Transverse Beam Dynamics

JUAS tutorial 2 (solutions)

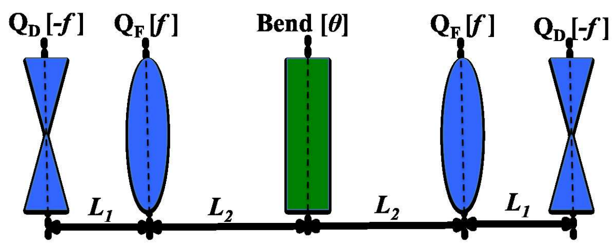
16 January 2013

1 Exercise: geometry of a storage ring, thin lens, tune, dispersion

Consider a proton synchrotron accumulator made of identical cells. The relevant ring parameters are given in the following table:

Proton kinetic energy	2 GeV
Cell type	Symmetric triplet$^{(*)}$
Ring circumference	960 m
Integrated quadrupole gradient ($\int Gdl$)	1.5 T

$^{(*)}$Note: A thin lens symmetric triplet cell consists of a thin lens defocusing quadrupole of focal length $-f$, followed with a drift space of length L_1, a thin lens focusing quadrupole of focal length f, a drift of length L_2, a thin lens dipole of horizontal bending angle θ, a drift of length L_2, a thin lens focusing quadrupole of focal length f, a drift of length L_1, and a thin lens defocusing quadrupole of focal length $-f$ (see Figure below).



Compute the focal length f of the quadrupoles. The proton rest mass is 938 MeV.

Assuming a longitunally constant gradient for the quadrupole $\int Gdl = G \cdot l$, where l is the length of the quadrupole

The quadrupole strength:

$$k = \frac{G}{B\rho} = \frac{1.5\text{T}}{(B\rho)l}$$

Knowing $f = \frac{1}{kl}$, we obtain

$$f = \frac{B\rho}{1.5\text{T}},$$

so we calculate the particle momentum in order to calculate the rigidity $B\rho = \frac{p}{e}$.

We have the information of the kinetic energy, therefore we can use the relativistic formula to obtain the momentum pc [GeV]

The total energy is given by

$$E = \sqrt{m_0^2c^4 + p^2c^2},$$

where $m_0 = 938$ MeV is the rest mass of the particle (in this case protons)

Knowing that the kinetic energy $E_k = E - m_0c^2 = \sqrt{m_0^2c^4 + p^2c^2} - m_0c^2$, and after some trivial algebra we obtain:

$$p = 2.78GeV/c$$

$$B\rho \approx \frac{1}{0.3}p[\text{GeV/c}]{=}9.27Tm$$

and with this information we can finally obtain the focal length:

$$f = \frac{B\rho}{1.5\text{T}} = 6.18m$$

Given the numerical values $L_1 = 1.5$ m and $L_2 = 6$ m:

- Compute the horizontal and vertical transfer matrices of a triplet cell (take into account that the sign of the focal length changes when going from horizontal to vertical plane).

Let´s go to calculate first the bending angle of the dipole:

The length of the cell is $L_{cell} = 2L_1 + 2L_2 = 15$ m

Knowing that the circumference of the machine is 960 m and that we have a thin dipole per cell:

Number of cells:$N = \frac{960}{15} = 64$

and therefore:

$\theta = \frac{2\pi}{64} = 0.098$ rad.

Having the information L_1, L_2, f and θ we can calculate the elements of the matrix of the triplet cell (Use the expression from the **Hint**):

$$M_{triplet} = \begin{pmatrix} \frac{f^3+2L_1^2L_2-2L_1f(L_1+L_2)}{f^3} & \frac{2(f-L_1)(L_1f+L_2f-L_1L_2)}{f^2} & (L_1+L_2-\frac{L_1L_2}{f})\theta \\ \frac{2L_1(L_1L_2-L_1f-f^2)}{f^4} & \frac{f^3+2L_1^2L_2-2L_1f(L_1+L_2)}{f^3} & \frac{(f^2+L_1f-L_1L_2)}{f^2}\theta \\ 0 & 0 & 1 \end{pmatrix}$$

For the horizontal plane:

$$M_{triplet(H)} = \begin{pmatrix} 0.525 & 9.153 & 0.592 \\ -0.079 & 0.525 & 0.099 \\ 0 & 0 & 1 \end{pmatrix}$$

For the vertical plane:

Here we have to take into account that in our example we consider only horizontal bending magnets, and no bend in vertical ($\theta = 0$ in vertical). In addition we have to consider $f \to -f$,

$$M_{triplet(V)} = \begin{pmatrix} 0.296 & 22.26 & 0 \\ -0.04 & 0.296 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Compute the horizontal and vertical machine tunes.

In this example we are considering the propagation of 3D vectors $(x, x', \Delta p/p_0)$:

$$\begin{pmatrix} x \\ x' \\ \Delta p/p_0 \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \\ \Delta p/p_0 \end{pmatrix}_0 = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p_0 \end{pmatrix}_0$$

REMEMBER: In terms of the betatron functions these elements of the matrix M between two points can be defined as follows:

$$M = \begin{pmatrix} \sqrt{\frac{\beta_x}{\beta_{x0}}}(\cos\phi_x + \alpha_{x0}\sin\phi_x) & \sqrt{\beta_x\beta_{x0}}\sin\phi_x & D_x \\ \frac{(\alpha_{x0}-\alpha_x)\cos\phi_x-(1+\alpha_{x0}\alpha_x)\sin\phi_x}{\sqrt{\beta_x\beta_{x0}}} & \sqrt{\frac{\beta_{x0}}{\beta_x}}(\cos\phi_x - \alpha_x\sin\phi_x) & D'_x \\ 0 & 0 & 1 \end{pmatrix}$$

The terms $m_{13} = D_x$ and $m_{23} = D'_x$, are the horizontal dispersion and the derivative of the horizontal dispersion over s, respectively.

We are dealing with a symmetric cell, so:

$\beta_x(s_0 + L_{cell}) = \beta_x(s_0) = \beta_{x0}$

$\alpha_x(s_0 + L_{cell}) = \alpha_x(s_0) = \alpha_{x0}$

$D_x(s_0 + L_{cell}) = D_x(s_0)$

These periodicity conditions are valid for the case of our triplet, FODO cells, and other symmetric cells. The Twiss parameters and the dispersion at the entrance of the cell are equal to those at the exit of the cell. Considering this we can rewrite the transport matrix of our triplet cell as:

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\mu_x + \alpha_{x0}\sin\mu_x & \beta_{x0}\sin\mu_x & D_x \\ -\frac{(1+\alpha_{x0}^2)\sin\mu_x}{\beta_{x0}} & \cos\mu_x - \alpha_{x0}\sin\mu_x & D'_x \\ 0 & 0 & 1 \end{pmatrix}$$

where μ_x is the horizontal phase advance of the cell (Remember that we are calculating first the transport in the horizontal phase, and we have to proceed in a similar way with the vertical case).

Since we have calculated the matrix elements corresponding to our triplet cell, then we can calculate the phase advance comparing the elements $M_{triplet(H)} = M$:

$$\cos\mu_x = \frac{1}{2}(m_{11} + m_{22}) = 0.525 \longrightarrow \mu_x = 1.018\text{rad}$$

In order not to get confused with the concepts, it is necessary to remember that in circular machines the transfer matrix for a complete turn can also be written as:

$$\begin{pmatrix} \cos\tilde{\mu_x} + \alpha_{x0}\sin\tilde{\mu_x} & \beta_{x0}\sin\tilde{\mu_x} & D_x \\ -\frac{(1+\alpha_{x0}^2)\sin\tilde{\mu_x}}{\beta_{x0}} & \cos\tilde{\mu_x} - \alpha_{x0}\sin\tilde{\mu_x} & D'_x \\ 0 & 0 & 1 \end{pmatrix}$$

But in this case $\tilde{\mu_x}$ is the phase advance after one turn.

The tune is defined as the number of betatron oscillations per turn:

$$Q_{x,y} \equiv \frac{\tilde{\mu}_{x,y}}{2\pi} = \frac{N\mu_{x,y}}{2\pi}$$

where $N = 64$ is the number of periodic cells in the total machine.
The horizontal tune is

$$Q_x = \frac{N\mu_x}{2\pi} = \frac{64 \cdot 1.018}{2\pi} = 10.37$$

If we proceed in a similar way for the vertical plane with the transport matrix $M_{triplet(V)}$, we obtain:

$$Q_y = \frac{N\mu_y}{2\pi} = \frac{64 \cdot 1.27}{2\pi} = 12.94$$

- Compute the horizontal and vertical betatron functions at the entrance of a triplet cell.

$$\beta_{x0}\sin\mu_x = m_{12} = 9.153 \longrightarrow \beta_{x0} = 10.75$$

$$\beta_{y0}\sin\mu_y = m_{12} = 22.26 \longrightarrow \beta_{y0} = 23.31$$

- Compute the horizontal and vertical dispersion functions at the entrance of a triplet cell.

$$D_x = m_{13} = 0.592$$

$$D_y = 0$$

Hint:
The 3×3 horizontal transfer matrix for one symmetric triplet cell is (in the thin lens approximation):

$$M_{triplet} = \begin{pmatrix} \frac{f^3+2L_1^2L_2-2L_1f(L_1+L_2)}{f^3} & \frac{2(f-L_1)(L_1f+L_2f-L_1L_2)}{f^2} & (L_1+L_2-\frac{L_1L_2}{f})\theta \\ \frac{2L_1(L_1L_2-L_1f-f^2)}{f^4} & \frac{f^3+2L_1^2L_2-2L_1f(L_1+L_2)}{f^3} & \frac{(f^2+L_1f-L_1L_2)}{f^2}\theta \\ 0 & 0 & 1 \end{pmatrix}$$

for the transport of a vector $\begin{pmatrix} x \\ x' \\ \Delta p/p_0 \end{pmatrix}$, where $\Delta p/p_0$ is the momentum offset with respect to the design momentum p_0.

2 Exercise: emittance

From the solution of the trajectory equation,

$$x = \sqrt{\beta\epsilon}\cos[\phi + \phi_0] \tag{1}$$

- Derive the following relation:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon$$

where ϵ is the emittance, and β, α, and γ are the so-called Twiss parameters:

$$\alpha \equiv -\tfrac{\beta'}{2} \text{ , and } \gamma \equiv \tfrac{1+\alpha^2}{\beta}$$

(DO NOT CONFUSE this γ with the relativistic Lorentz factor!).
we make the first derivative x' (when you do the derivative remember that β, α and ϕ depend on s):

$$x' = -\sqrt{\frac{\epsilon}{\beta}}\left(\alpha\cos[\phi + \phi_0] + \sin[\phi + \phi_0]\right) \tag{2}$$

From Eq. (1):

$$\cos[\phi + \phi_0] = \frac{x}{\sqrt{\beta\epsilon}}$$

Substituting this into Eq. (2) we get:

$$\sin[\phi + \phi_0] = \frac{\sqrt{\beta}x'}{\sqrt{\epsilon}} + \frac{\alpha x}{\sqrt{\beta\epsilon}}$$

If we now use the trigonometric relation $\sin^2\Theta + \cos^2\Theta = 1$ we obtain:

$$\frac{x^2}{\beta} + \left(\frac{\alpha}{\sqrt{\beta}}x + \sqrt{\beta}x'\right)^2 = \epsilon$$

Finally, taking into account the definition

$$\gamma \equiv \frac{1+\alpha^2}{\beta}$$

we arrive to

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon$$