\frac{\delta m}{m} + 2\frac{\delta v}{v_0}\right]}_{\text{Variable increments depending on the excursion } x \text{ inside the apparatus.}}\right)$$

Transverse motion in an electrostatic bend continued

- It is now necessary to evaluate the field between two cylindrical plates biased at $\pm V/2$ with respect to the screening enclosure. Neglecting any fringe fields, the equi-potential surfaces will be concentric with the electrode surfaces, so that,



$$E_0 \ell = E_\rho (\ell + \Delta \ell)$$

$$\theta = \frac{\ell}{\rho_0} = \frac{(\ell + \Delta \ell)}{(\rho_0 + x)}$$

$$E_0 \rho_0 = E_\rho (\rho_0 + x)$$

$$E_x \equiv E_\rho = E_0 \frac{\rho_0}{(\rho_0 + x)} \qquad (6)$$

- Substituting for the field from (6) and *q/m* from (3) gives,

$$\frac{d^2 x}{ds^2} - \frac{1}{(\rho_0 + x)} = -\frac{1}{(\rho_0 + x)} \left(1 - \left[\frac{\Delta m}{m} + 2 \frac{\Delta v}{v_0} \right] - \left[\frac{\delta m}{m} + 2 \frac{\delta v}{v_0} \right] \right) \quad (7)$$

Treating the constant increments

- **The constant increments are defined by the incoming (and outgoing) beam with respect to the ideal central orbit and can be re-expressed using the beam parameters outside the apparatus. It follows that,**

$$\frac{\Delta m}{m} = \frac{\beta^2}{\left(1-\beta^2\right)}\frac{\Delta v}{v}$$

- **so that,**

$$\frac{\Delta m}{m} + 2\frac{\Delta v}{v_0} = \frac{\left(2-\beta^2\right)}{\beta^2}\frac{\Delta m}{m}$$

- **The mass variation $\Delta m/m$ is the same as the total energy variation $\Delta E/E$, which in turn can be expressed in terms of $\Delta p/p$ so,**

$$\frac{\Delta m}{m} + 2\frac{\Delta v}{v_0} = \frac{\Delta p}{p}\left(2-\beta^2\right) \qquad (8)$$

Treating the variable increments

- **The variable increments are zero unless a betatron oscillation or energy mismatch takes the ion away from the central orbit. When this happens, the ion absorbs or releases energy to the electrostatic field. The energy change alters the ion's velocity, which alters its transit time and hence the deflection. Since there are no azimuthal forces (central force approximation), the conservation of angular momentum can be used to relate the ideal central orbit particle to the off-axis particle,**

$$m\rho_0 v_0 = (m+\delta m)(\rho_0 + x)(v_0 + \delta v)$$

- **When expanded and truncated to first order, this gives a relation between the excursion and the incremental changes,**

$$\frac{x}{\rho_0} = -\frac{\delta m}{m} - \frac{\delta v}{v_0} = -\frac{1}{\left(1-\beta^2\right)}\frac{\delta v}{v}$$

- **Further manipulation gives,**

$$\frac{\delta m}{m} + 2\frac{\delta v}{v_0} = -\frac{x}{\rho_0}\left(2-\beta^2\right) \qquad (9)$$

Transverse motion equation

- The combination of (7), (8) and (9) finally gives the radial motion equation,

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + \frac{1}{{\rho_0}^2}\left(2-\beta^2\right)x = \frac{1}{\rho_0}\frac{\Delta p}{p}\left(2-\beta^2\right) \quad (10)$$

- ***This only differs from the equivalent magnetic equation for a pure dipole by the factor (2-β^2).*** **So apart from this additional term, the same general solutions can be used to construct the transfer matrices.**
- **At relativistic energies (as $\beta \to 1$), the difference between the magnetic and electric motion equations disappears.**
- **Comparing magnetic and electric bends shows that:**

  At low energies, the (1+γ) term in the numerator of the rigidity improves the efficiency of electrostatic bends. The convenience of being able to shape the field with simple mechanical surfaces, calculate the field from simple mechanical dimensions, the low power consumption, the absence of hysteresis and the absence of heating leads to these devices being widely used.

Matrices for on-momentum ions

- The transfer matrix of a focusing element (K>0) is:

$$\begin{pmatrix} \cos\left(\sqrt{|K|}\ell\right) & \frac{1}{\sqrt{|K|}}\sin\left(\sqrt{|K|}\ell\right) \\ -\sqrt{|K|}\sin\left(\sqrt{|K|}\ell\right) & \cos\left(\sqrt{|K|}\ell\right) \end{pmatrix} \quad (11)$$

- The transfer matrix of a defocusing element (K<0) is:

$$\begin{pmatrix} \cosh\left(\sqrt{|K|}\ell\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}\ell\right) \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}\ell\right) & \cosh\left(\sqrt{|K|}\ell\right) \end{pmatrix} \quad (12)$$

where,

$$K_x = \frac{1}{{\rho_0}^2}\left(2-\beta^2\right), \quad K_{\mathrm{y}} = 0 \qquad (13)$$

- The transfer matrix of a drift space (K=0) is:

$$\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \qquad (14)$$

Solutions of the motion equation including momentum

- **As for the magnetic case z can be x or y,**

$$\begin{pmatrix} z \\ z' \\ \Delta p/p \end{pmatrix}_2 = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ z' \\ \Delta p/p \end{pmatrix}_1 \qquad (15)$$

- **The terms m_{11}, m_{12}, m_{21} and m_{22} have the forms from the earlier slide. The new terms have the forms,**

For $K>0$:

$$m_{13} = \pm\left(2-\beta^2\right)\frac{1}{\rho}\frac{1}{|K|}\left[1-\cos\left(\sqrt{|K|}\ell\right)\right]$$

$$m_{23} = \pm\left(2-\beta^2\right)\frac{1}{\rho}\frac{1}{\sqrt{|K|}}\sin\left(\sqrt{|K|}\ell\right) \qquad (16)$$

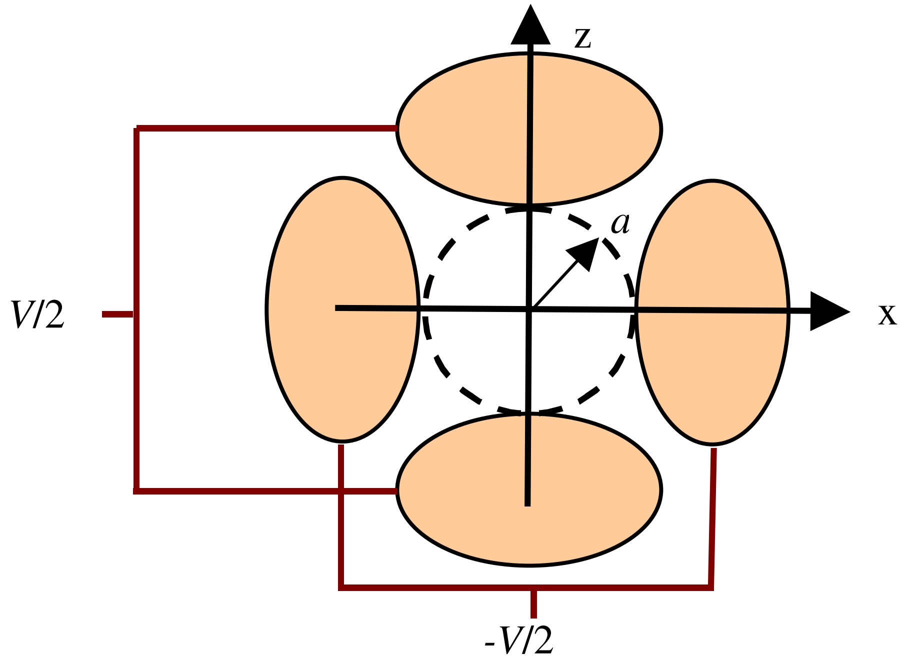
For $K<0$:

$$m_{13} = \pm\left(2-\beta^2\right)\frac{1}{\rho}\frac{1}{|K|}\left[\cosh\left(\sqrt{|K|}\ell\right)-1\right]$$

$$m_{23} = \pm\left(2-\beta^2\right)\frac{1}{\rho}\frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}\ell\right) \qquad (17)$$

[Upper sign for horizontal and lower sign for vertical bending. *K* values for cylindrical, spherical and toroidal electrodes are in the Formula Book.]

Transverse motion in an electrostatic quadrupole



- **Invoking the 'small-angle' approximation gives,**

$$mv^2 \frac{\mathrm{d}^2 x}{\mathrm{d}s^2} = qE_\mathrm{x}$$

$$mv^2 \frac{\mathrm{d}^2 y}{\mathrm{d}s^2} = qE_\mathrm{y}$$

[Basically we are following the derivation in Lecture 2]

- **The field components E_x and E_y are derived from the potential,**

$$\boxed{\Phi = -\frac{V}{2a^2}\left(x^2 - y^2\right) \quad (18)}$$

Electrostatic quadrupole continued

- **The field components are:**

$$E_x = -\frac{\partial \Phi}{\partial x} = \frac{V}{a^2}x$$

$$E_y = -\frac{\partial \Phi}{\partial y} = \frac{V}{a^2}y$$

- **After some substitutions:**

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{Vq}{a^2 m v^2}x = 0$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}s^2} + \frac{Vq}{a^2 m v^2}y = 0$$

where,

$$\boxed{K_{\substack{x\\y}} = \frac{Vq}{a^2 m v^2} = \mp \frac{1}{|E_0 \rho_0|}\frac{V}{a^2} \qquad (19)}$$

and the matrices (11) and (12) also apply.

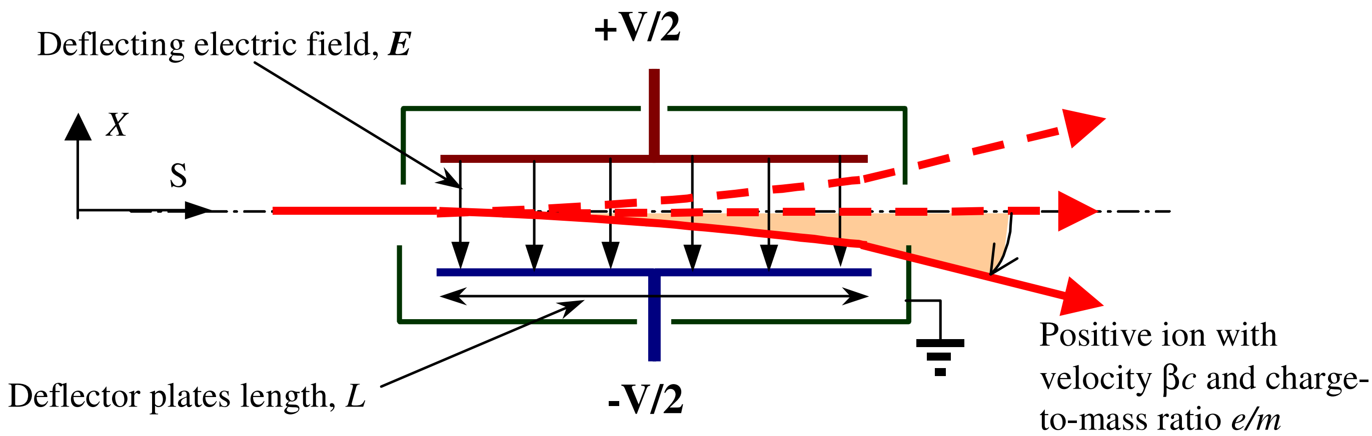
Electrostatic quadrupole continued

- **The main difference between magnetic and electrostatic quadrupoles is the way in which the *K*-factor is expressed:**

  - **For electrostatic lenses, we use the voltage across the electrodes because the potential is well defined and a simple calculation can lead to an accurate knowledge of the field.**

  - **Whereas for magnets, the corresponding pole tip field is difficult to measure and the current in the coil is related in a non-linear manner to this field. For these reasons, the gradient on the axis is preferred.**

Straight parallel-plate deflectors



- **Straight parallel-plate deflectors are frequently used for switching.**
- **Although they look simple, the detailed analysis is, in fact, more complicated.**
- **First the central orbit is not circular. It is parabolic and that is already an approximation.**

$$X_{\text{C.O.}} = -\frac{S^2}{2\rho_0}$$

where *X* and *S* are survey coordinates not local beam coordinates.

- **The vertical motion is treated as if it were a drift space and the horizontal motion can be approximated as,**

$$x = x_0 + x_0' s + \left[\frac{\Delta p}{p}\left(2-\beta^2\right)-1\right]\frac{s^2}{2\rho_0}$$

Energy, Parameter space

- The electrostatic devices in this lecture are mathematically valid for all energies. However, the lecture is more appropriate for energies above a few MeV.

- At lower non-relativistic energies (keV), you may come across cylindrically symmetric lenses such as the *Einzel* electrostatic lens and the *Glaser* magnetic solenoid lens, e.g. in electron microscopes and electron guns.

- For the Einzel lens, it is usual to use a cylindrical co-ordinate system and to include the energy changes (which will be proportionally larger) into the equation of motion in a more basic way.

Summary

- We have seen how the transverse motion in an electrostatic bend is affected by energy being exchanged with the bending field.

- The resultant equations for the bend are basically the same as those of the magnetic case with an additional multiplier $(2-\beta^2)$.

- The case of the quadrupole was treated according to the 'small angle' approximation. This neglects the energy exchanges with the field and the basic physics is then identical.

- We have also seen that to define the strength of an electrostatic lens, it is customary to refer to the voltage on the electrodes and the radius of the inscribed circle.

- The equations presented are entirely consistent with the matrix approach used in accelerator theory.

- A large part of lens theory that applies to low energies (keV) and uses cylindrically symmetric Einzel and Glaser lenses has been omitted.