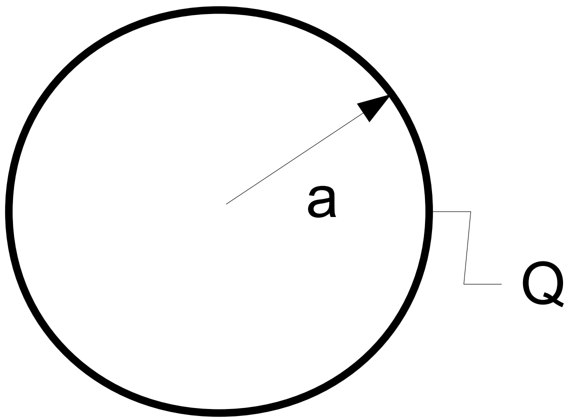
Electromagnetism: Tutorial

Tut-Ex 1

Given is a conducting hollow sphere carrying a charge Q. What is the field inside and outside and what is the stored energy?

$$\oiint \vec{D}\cdot d\vec{F} = \iiint \rho \, dV$$



$$\oiint \vec{D}\cdot d\vec{F} = \begin{cases} 0 & r \leq a \\ Q & r \geq a \end{cases} \quad \rightarrow \quad \vec{E} = \begin{cases} 0 \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\vec{e}_r \end{cases}$$

Stored energy

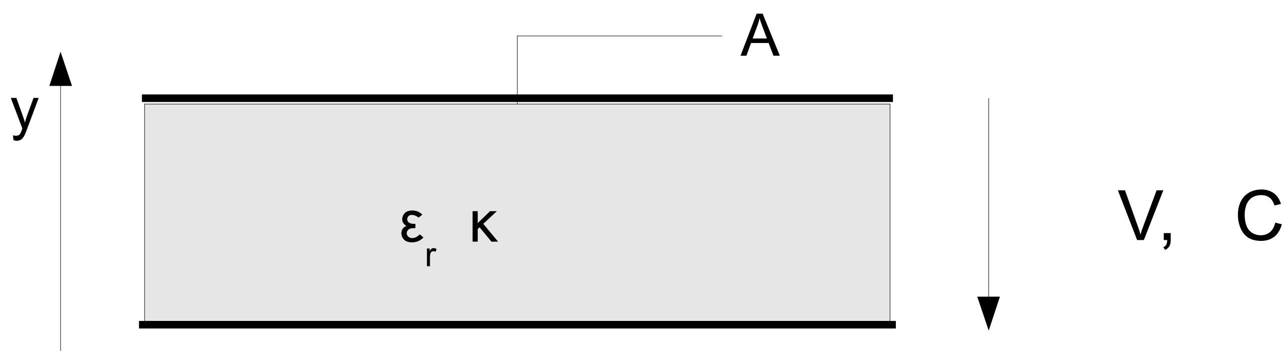
$$W_e = \frac{1}{2} \iiint \vec{E} \cdot \vec{D} \, dV = \frac{Q^2}{8\pi\epsilon_0} \oint_a^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon_0 a}$$

$$W_e = \frac{1}{2} \iiint \rho \Phi \, dV = \frac{1}{2} \oiint \rho_S \Phi \, dF$$

$$= \frac{1}{2} \oiint \frac{Q}{4\pi a^2} \frac{Q}{4\pi\epsilon_0 a} dF = \frac{Q^2}{8\pi\epsilon_0 a}$$

Tut-Ex 2

A capacitor C is filled with a lossy dielectric and charged to a voltage V. What is the time constant for discharge?



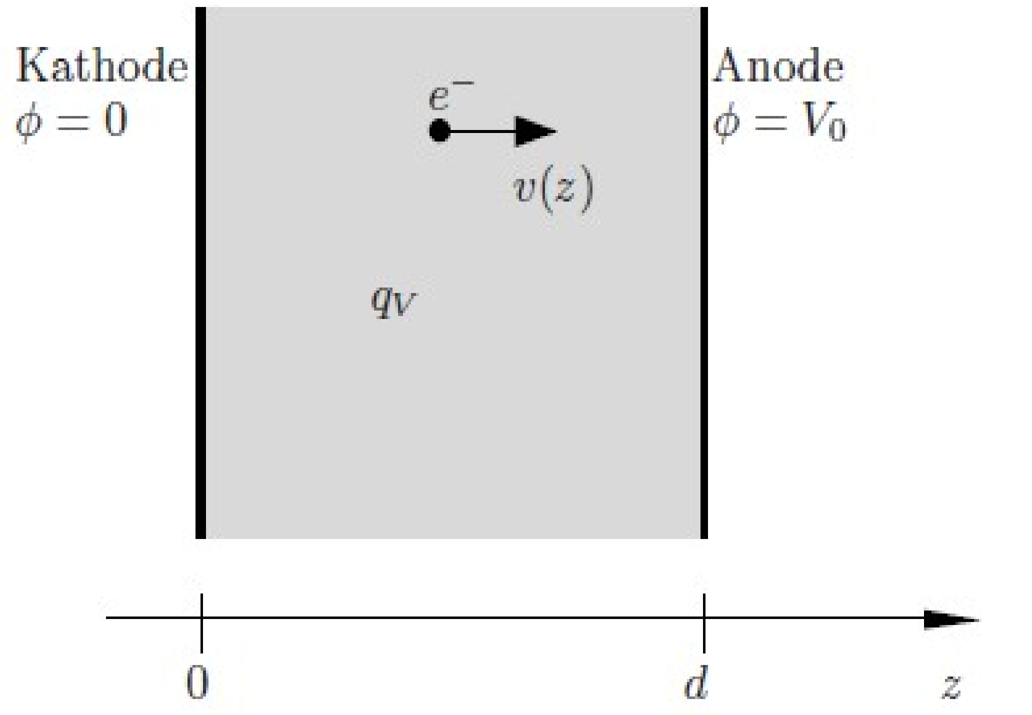
$$C\ carries\ a\ surface\ charge\ q_S = CV/A = D$$

$$\vec{\nabla}\cdot \quad \vec{\nabla}\times\vec{H} = J + \frac{\partial \vec{D}}{\partial t} \quad \rightarrow \quad \frac{\kappa}{\epsilon}\vec{\nabla}\cdot\vec{D} + \frac{d}{dt}\vec{\nabla}\cdot\vec{D} = 0$$

$$\frac{\kappa}{\varepsilon}\vec{D} + \frac{d\vec{D}}{dt} = 0 \quad \rightarrow \quad \vec{D} = \vec{D}_0 e^{-t/T_\varepsilon}, \quad T_\varepsilon = \frac{\varepsilon}{\kappa}$$

Tut-Ex 3

Given is a 1-dimensional planar diode. What is the current density at saturation?



Poisson equation:

$$\frac{d^2\Phi(z)}{dz^2} = \frac{e\, n_e(z)}{\epsilon_0}$$

at saturation:

$$E(z=0) = \frac{d\phi}{dz} = 0$$

$$\textit{kinetic energy}: \quad e\Phi(z) = \frac{1}{2} m_0 v^2(z)$$

$$\textit{current density}: \quad J = e\, n_e(e)\, v(z) = e\, n_e(z) \sqrt{\frac{2e\Phi(z)}{m_0}}$$

$$\textit{Poisson equ.}: \quad \frac{d^2\Phi}{dz^2} = \sqrt{\frac{m_0}{2e\Phi}}\, \frac{J}{\epsilon_0}$$

$$\textit{multiplying with } 2\, d\Phi/dz \textit{ and integrating}$$

$$\left(\frac{d\Phi}{dz}\right)^2 = \frac{4}{\epsilon_0} J \sqrt{\frac{m_0}{2e}} \sqrt{\Phi} + C$$

$$saturation\ at\ z=0:$$

$$\Phi=0, \quad E=d\Phi/dz=0 \quad \rightarrow \quad C=0$$

$$\frac{d\Phi}{dz}=2\left(\frac{J}{\epsilon_0}\right)^{1/2}\left(\frac{m_0}{2e}\Phi\right)^{1/4}$$

$$integration:$$

$$\frac{4}{3}\Phi^{3/4}=2\left(\frac{J}{\epsilon_0}\right)^{1/2}\left(\frac{m_0}{2e}\right)^{1/4} z+C$$

$$boundary\ conditions: \Phi(z=0)=0, \quad \Phi(z=d)=V_0$$

$$J=\frac{4}{9}\epsilon_0\sqrt{2\frac{e}{m_0}}\ \frac{V_0^{3/2}}{d^2} \qquad Child-Langmuir\ law$$

Tut-Ex 4

Derive the magnetic vector potential for a given current density.

$$\vec{\nabla}\cdot\vec{B}=0 \quad \rightarrow \quad \vec{B}=\vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\times\vec{B}=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\vec{\nabla}^2\vec{A}=\mu\vec{J}$$

$$\vec{\nabla}\cdot\vec{A}=0 \quad \rightarrow \quad \vec{\nabla}^2\vec{A}=-\mu\vec{J}$$

$$\vec{\nabla}^2 A_i=-\mu J_i, \quad i=x,y,z \qquad (1)$$

$$Poisson\ \ equation: \qquad \vec{\nabla}^2\Phi=-\frac{\rho}{\epsilon}$$

$$Potential\ of\ \ point\ charge: \qquad \Phi=q/4\pi\epsilon r$$

Potential of charge distribution:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon} \iiint \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} dV' \qquad (2)$$

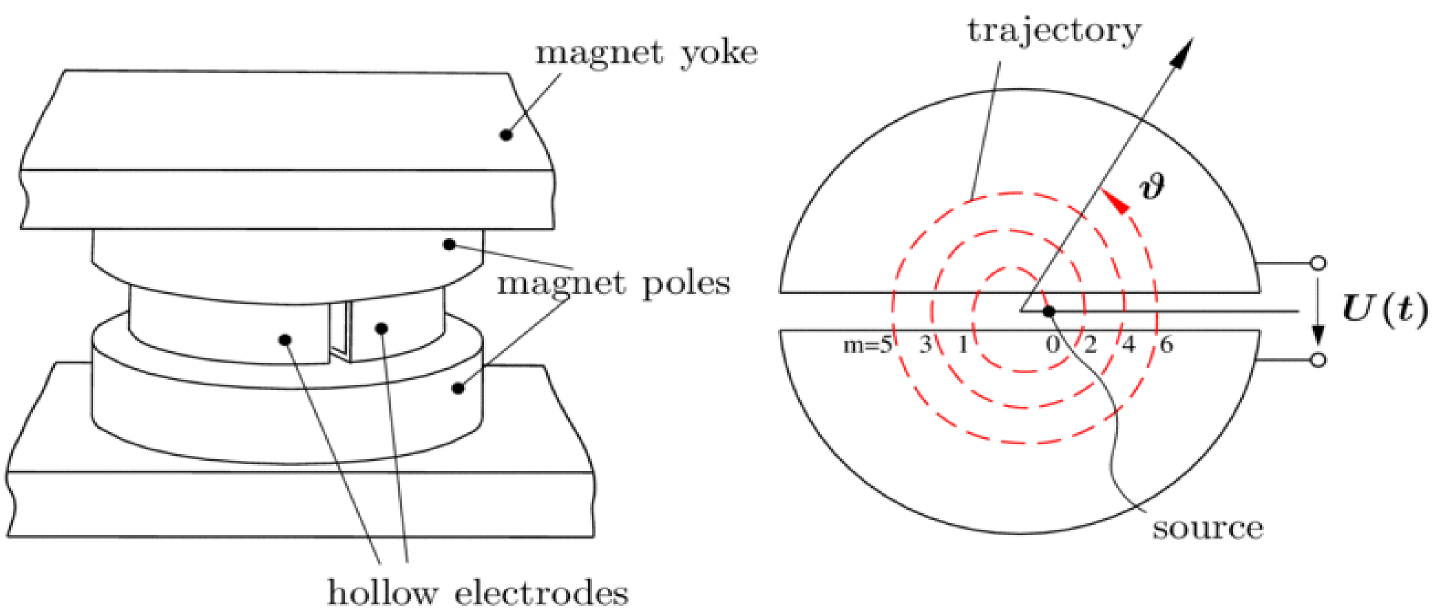
cartesian components of (1) *fulfill similar equation as* (2)

$$\Phi \rightarrow A_i, \quad \frac{1}{\epsilon} \rightarrow \mu, \quad \rho \rightarrow J_i$$

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi} \iiint \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} dV'$$

Tut-Ex 5

Given is a non-relativistic cyclotron with a constant magnetic induction B and maximum radius R.
What is the end energy and the total accelerating voltage?



centrifugal force equals Lorentz force:

$$\frac{m_0 v^2}{r} = q\,v\,B \quad \rightarrow \quad v_{max} = \frac{qB}{m_0} R$$

maximum kinetic energy:

$$E_{kin}=\frac{1}{2}m_0 v_{max}^2=\frac{1}{2}\frac{q^2}{m_0}B^2 R^2$$

total experienced voltage:

$$V=\frac{E_{kin}}{q}=\frac{1}{2}\frac{q}{m_0}B^2 R^2$$

cyclotron frequency and RF − frequency:

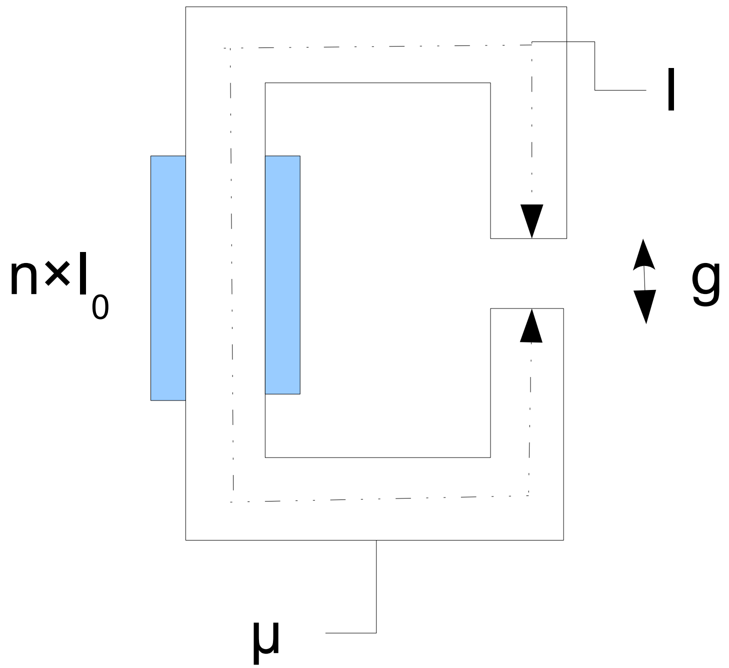
$$\omega_C=2\pi f=\frac{2\pi}{T}=\frac{v}{r}=\frac{qB}{m_0}, \qquad \omega_{RF}=2\omega_C$$

Example : R=0.5m, B=1.5T, deuterium (p, n) with q=e, m_0=3.34 10^{-27}kg

E_{kin}=13.5 MeV, f_{RF}=23 MHz

Tut-Ex 6

A long dipole magnet is excited by a coil with n windings and current I_0. Calculate the magnetic field in the air gap.



$$\oint \vec{H}\cdot d\vec{s} = \iint \vec{J}\cdot d\vec{A}$$

$$H_i l + H_o g = n I_0$$

$$B_i = B_o \;\rightarrow\; \mu H_i = \mu_0 H_o$$

$$H_o = \frac{n I_0}{g + \mu_0 l/\mu} \approx \frac{n I_0}{g}$$

Tut-Ex 7

Give the E- and H-field of a z-polarized plane wave which propagates in x-direction.
What is the time-averaged radiated power density?

$$\vec{E} = E_0 \mathrm{e}^{i(\omega t - kx)} \vec{e}_z, \qquad k = \frac{\omega}{c}$$

$$Z\vec{H} = \vec{e}_x \times \vec{E} = -E_0 \mathrm{e}^{i(\omega t - kx)} \vec{e}_y, \qquad Z = \sqrt{\frac{\mu}{\epsilon}}$$

$$\vec{S}_c = \frac{1}{2} \vec{E} \times \vec{H}^* = \frac{1}{2Z} |E_0|^2 \vec{e}_x$$

Tut-Ex 8

Derive the longitudinal vector potential for TM-waves in a rectangular waveguide.
What is the equation for the separation constants?

$$\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_z}{\partial z^2}+k^2 A_z=0, \quad k=\frac{\omega}{c}$$

$$Bernoulli \;\; ansatz: \quad A_z(x,y,z)=X(x)Y(y)Z(z)$$

$$\underbrace{\frac{1}{X}\frac{d^2 X}{dx^2}}_{-k_x^2}+\underbrace{\frac{1}{Y}\frac{d^2 Y}{dy^2}}_{-k_y^2}+\underbrace{\frac{1}{Z}\frac{d^2 Z}{dz^2}}_{-k_z^2}+k^2=0$$

$$Dispersion \;\; relation\,(equ.\,of \;\; separation\,constants):$$

$$k^2=k_x^2+k_y^2+k_z^2$$

$$\frac{d^2X}{dx^2}+k_x^2X=0 \quad \rightarrow \quad X=C_1\cos(k_x x)+C_2\sin(k_x x)$$

$$=\begin{Bmatrix}\cos(k_x x)\\ \sin(k_x x)\end{Bmatrix} \; or \begin{Bmatrix}e^{ik_x x}\\ e^{-ik_x x}\end{Bmatrix}$$

for $Y(y), Z(z)$ *correspondingly.*

general solution

$$A_z(x,y,z)=\begin{Bmatrix}\cos(k_x x)\\ \sin(k_x x)\end{Bmatrix}\begin{Bmatrix}\cos(k_y y)\\ \sin(k_y y)\end{Bmatrix}\begin{Bmatrix}e^{ik_z z}\\ e^{-ik_z z}\end{Bmatrix}e^{i\omega t}$$

TM − waves: $\quad \vec{H}=\vec{\nabla}\times A\vec{e}_z$

$$E_z=-\frac{1}{i\omega\epsilon}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_z}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial\varphi^2}\right]=\frac{k_x^2+k_y^2}{i\omega\epsilon}A_z\sim A_z$$

$$E_z(x=0,a)=E_z(y=0,b)=0:$$

$$A_z=C\sin(k_x x)\sin(k_y y)e^{i(\omega t-k_z z)},\quad k_x=m\frac{\pi}{a},\quad k_y=n\frac{\pi}{b}$$

$$k^2=\left(\frac{\omega}{c}\right)^2=\left(m\frac{\pi}{a}\right)^2+\left(n\frac{\pi}{b}\right)^2+k_z^2$$

Tut-Ex 9

Give the longitudinal wavelength and phase and group velocity of a TE_{10}-mode in a rectangular waveguide.

$$m=1,\quad n=0:$$

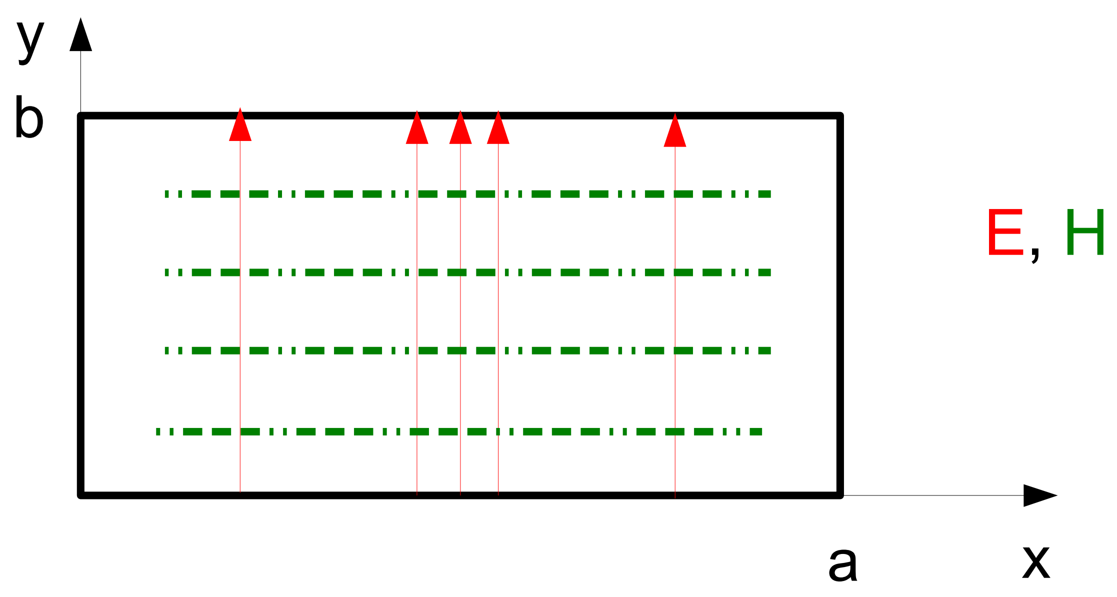
$$k_z=\frac{2\pi}{\lambda_z}=\sqrt{k^2-\left(\frac{\pi}{a}\right)^2}\quad\rightarrow\quad\lambda_z=\frac{\lambda_0}{\sqrt{1-(\lambda_0/2a)^2}}$$

$$v_{ph} = \frac{\omega}{\beta} = \frac{\omega}{k_z} = \frac{c}{\sqrt{1-(\lambda_0/2a)^2}} \qquad > c$$

$$v_g = \frac{\partial\omega}{\partial\beta} = \frac{\partial\omega}{\partial k_z} = c\sqrt{1-(\lambda_0/2a)^2} \qquad < c$$

$$\rightarrow \qquad v_{ph} v_g = c^2$$

TE_{10}-mode



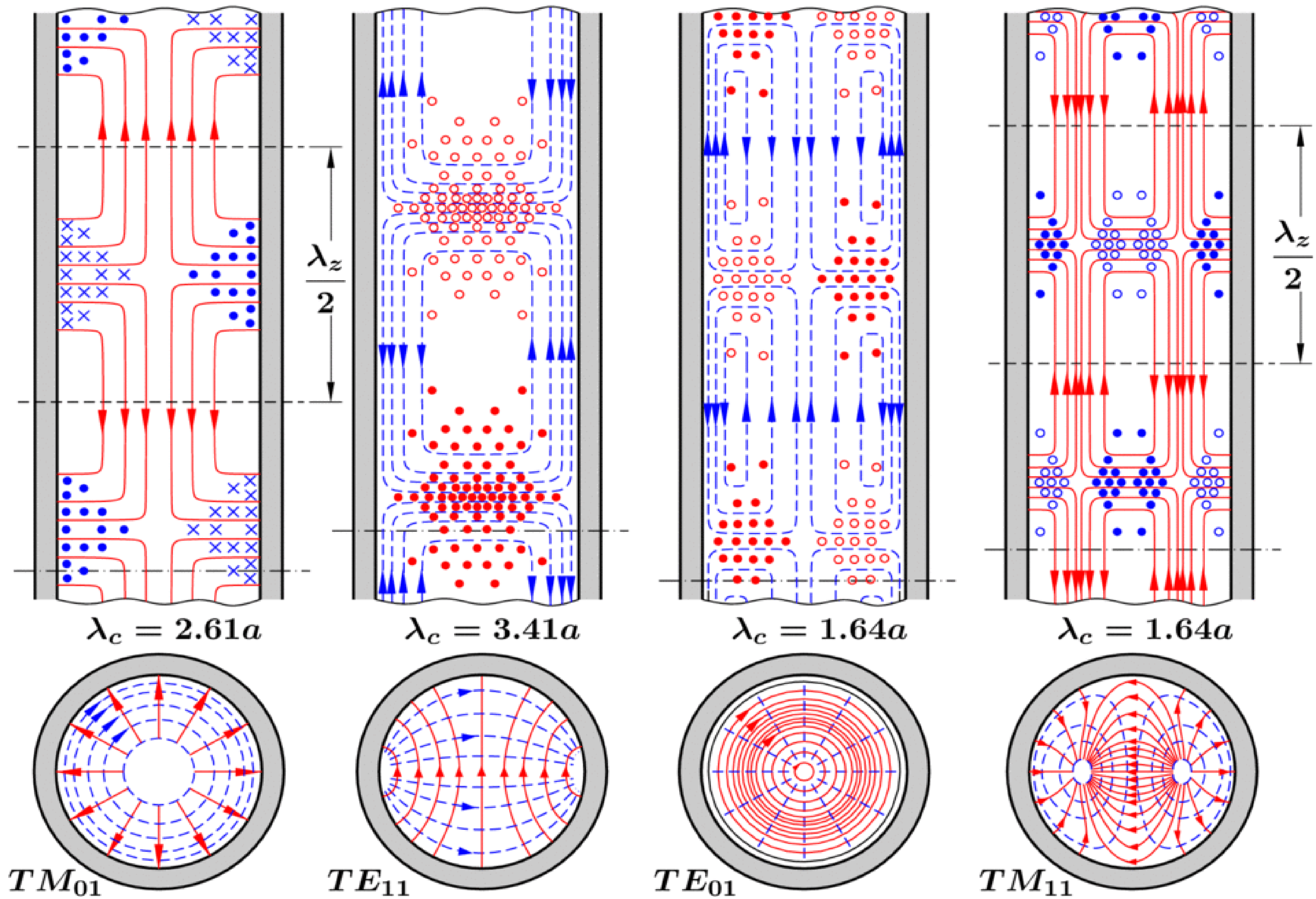
Tut-Ex 10

What is the lowest mode in a circular waveguide?
Show the field pattern.
In which frequency range is mono-mode operation possible?

$$TE_{11}-mode: \quad K_{11}=\frac{j_{11}'}{a}=\sqrt{k^2-k_z^2}, \quad j_{11}'=1.841$$

$$TM_{01}-mode: \quad k_c=\frac{j_{01}}{a}, \quad j_{01}=2.405$$

$$f_{c11}=j_{11}'c/2\pi a \quad \rightarrow \quad f_{C01}=j_{01}c/2\pi a$$



$\frac{\lambda_z}{2}$
$\frac{\lambda_z}{2}$
$\lambda_c = 2.61a$
$\lambda_c = 3.41a$
$\lambda_c = 1.64a$
$\lambda_c = 1.64a$
TM_{01}
TE_{11}
TE_{01}
TM_{11}

Tut-Ex 11

Calculate the accelerating voltage, shunt impedance and R-upon-Q of a TM_{010}-mode pill-box cavity.

$$on\,axis:$$

$$E_z = -i\frac{2}{\omega\epsilon}\left(\frac{j_{01}}{a}\right)^2 D_{010}\,e^{i\omega t}$$

$$V_m = \left|\int_0^g a_m \vec{e}_m\cdot\vec{e}_z\,e^{i\omega t}\,dz\right| = \left|\int_0^g E_z\,e^{i\omega z/v}\,dz\right|, \qquad z = vt$$

$$= \frac{2}{\omega\epsilon}\left(\frac{j_{01}}{a}\right)^2 D_{010}\,g\,T\,, \quad T = \frac{\sin(\omega g/2v)}{\omega g/2v}$$

$$\bar{P}_d = \frac{4\pi}{\kappa \delta_s} j_{01}^2 \left(1 + \frac{g}{a}\right) |D_{010}|^2 J_1^2(j_{01})$$

$$R_{sh} = \frac{V^2}{\bar{P}_d} = \frac{4}{\pi} \frac{Z}{\delta_S} \frac{g^2}{a+g} \frac{T^2}{j_{01} J_1^2(j_{01})}$$

$$\bar{W} = \frac{2\pi g}{\omega^2 \epsilon} \frac{j_{01}^4}{a^2} |D_{010}|^2 J_1^2(j_{01})$$

$$\frac{R_{sh}}{Q_0} = \frac{V^2}{\omega \bar{W}} = \frac{2}{\pi} Z \frac{g}{a} \frac{T^2}{j_{01} J_1^2(j_{o1})}$$