Space Charge Effects and Instabilities

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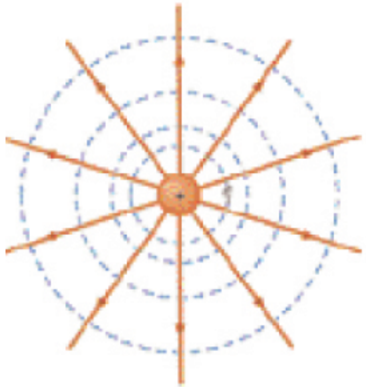


JUAS, 24-25 January 2013

EQUATION OF MOTION

Charged particles in a transport channel or in a circular/linear accelerator are accelerated, guided and confined by external electromagnetic fields. The motion of a single charge is governed by the Lorentz force through the equation:

$$\frac{d\left(m_0 \gamma \boldsymbol{v}\right)}{dt} = \boldsymbol{F}_{e.m.}^{ext} = e\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)$$

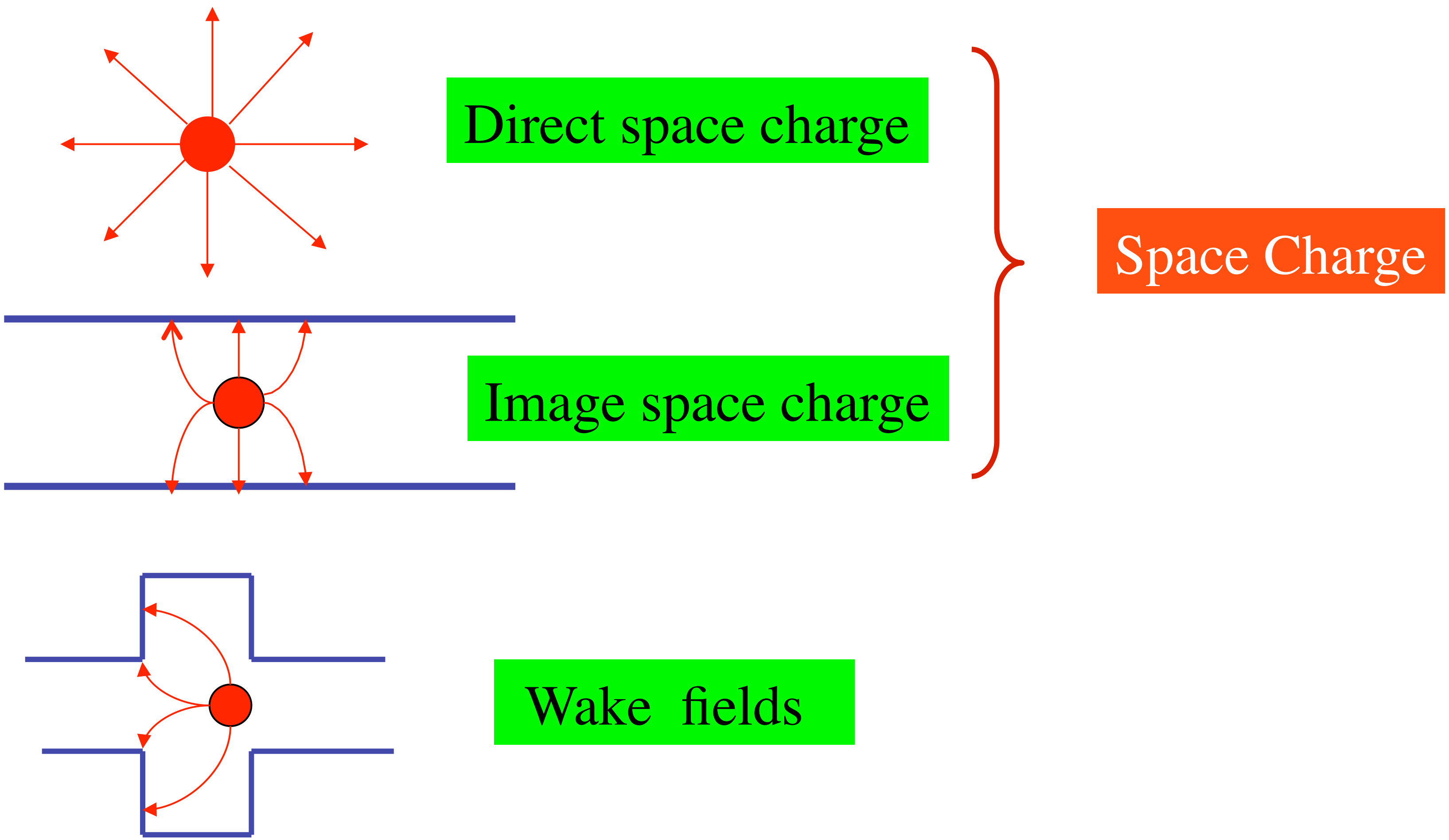
Where m_0 is the rest mass, γ is the relativistic factor and v is the particle velocity.

Acceleration is usually provided by the electric field of RF cavities. Magnetic fields are produced in the bending magnets for guiding the charges on the reference trajectory (orbit), in the quadrupoles for the transverse confinement, in the sextupoles for the chromaticity correction.

However, there is another source of e.m. fields, the beam itself…

SPACE CHARGE AND WAKE FIELDS

In a real accelerator, there is another important source of e.m. fields to be considered, the beam itself, which circulating inside the pipe, produces additional e.m. fields:



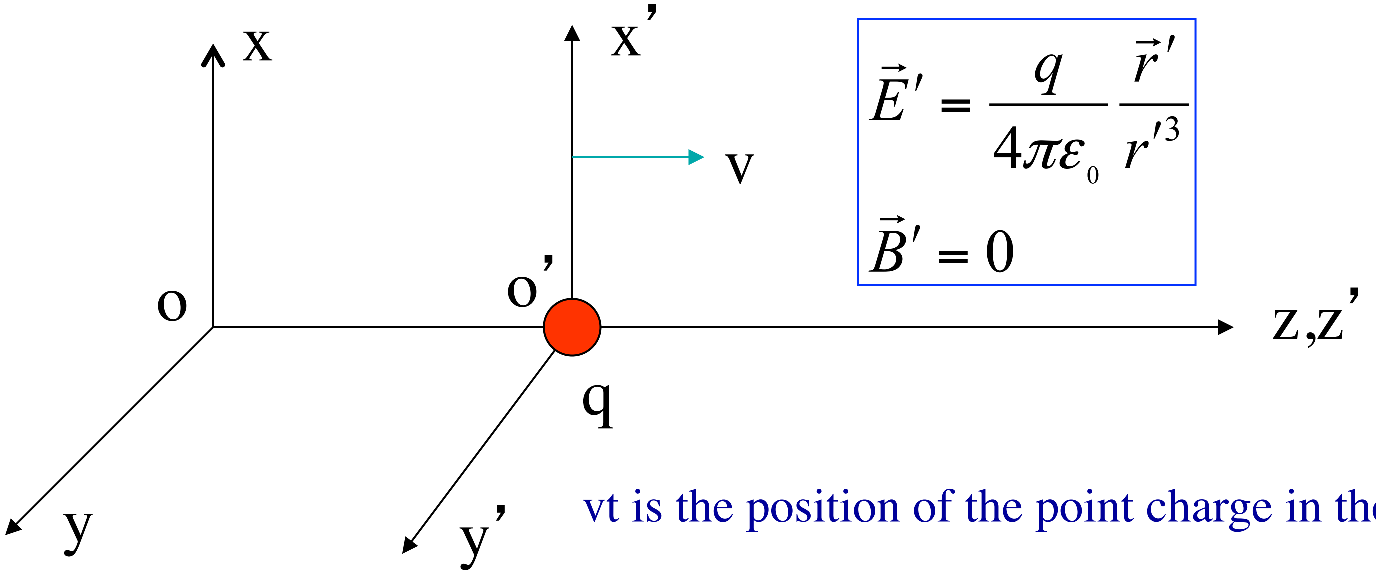
These fields depend on:

- the total current
- the geometry and the beam pipe
- the surrounding material.

They are responsible of many phenomena of beam dynamics:

- betatron tune shift
- synchrotron tune shift
- energy loss
- energy spread and emittance degradation
- instabilities.

Fields of a point charge with uniform motion



$$\vec{E}' = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}'}{r'^3}$$

$$\vec{B}' = 0$$

vt is the position of the point charge in the system O.

$$E'_x = \frac{q}{4\pi\varepsilon_0}\frac{x'}{r'^3} \qquad E'_y = \frac{q}{4\pi\varepsilon_0}\frac{y'}{r'^3} \qquad E'_z = \frac{q}{4\pi\varepsilon_0}\frac{z'}{r'^3}$$

- In O' the charge is at rest
- The electric field is radial with spherical symmetry
- The magnetic field is zero

Relativistic transforms of the fields and coordinates from O' to O

$$\begin{cases} E_x = \gamma(E'_x + vB'_y) \\ E_y = \gamma(E'_y - vB'_x) \\ E_z = E'_z \end{cases} \quad \begin{cases} B_x = \gamma(B'_x - vE'_y/c^2) \\ B_y = \gamma(B'_y + vE'_x/c^2) \\ B_z = B'_z \end{cases} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\begin{cases} x' = x \\ y' = y \\ z' = \gamma(z - vt) \end{cases}$$

$$r' = \left(x'^2 + y'^2 + z'^2\right)^{1/2}$$

$$r' = \left[x^2 + y^2 + \gamma^2 (z - vt)^2\right]^{1/2}$$

$$E_x = \gamma E'_x = \frac{q}{4\pi\varepsilon_0}\frac{\gamma x'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma x}{\left[x^2 + y^2 + \gamma^2(z - vt)^2\right]^{3/2}}$$

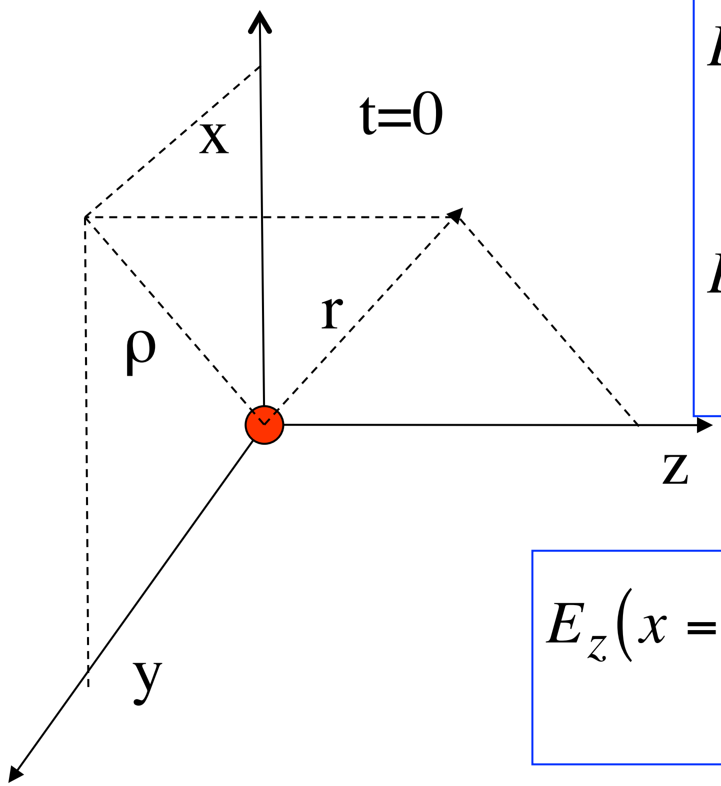
$$E_y = \gamma E'_y = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y}{\left[x^2 + y^2 + \gamma^2(z - vt)^2\right]^{3/2}}$$

$$E_z = E'_z = \frac{q}{4\pi\varepsilon_0}\frac{z'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma(z - vt)}{\left[x^2 + y^2 + \gamma^2(z - vt)^2\right]^{3/2}}$$

The field pattern is moving with the charge and it can be observed at t=0.

$$\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma\vec{r}}{\left[x^2 + y^2 + \gamma^2 z^2\right]^{3/2}}$$

The fields have lost the spherical symmetry but still keep a symmetry with respect the z-axis.



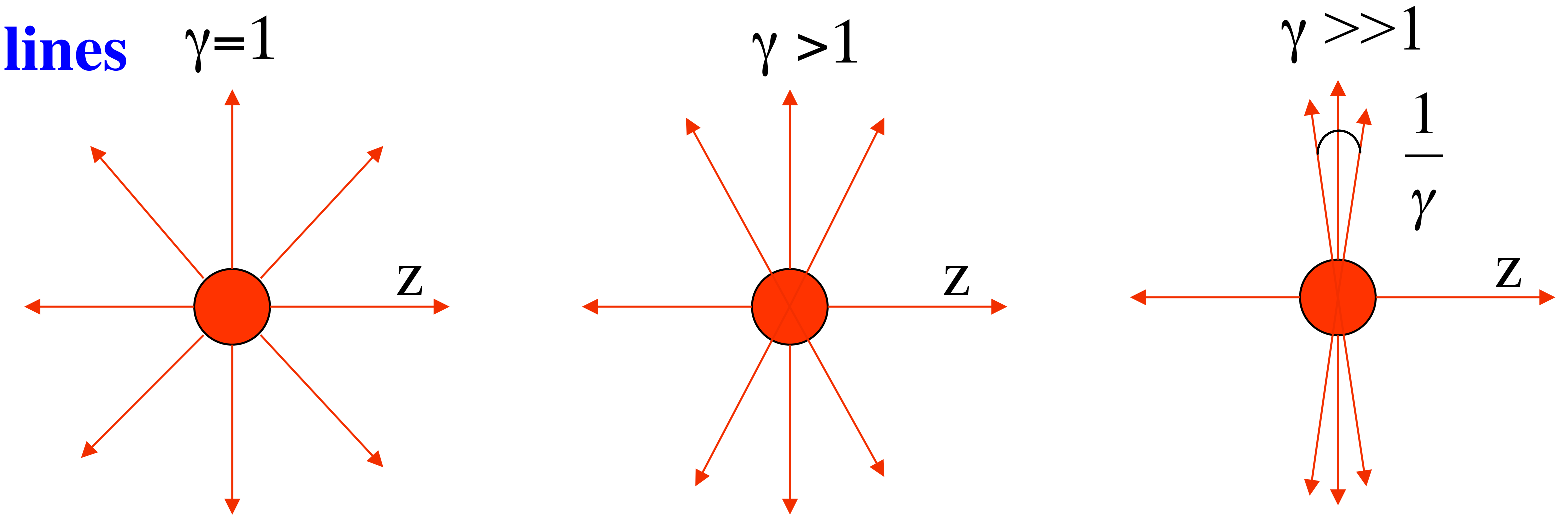
$$E_x(z=0) = \frac{q}{4\pi\varepsilon_0}\frac{\gamma x}{\left[x^2+y^2\right]^{3/2}}$$

$$E_y(z=0) = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y}{\left[x^2+y^2\right]^{3/2}}$$

$$\vec{E}_\rho = \frac{q}{4\pi\varepsilon_0}\frac{\gamma\vec{\rho}}{\rho^3}$$

$$E_z(x=y=0) = \frac{q}{4\pi\varepsilon_0}\frac{\gamma z}{\left[\gamma^2 z^2\right]^{3/2}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\gamma^2 z^2}$$

Field lines



$$E_x = \gamma E'_x = \frac{q}{4\pi\varepsilon_0}\frac{\gamma x'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma x}{\left[x^2+y^2+\gamma^2(z-vt)^2\right]^{3/2}}$$

$$E_y = \gamma E'_y = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma y}{\left[x^2+y^2+\gamma^2(z-vt)^2\right]^{3/2}}$$

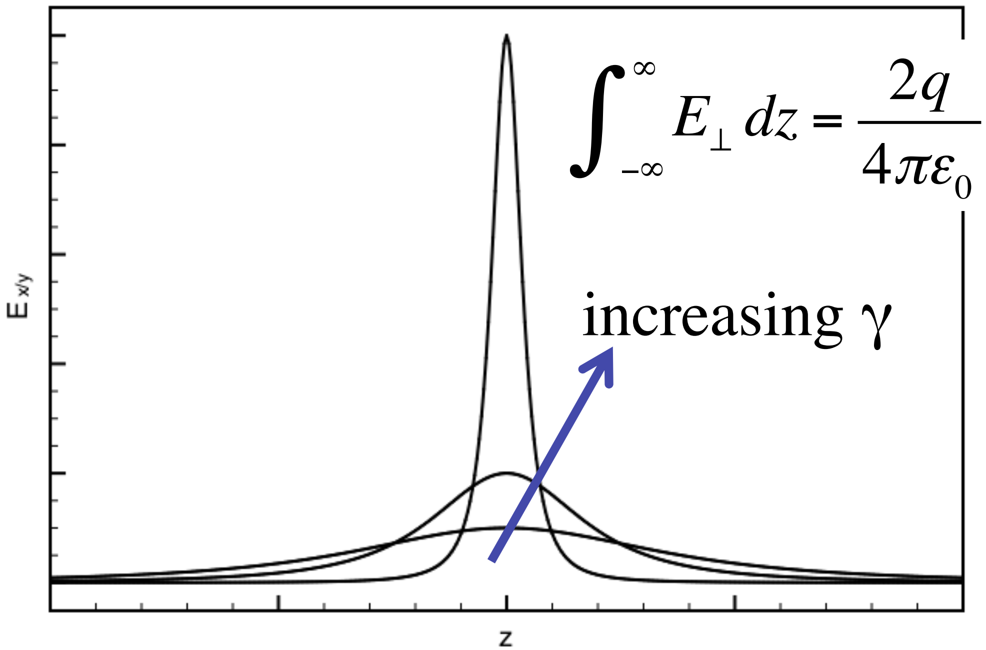
$$E_z = E'_z = \frac{q}{4\pi\varepsilon_0}\frac{z'}{r'^3} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma(z-vt)}{\left[x^2+y^2+\gamma^2(z-vt)^2\right]^{3/2}}$$

$\gamma \to \infty$

$$\begin{cases} E_x = ? \\ E_y = ? \\ E_z \to 0 \end{cases}$$

$z \neq vt \Rightarrow E_\perp = 0$ (1)

$z = vt \Rightarrow E_\perp = \infty$ (2)



$$\int_{-\infty}^{\infty} E_\perp \, dz = \frac{2q}{4\pi\varepsilon_0}\frac{1}{\rho}$$ independent on γ (3)

$$\Rightarrow \quad E_\perp = \frac{2q}{4\pi\varepsilon_0}\frac{1}{\rho}\delta(z-vt)$$

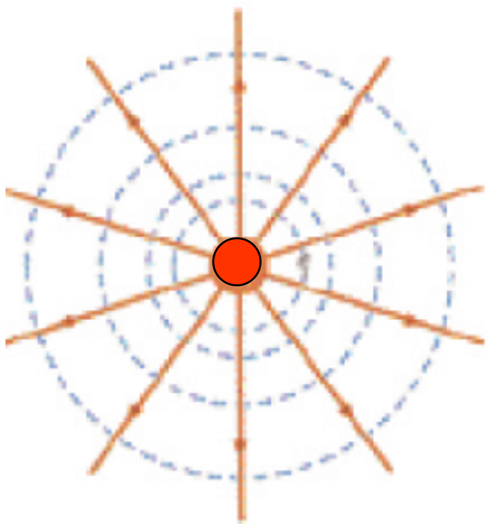
B is transverse to the motion direction

$$E_x = \gamma(E'_x + vB'_y) \qquad B_x = \gamma(B'_x - vE'_y/c^2)$$
$$E_y = \gamma(E'_y - vB'_x) \qquad B_y = \gamma(B'_y + vE'_x/c^2)$$
$$E_z = E'_z \qquad B_z = B'_z$$

$$B_z = 0$$
$$B_x = -vE_y/c^2$$
$$B_y = vE_x/c^2$$

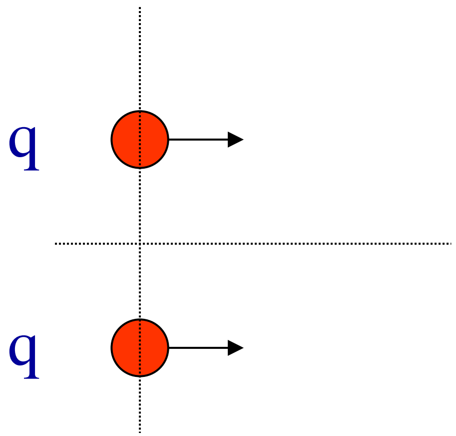
$$\vec{B}_\perp = \frac{\vec{v} \times \vec{E}}{c^2}$$

$$B_\theta = \frac{vE_\rho}{c^2} = \frac{\beta E_\rho}{c}$$



$\gamma \to \infty$

Two point charges with same velocity on parallel trajectories



In the rest frame O'

$$F'_\rho = \frac{1}{4\pi\varepsilon_0}\frac{qq}{\rho'^2}$$

In the moving frame O

Relativistic transform

$$F_\rho = \frac{1}{\gamma}F'_\rho = \frac{1}{4\pi\varepsilon_0}\frac{qq}{\gamma\rho^2}$$

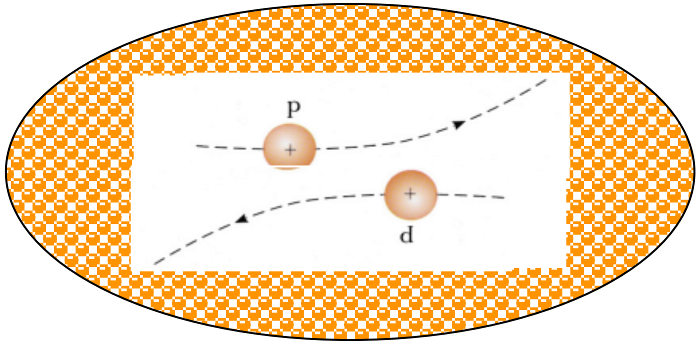
Lorentz force

$$F_\rho = q(E_\rho - vB_\theta) = q(E_\rho - \beta^2 E_\rho) = \frac{q}{\gamma^2}E_\rho = \frac{1}{4\pi\varepsilon_0}\frac{qq}{\gamma\rho^2}$$

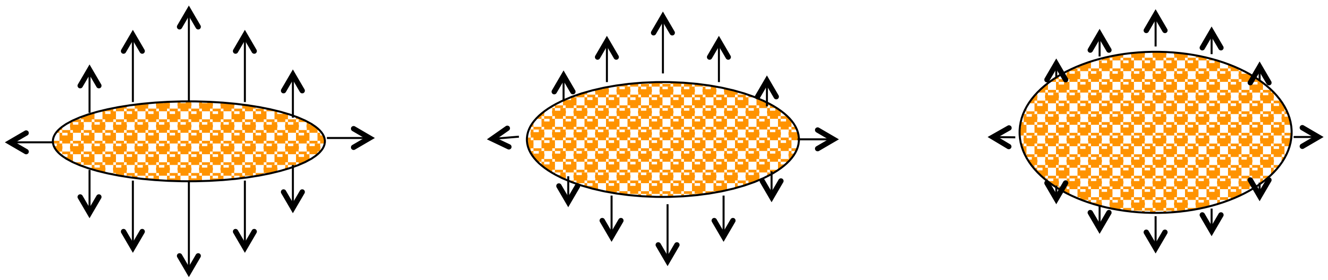
Space Charge: What does it mean?

It is the net effect of the **Coulomb** interactions in a multi-particle system, and it can be classified into two regimes:

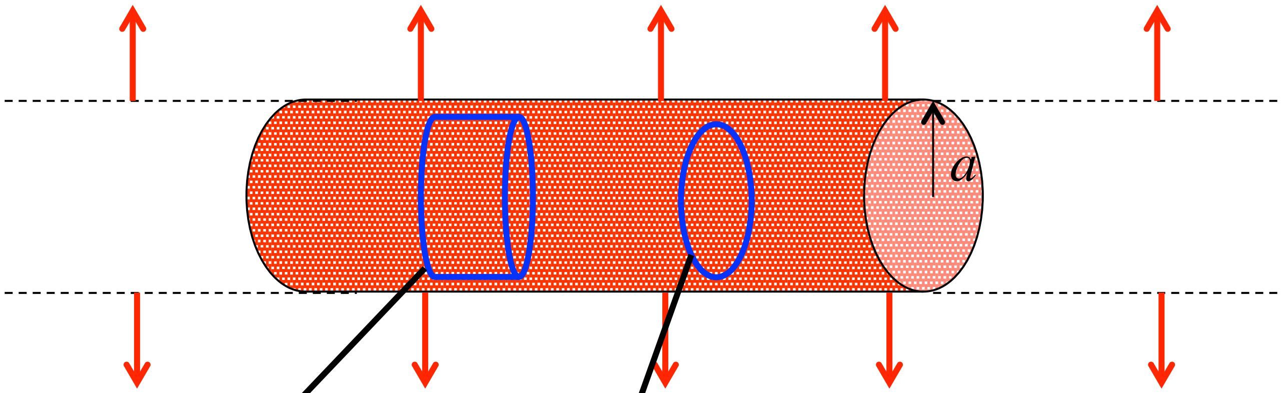
1) **Collisional Regime** ==> dominated by **binary collisions** between particles ==> **Single Particle Effects**



2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compared to the average separation of the particles ==> **Collective Effects**



Example 1. Relativistic Uniform Cylindrical Beam



$$J = \frac{I}{\pi a^2} \qquad \rho = \frac{I}{\pi a^2 v}$$

$$J = \beta c \rho$$

$$I = J\pi a^2 = \beta c \lambda_0$$

Gauss' s law

$$\int \varepsilon_0 E \cdot dS = \int \rho dV$$

Linear

$$2\pi r l \varepsilon_0 E_r = \rho \pi r^2 l$$

$$E_r = \frac{\rho r}{2\varepsilon_0} = \frac{Ir}{2\pi\varepsilon_0 a^2 v} \quad \text{for} \quad r \le a$$

$$B_\theta = \frac{\beta}{c} E_r$$

Ampere' s law

$$\oint B \cdot dl = \mu_0 \int J \cdot dS$$

$$2\pi r B_\theta = \mu_0 J \pi r^2$$

$$B_\theta = \frac{\mu_0 J r}{2} = \mu_0 \frac{Ir}{2\pi a^2} \quad \text{for} \quad r \le a$$

$$\lambda_o = \rho\pi a^2$$

$$\lambda(r) = \lambda_o (r/a)^2$$

$$J = \beta c \rho$$

$$I = J\pi a^2 = \beta c \lambda_0$$

for $r < a$

$$E_r(r) = \frac{\lambda_0 \, r}{2\pi\varepsilon_0 a^2}$$

$$B_\theta(r) = \frac{\beta}{c} E_r(r) = \frac{\lambda_0 \beta}{2\pi\varepsilon_0 c} \frac{r}{a^2}$$

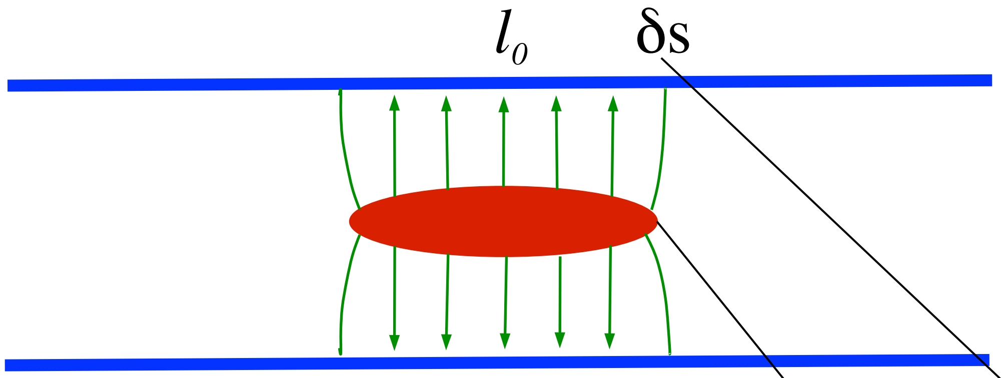
Lorentz Force

$$F_r = e\left(E_r - \beta c B_\theta\right) = e\left(1 - \beta^2\right) E_r = \frac{eE_r}{\gamma^2}$$

- has only **radial** component
- is a **linear** function of the transverse coordinate

The attractive magnetic force, which becomes significant at high velocities, tends to compensate the repulsive electric force. Therefore, space charge defocusing is primarily a non-relativistic effect.

Relativistic Uniform Cylindrical Beam – finite length

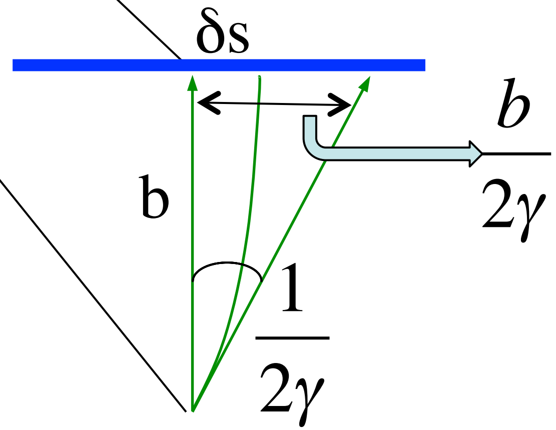


$$\delta s \cong \frac{b}{2\gamma}$$

$$l_0 >> \delta s$$

$$\gamma >> \frac{b}{2 l_0}$$

Beam pipe radius b
Bunch length l_0
Widening at the wall δs



e.g.:
$b = 1$ cm
$l_0 = 100\ \mu$m

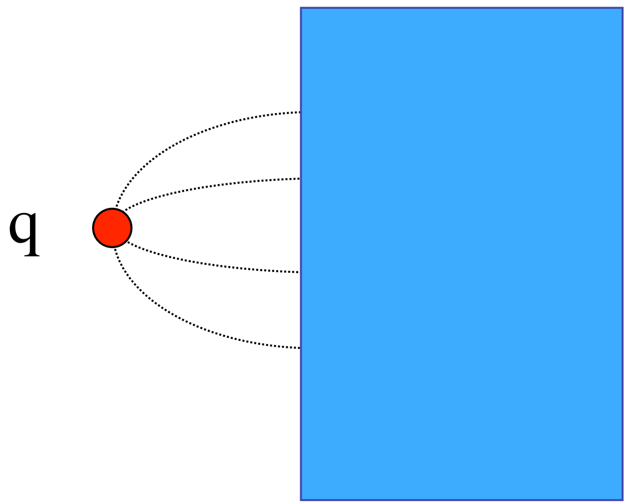
$\Rightarrow$ $\gamma >> 500$

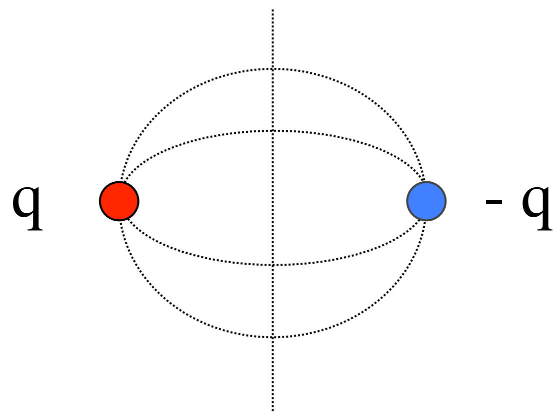
Space charge with image charges/currents

Static Fields: conducting or magnetic screens

Let us consider a point charge q close to a **conducting screen**.

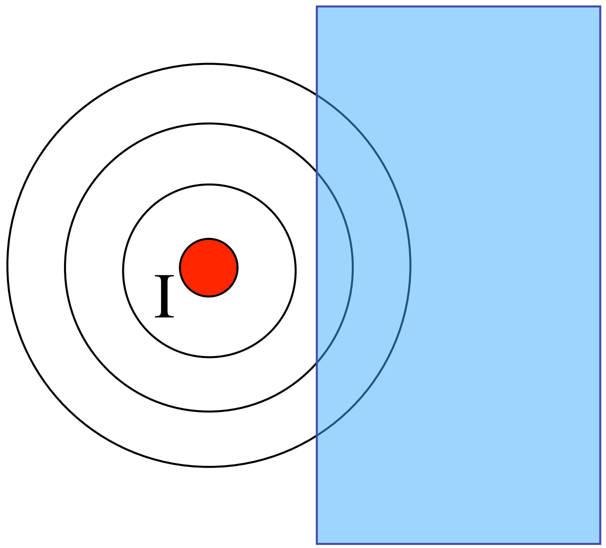
The electrostatic field can be derived through the "image method". Since the metallic screen is an equi-potential plane, it can be removed provided that a "virtual" charge is introduced such that the potential is constant at the position of the screen



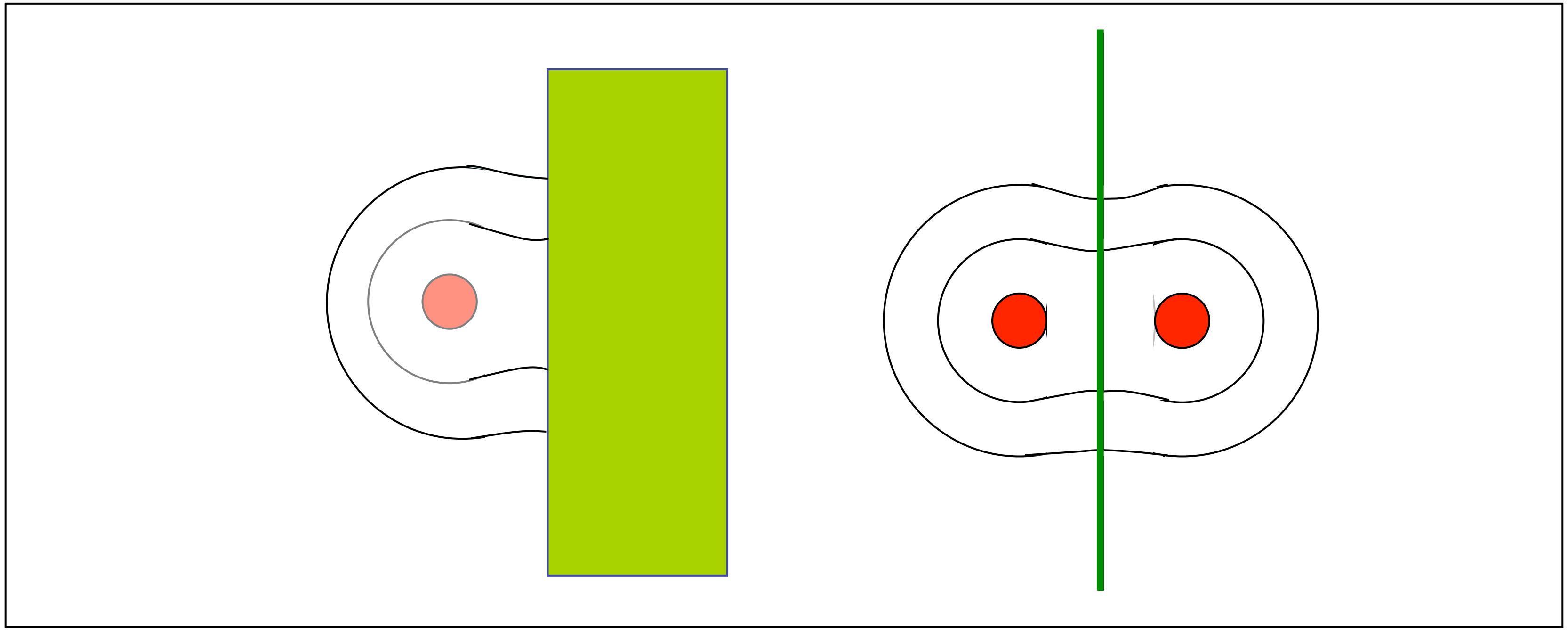


A constant current in the free space produces circular magnetic field.

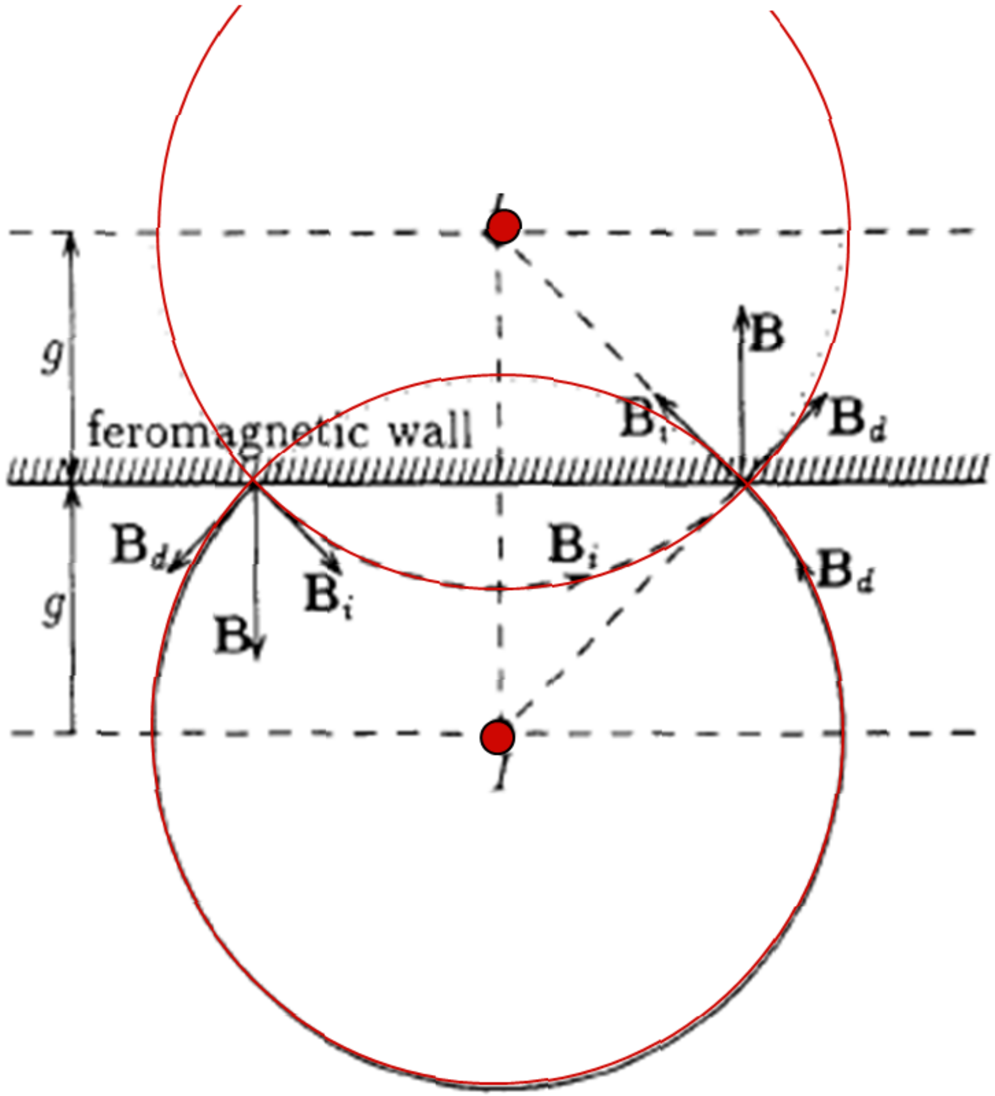
If $\mu_r \approx 1$, the material, even in the case of a good conductor, does not affect the field lines.



For **ferromagnetic type**, with $\mu_r >> 1$, the very high magnetic permeability makes the tangent magnetic field zero at the boundary so that the magnetic field is perpendicular to the surface, just like the electric field lines close to a conductor.



In analogy with the image method we get the magnetic field, in the region outside the material, as superposition of the fields due to two symmetric equal currents flowing in the same direction.



Satisfying a magnetic boundary condition by an image current.

A. Hofmann

Time-varying Fields

Static electric fields vanish inside a conductor for any finite conductivity, while magnetic fields pass through unless of an high permeability.

This is no longer true for time changing fields, which can penetrate inside the material in a region δ_w called skin depth. Inside the conducting material we write the following Maxwell equations:

$$\begin{cases} \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial \mathbf{t}} \\ \nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial \mathbf{t}} \end{cases} \qquad \begin{cases} \mathbf{B} = \mu \mathbf{H} \\ \mathbf{D} = \varepsilon \mathbf{E} \\ \mathbf{J} = \sigma \mathbf{E} \end{cases} \Longrightarrow$$

Constitutive relations

Copper $\sigma = 5.8\ 10^7\ (\Omega\mathrm{m})^{-1}$
Aluminium $\sigma = 3.5\ 10^7\ (\Omega\mathrm{m})^{-1}$
Stainless steel $\sigma = 1.4\ 10^6\ (\Omega\mathrm{m})^{-1}$

Consider a plane wave (H_y, E_x) propagating in the material

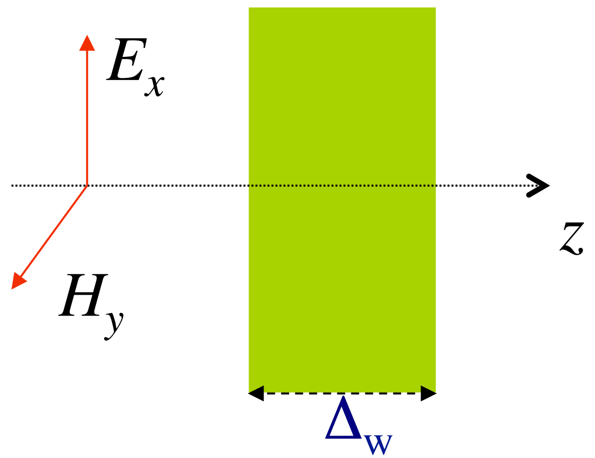
$$\frac{\partial^2 E_x}{\partial z^2} - \varepsilon\mu \frac{\partial^2 E_x}{\partial t^2} - \sigma\mu \frac{\partial E_x}{\partial t} = 0$$

(the same equation holds for H_y). Assuming that fields propagate in the z-direction with the law:

$$H_y = \tilde{H}_0 e^{i\omega t - kz}$$

$$E_x = \tilde{E}_0 e^{i\omega t - kz}$$

$$(k^2 + \varepsilon\mu\omega^2 - i\omega\mu\sigma)\tilde{E}_o e^{i\omega t - kz} = 0$$

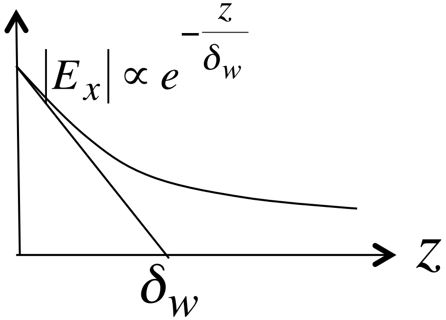


We say that the material behaves like a conductor if $\sigma >> \omega\varepsilon$ thus:

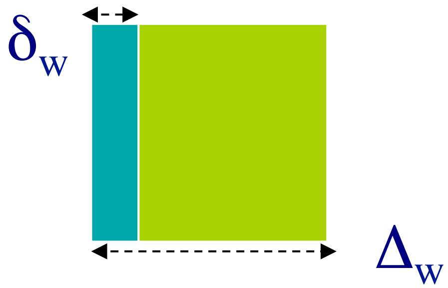
$$k \cong (1+i)\sqrt{\frac{\sigma\mu\omega}{2}} \qquad \Re(k) = \sqrt{\frac{\sigma\mu\omega}{2}}$$

$\Rightarrow$ Exponential decay

Fields propagating along "z" are attenuated.
The attenuation constant measured in meters is called skin depth δ_w:



$$\delta_w \cong \frac{1}{\Re(k)} = \sqrt{\frac{2}{\omega\sigma\mu}}$$

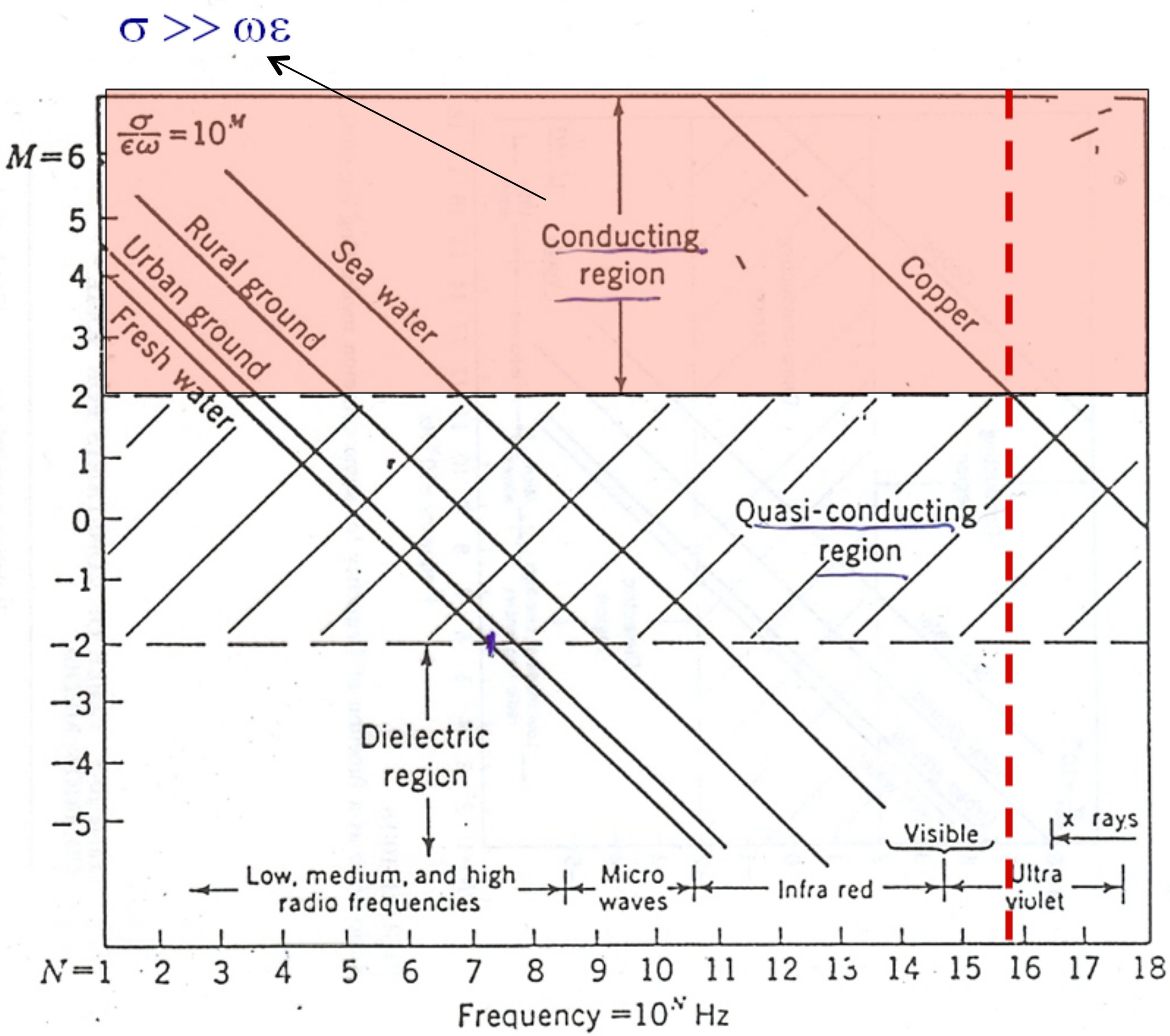


The skin depth depends on the material properties and on the frequency. Fields pass through the conductor wall if the skin depth is larger than the wall thickness Δ_w. This happens at relatively low frequencies.

At higher frequencies, for a good conductor $\delta_w << \Delta_w$ and both electric and magnetic fields vanish inside the wall.

For the copper $\quad \delta_w \cong \frac{6.6}{\sqrt{f(Hz)}}(cm); \qquad \omega = 2\pi f$

For a pipe 2mm thick, the fields pass through the wall up to 1 kHz.
(Skin depth of Aluminium is larger by a factor 1.28)



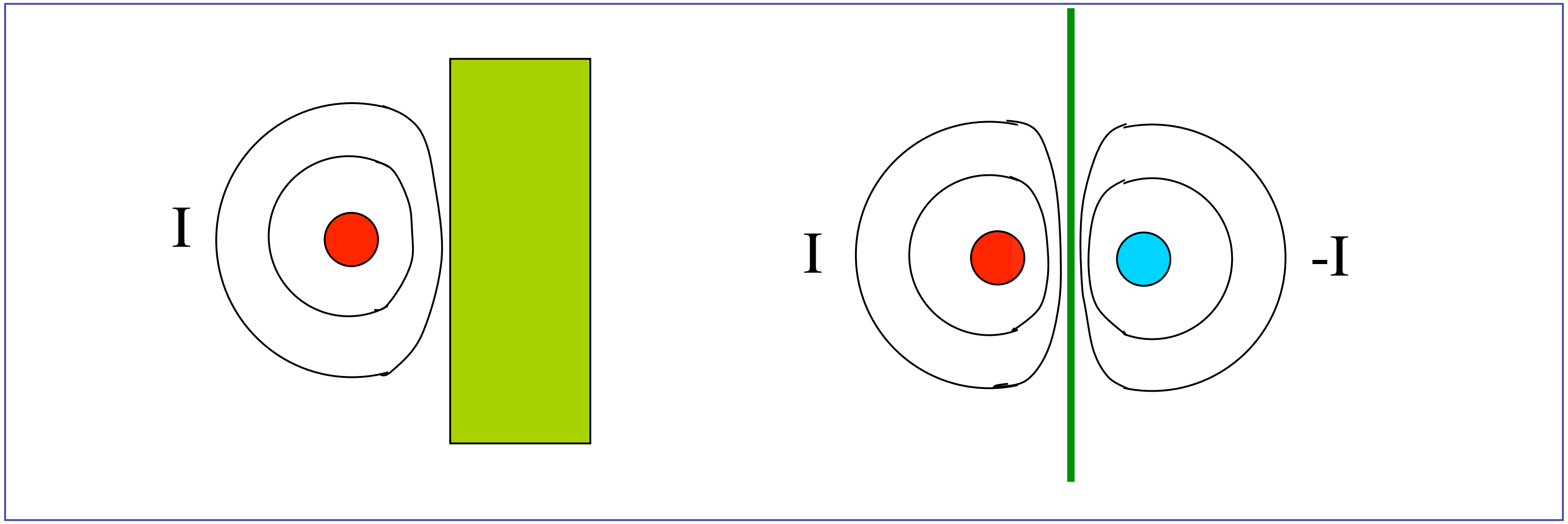
Note that copper behaves like a conductor at frequencies far above the microwave region. On the other hand, fresh water acts like a dielectrics at frequencies above about 10MHz

Ratio σ/ωε as a function of frequency f for some common media (log-log plot)

•Compare the wall thickness and the skin depth (region of penetration of the e.m. fields) in the conductor.

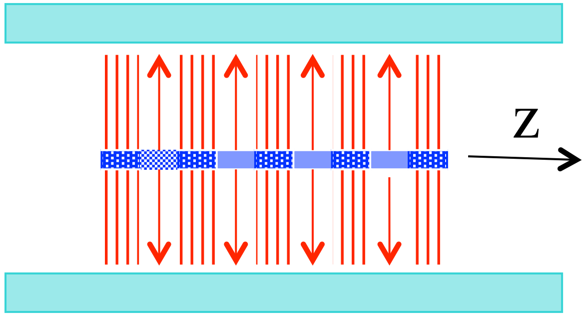
• If the fields penetrate and pass through the material, they can interact with bodies in the outer region.

• If the skin depth is very small (rapidly varying fields), fields do not penetrate, the electric field lines are perpendicular to the wall, as in the static case, while the magnetic field lines are tangent to the surface.

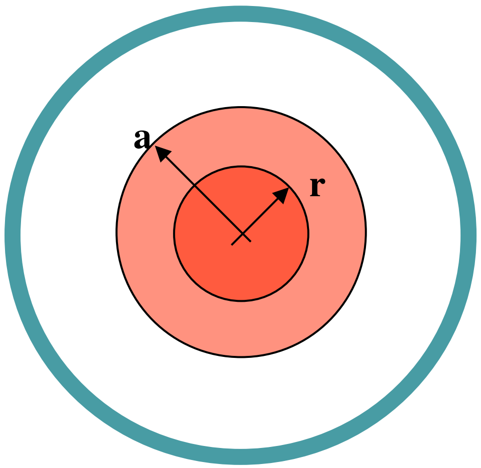


Example 2: Circular Perfectly Conducting Pipe

(Uniform Beam at Center)



In the case of cylindrical charge distribution, and $\gamma \to \infty$, the electric field lines are perpendicular to the direction of motion. The transverse fields intensity can be computed as in the static case, applying the Gauss and Ampere laws.



$$\lambda(r) = \lambda_0 \left(\frac{r}{a}\right)^2 ; \int_S E_r (2\pi r)\Delta z = \frac{\lambda(r)\Delta z}{\varepsilon_0}$$

$$E_r = \frac{\lambda(r)}{2\pi\varepsilon_0 r}; \quad \mathrm{B}_\theta = \frac{\beta}{c} E_r$$

$$E_r(\boldsymbol{r}) = \frac{\lambda_0}{2\pi\, \varepsilon_0} \frac{r}{a^2}; \quad B_\theta(\boldsymbol{r}) = \frac{\lambda_0 \beta}{2\pi\varepsilon_0 c} \frac{r}{a^2}$$

$$F_\perp(r) = e(E_r - \beta c\, B_\theta) = \frac{e}{\gamma^2} E_r$$

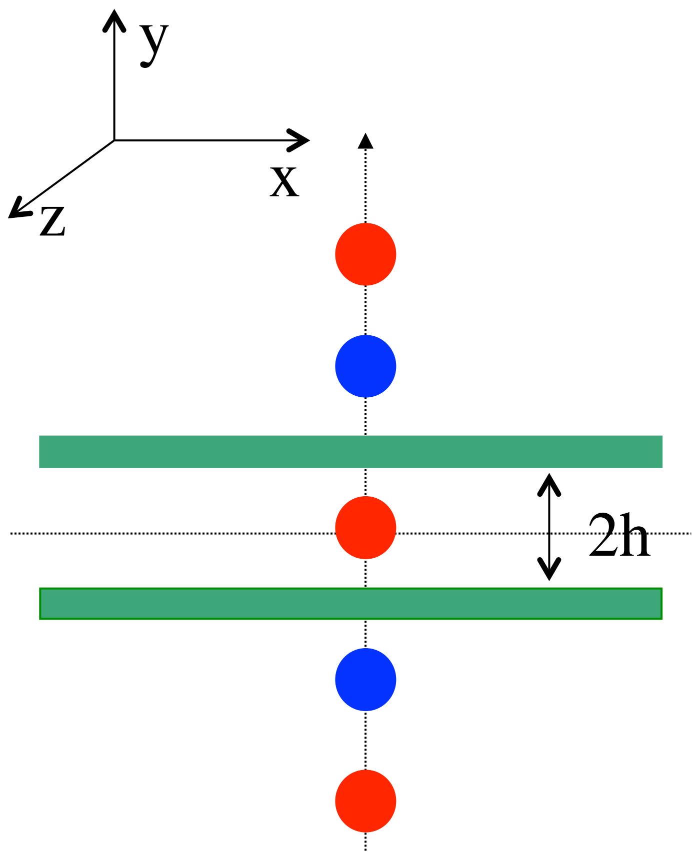
- Due to the symmetry, the transverse fields produced by an ultra-relativistic charge inside the pipe are the same as in the free space.

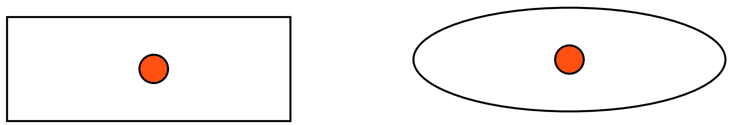
- For a distribution with cylindrical symmetry, in the ultra-relativistic regime, there is a cancellation of the electric and magnetic forces.

- The uniform beam produces exactly the same forces as in the free space.

- This result does not depend on the longitudinal distribution of the beam. In this case one should consider the local charge density $\lambda(z)$

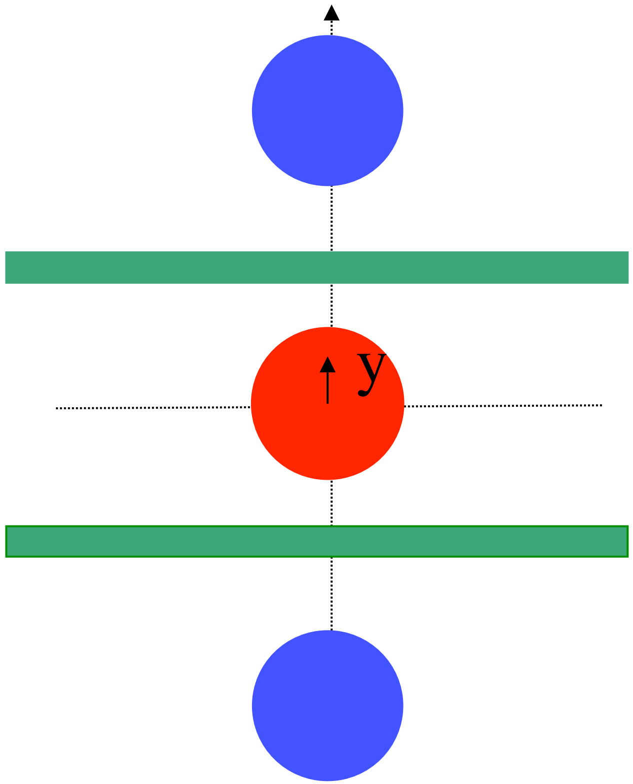
Parallel Plates (Beam at Center)





In some cases, the beam pipe cross section is such that we can consider only the surfaces closer to the beam, which behave like two parallel plates. In this case, we use the image method to a charge distribution of radius a between two conducting plates $2h$ apart. By applying the superposition principle we get the total image field at a position y inside the beam.

$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\,\varepsilon_0}\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{2nh+y} - \frac{1}{2nh-y}\right]$$



$$E_y^{im}(z,y) = \frac{\lambda(z)}{2\pi\,\varepsilon_0}\sum_{n=1}^{\infty}(-1)^n\frac{-2y}{(2nh)^2 - y^2} \cong \frac{\lambda\,(z)}{4\pi\,\varepsilon_o h^2}\frac{\pi^2}{12}y$$

Where we have assumed $h>>a>y$.

For d.c. or slowly varying currents, the boundary condition imposed by the conducting plates does not affect the magnetic field.

There is no magnetic field which can compensate the electric field due to the "image" charges.

$$F_y(y) = \frac{e}{\gamma^2}E_y^{dir} + eE_y^{im} = \frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\;\varepsilon_0}\frac{y}{a^2} + \frac{e\lambda\;(z)}{4\pi\;\varepsilon_o h^2}\frac{\pi^2}{12}y$$

From the divergence equation we derive also the other transverse component:

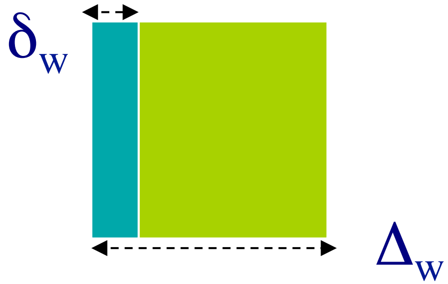
$$\frac{\partial}{\partial x}E_x^{im} = -\frac{\partial}{\partial y}E_y^{im} \Rightarrow E_x^{im}(z,x) = \frac{-\lambda(z)}{4\pi\,\varepsilon_0 h^2}\frac{\pi^2}{12}x$$

Including also the direct space charge force, we get:

$$F_x(z,x) = \frac{e\lambda(z)x}{2\pi\,\varepsilon_0}\left(\frac{1}{a^2\gamma^2} - \frac{\pi^2}{24h^2}\right)$$

$$F_y(z,y) = \frac{e\lambda(z)y}{2\pi\,\varepsilon_0}\left(\frac{1}{a^2\gamma^2} + \frac{\pi^2}{24h^2}\right)$$

Therefore, for γ>>1, and for d.c. or slowly varying currents the cancellation effect applies only for the direct space charge forces. There is no cancellation of the electric and magnetic forces due to the "image" charges.



Parallel Plates (Beam at Center) a.c. currents

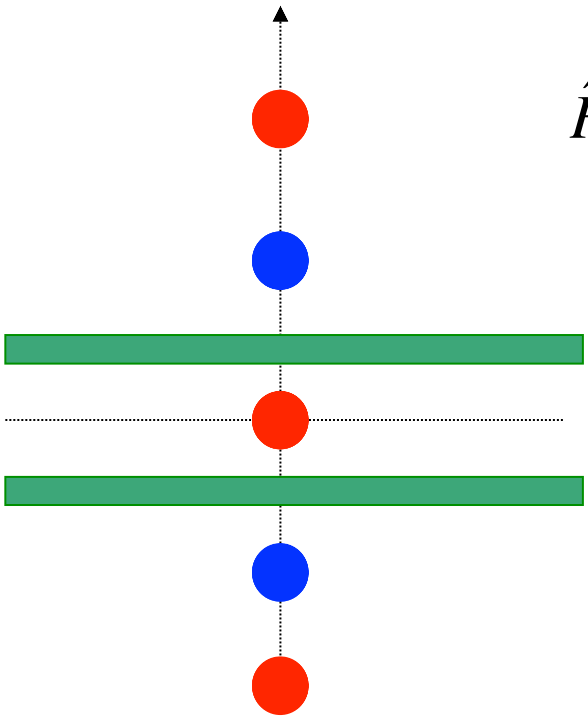
Usually, the frequency beam spectrum is quite rich of harmonics, especially for bunched beams.

It is convenient to decompose the current into a d.c. component, I, for which $\delta_w \gg \Delta_w$, and an a.c. component, $\hat{I}$, for which $\delta_w \ll \Delta_w$.

While the d.c. component of the magnetic field does not perceive the presence of the material, its a.c. component is obliged to be tangent at the wall.

We can see that this current produces a magnetic field able to cancel the effect of the electrostatic force.

$$\hat{E}_y(z,y) = \frac{\hat{\lambda}(z)y}{\pi\,\varepsilon_0}\frac{\pi^2}{48h^2}; \quad \hat{B}_x(z,y) = -\frac{\beta}{c}\hat{E}_y(z,y)$$



$$\hat{F}_y(z,y) = \frac{e\hat{\lambda}(z)y}{\pi\,\varepsilon_0\gamma^2}\frac{\pi^2}{48h^2} \qquad \left(\hat{I} = \beta c\hat{\lambda}\right)$$

$$\hat{F}_y(z,x) = \frac{e\hat{\lambda}(z)y}{2\pi\,\varepsilon_0\gamma^2}\left(\frac{1}{a^2} + \frac{\pi^2}{24h^2}\right)$$

$$\hat{F}_x(z,x) = \frac{e\hat{\lambda}(z)x}{2\pi\,\varepsilon_0\gamma^2}\left(\frac{1}{a^2} - \frac{\pi^2}{24h^2}\right)$$

There is cancellation of the electric and magnetic forces.

Parallel Plates - General expression of the force

Taking into account all the boundary conditions for d.c. and a.c. currents, we can write the expression of the force as:

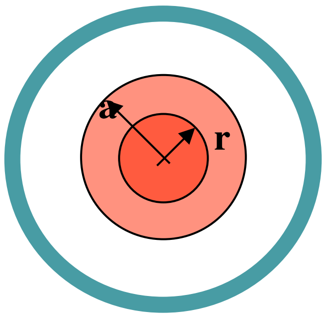
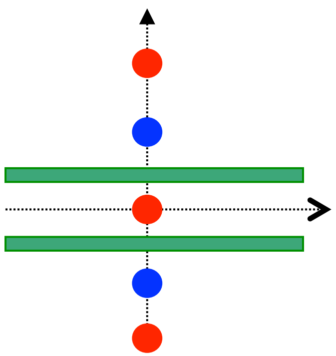
$$F_u = \frac{e}{2\pi\,\varepsilon_0}\left[\frac{1}{\gamma^2}\left(\frac{1}{a^2} \mp \frac{\pi^2}{24h^2}\right)\lambda \mp \beta^2\left(\frac{\pi^2}{24h^2} + \frac{\pi^2}{12g^2}\right)\bar{\lambda}\right]u$$

$$u = x, y$$

where λ is the total current divided by βc, $\bar{\lambda}$ its d.c. part, g the gap of a bending magnet, and we take the sign (+) if u=y, and the sign (–) if u=x.

-L. J. Laslett, LBL Document PUB-616, 1987, vol III

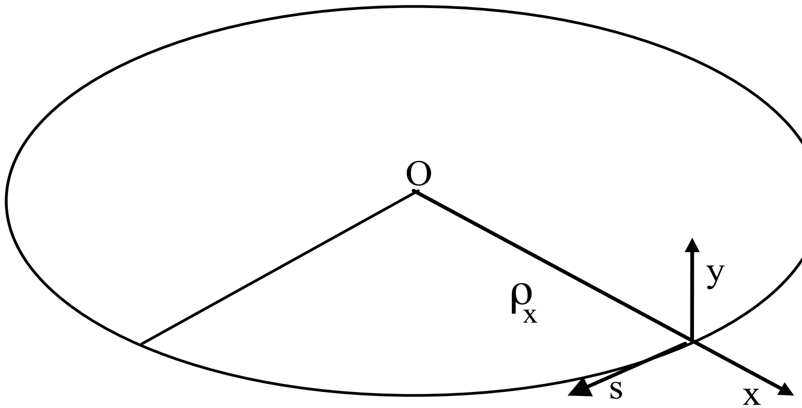
$$\lambda(z) = \lambda_o + \hat{\lambda}\cos(k_z z) \quad ; \quad k_z = 2\pi / l_w$$

	D.C.	A.C. $(\delta_w << \Delta_w)$
	$F_\perp(r) = \dfrac{e}{\gamma^2}\dfrac{\lambda(z)}{2\pi\,\varepsilon_0}\dfrac{r}{a^2}$	
	$F_x(z,x) = \dfrac{e\lambda_0 x}{2\pi\,\varepsilon_0}\left(\dfrac{1}{a^2\gamma^2} - \dfrac{\pi^2}{24h^2}\right)$ $F_y(z,x) = \dfrac{e\lambda_0 y}{2\pi\,\varepsilon_0}\left(\dfrac{1}{a^2\gamma^2} + \dfrac{\pi^2}{24h^2}\right)$	$\hat{F}_x(z,x) = \dfrac{e\hat{\lambda}(z)x}{2\pi\,\varepsilon_0\gamma^2}\left(\dfrac{1}{a^2} - \dfrac{\pi^2}{24h^2}\right)$ $\hat{F}_y(z,x) = \dfrac{e\hat{\lambda}(z)y}{2\pi\,\varepsilon_0\gamma^2}\left(\dfrac{1}{a^2} + \dfrac{\pi^2}{24h^2}\right)$

Space charge effects in circular accelerators

Self Fields and betatron motion

Consider a perfectly circular accelerator with radius ρ_x. The beam circulates inside the beam pipe. The transverse single particle motion in the linear regime, is derived from the equation of motion. Including the self field forces in the motion equation, we have



$$\frac{d\left(m\gamma\,\boldsymbol{v}\right)}{dt} = \boldsymbol{F}^{ext}\left(\vec{r}\right) + \boldsymbol{F}^{self}\left(\vec{r}\right) \qquad \frac{d\boldsymbol{v}}{dt} = \frac{\boldsymbol{F}^{ext}\left(\vec{r}\right) + \boldsymbol{F}^{self}\left(\vec{r}\right)}{m\gamma}$$

Following the same steps already seen in the "transverse dynamics" lectures, we write:

$$\vec{r} = (\rho_x + x)\hat{e}_x + y\hat{e}_y$$

$$\vec{v} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y + \omega_0(\rho_x + x)\hat{e}_z$$

$$\vec{a} = \left[\ddot{x} - \omega_0^2(\rho_x + x)\right]\hat{e}_x + \ddot{y}\hat{e}_y + \left[\dot{\omega}_0(\rho_x + x) + 2\omega_0\dot{x}\right]\hat{e}_z$$

For the motion along x:

$$\ddot{x} - \omega_0^2(\rho_x + x) = \frac{1}{m\gamma}\left(F_x^{ext} + F_x^{self}\right)$$

which is rewritten with respect to the azimuthal position $s = v_z t$:

$$\ddot{x} = v_z^2 x'' = \omega_0^2(\rho_x + x)^2 x''$$

$$x'' - \frac{1}{\rho_x + x} = \frac{1}{m v_z^2 \gamma}\left(F_x^{ext} + F_x^{self}\right)$$

We assume a small transverse displacement x, and only transverse quadrupole forces which keep the beam around the closed orbit:

$$F_x^{ext} \approx \left(\frac{\partial F_x^{ext}}{\partial x} \right)_{x=0} x \qquad\qquad x \ll \rho_x$$

Putting $v_z = \beta c$, we get

$$x'' + \left[\frac{1}{\rho_x^2} - \frac{1}{\beta^2 E_0} \left(\frac{\partial F_x^{ext}}{\partial x} \right)_{x=0} \right] x = \frac{1}{\beta^2 E_0} F_x^{self}(x)$$

where E_o is the particle energy. This equation expressed as function of "s" reads:

$$x''(s) + \left[\frac{1}{\rho_x^2(s)} + K_x(s) \right] x(s) = \frac{1}{\beta^2 E_o} F_x^{self}(x,s)$$

• In the analysis of the motion of the particles in presence of the self field, we will adopt a simplified model where particles execute simple harmonic oscillations around the reference orbit.

• This is the case where the focusing term is constant. Although this condition in never fulfilled in a real accelerator, it provides a reliable model for the description of the beam instabilities

$$x''(s) + K_x x(s) = \frac{1}{\beta^2 E_0} F_x^{self}(x)$$

Free betatron motion:

$$x''(s) + K_x x(s) = 0$$

$$\begin{cases} x(s) = A_x \cos\left[\sqrt{K_x}\, s - \varphi_x\right] \\ \lambda_\beta = \dfrac{2\pi}{\sqrt{K_x}} \\ Q_x = \dfrac{2\pi\rho_x}{\lambda_\beta} = \dfrac{2\pi\rho_x\sqrt{K_x}}{2\pi} = \rho_x\sqrt{K_x} \\ K_x = \left(\dfrac{Q_x}{\rho_x}\right)^2 \end{cases}$$

Perturbed motion:

$$x''(s) + \left(\frac{Q_x}{\rho_x}\right)^2 x(s) = \frac{1}{\beta^2 E_0} F_x^{self}(x,s)$$

Transverse Incoherent Effects

We take the linear term of the transverse force in the betatron equation:

$$F_x^{s.c.}(x,z) \cong \left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0} x \Rightarrow x'' + \left(\frac{Q_{x0}}{\rho_x}\right)^2 x = \frac{1}{\beta^2 E_0}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0} x$$

$$x'' + \left[\left(\frac{Q_{x0}}{\rho_x}\right)^2 - \frac{1}{\beta^2 E_0}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)_{x=0}\right] x = 0 \Rightarrow x'' + \left(\frac{Q_{x0} + \Delta Q_x}{\rho_x}\right)^2 x = 0$$

$$x'' + \left[\frac{Q_{x0}^2 + 2Q_{x0}\Delta Q_x + \cancel{\Delta Q_x^2}}{\rho_x^2}\right] x = 0 \Rightarrow x'' + \left[\frac{Q_{x0}^2}{\rho_x^2} + \frac{2Q_{x0}\Delta Q_x}{\rho_x^2}\right] x = 0$$

$$\Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)$$

Transverse Incoherent Effects

$$\Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\left(\frac{\partial\, F_x^{s.c.}}{\partial x}\right)$$

The shift of betatron wave number (tune shift) is negative since the space charge forces are defocusing on both planes (the betatron wavelength increases). Notice that the space charge force, and then the tune shift, is, in general, function of "z", therefore this expression represents a tune spread inside the beam. This is why we call it incoherent. Notice that if we include higher order terms in the transverse force, we cannot write the harmonic oscillator equation any more, and, in general, we get a tune shift that depends on the amplitude of the betatron oscillation.

Example 3: incoherent betatron tune shift for a uniform electron beam of radius *a*=100μm, length l_o=100μm, inside a circular perfectly conducting pipe (energy E_0=1GeV, N=10^{10}, ϱ_x=20m, Q_{xo}=4.15)

$$\boxed{\left(\frac{\partial F_x^{s.c.}}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{e\lambda_0 x}{2\pi\varepsilon_0\gamma^2 a^2}\right) = \frac{e\lambda_0}{2\pi\varepsilon_0\gamma^2 a^2}}$$

$$\boxed{\Delta Q_x = -\frac{\rho_x^2 N e^2}{4\pi\varepsilon_0 a^2\beta^2\gamma^2 E_0 Q_{x0} l_0}}$$

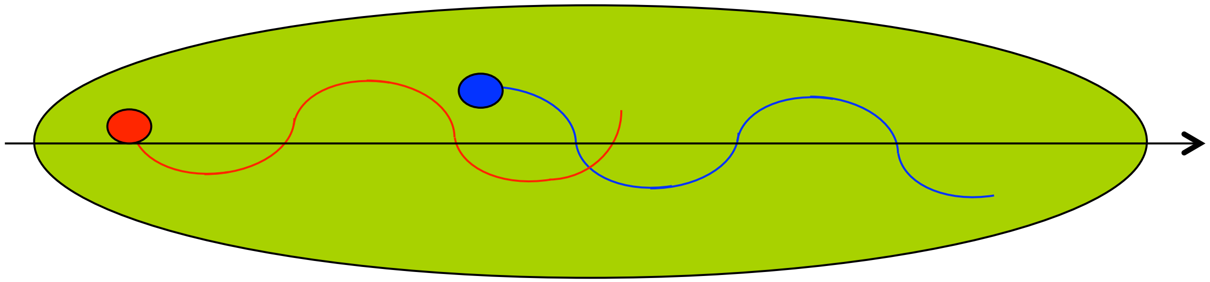
$$r_{e,p} = \frac{e^2}{4\pi\varepsilon_o m_o c^2} \quad (electrons: 2.82\ 10^{-15} m,\ protons: 1.53\ 10^{-18} m)$$

$$\boxed{\Delta Q_x = -\frac{\rho_x^2 N r_{e,p}}{a^2\beta^2\gamma^3 Q_{xo} l_o} \approx -0.36}$$

Remember that for real bunched beams the space charge forces depend on the longitudinal and radial position of the charge.

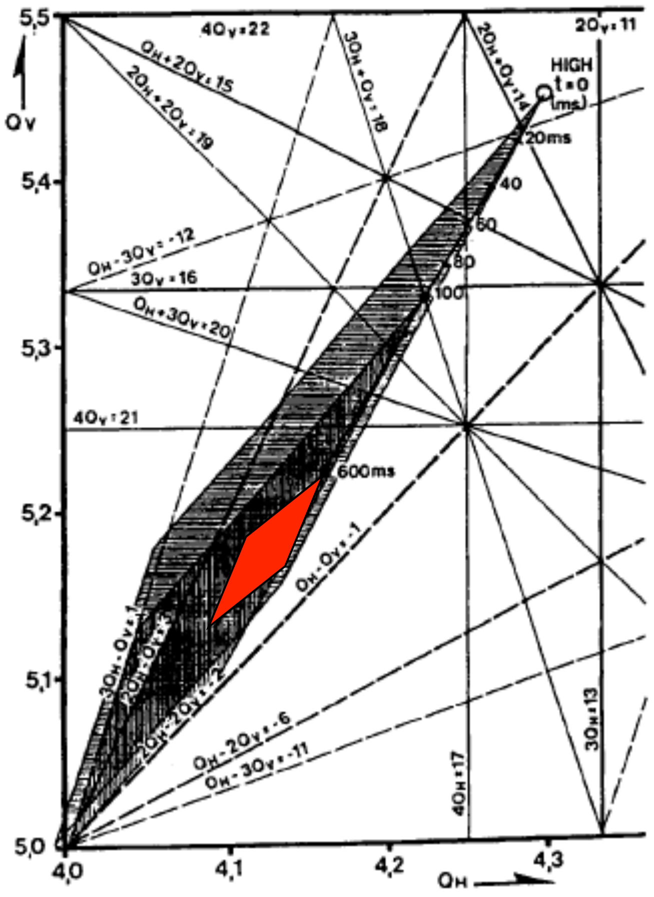
Shift and Spread of the Incoherent Tunes

If the beam is located at the centre of symmetry of the pipe, the e.m. forces due to space charge and images cannot affect the motion of the centre of mass (coherent), but change the trajectory of individual charges in the beam (incoherent).



CONSEQUENCES: in circular accelerators the values of the betatron wave numbers should not be close to rational numbers in order to avoid the crossing of linear and non-linear resonances where the beam becomes unstable. The spread induced by the space charge force can make hard to satisfy this basic requirement.

Example from A. Hofmann in CAS 1992 (General Course - Jyväskylä Finland)

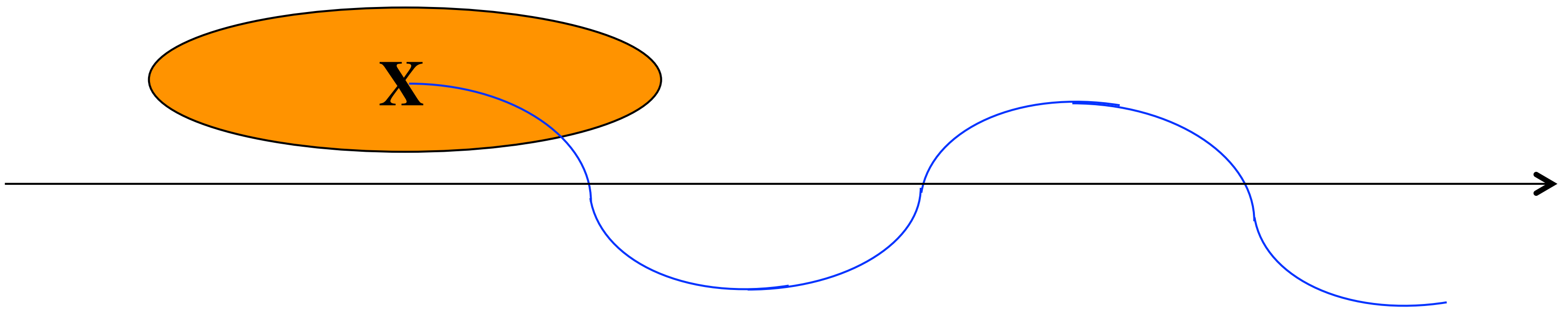


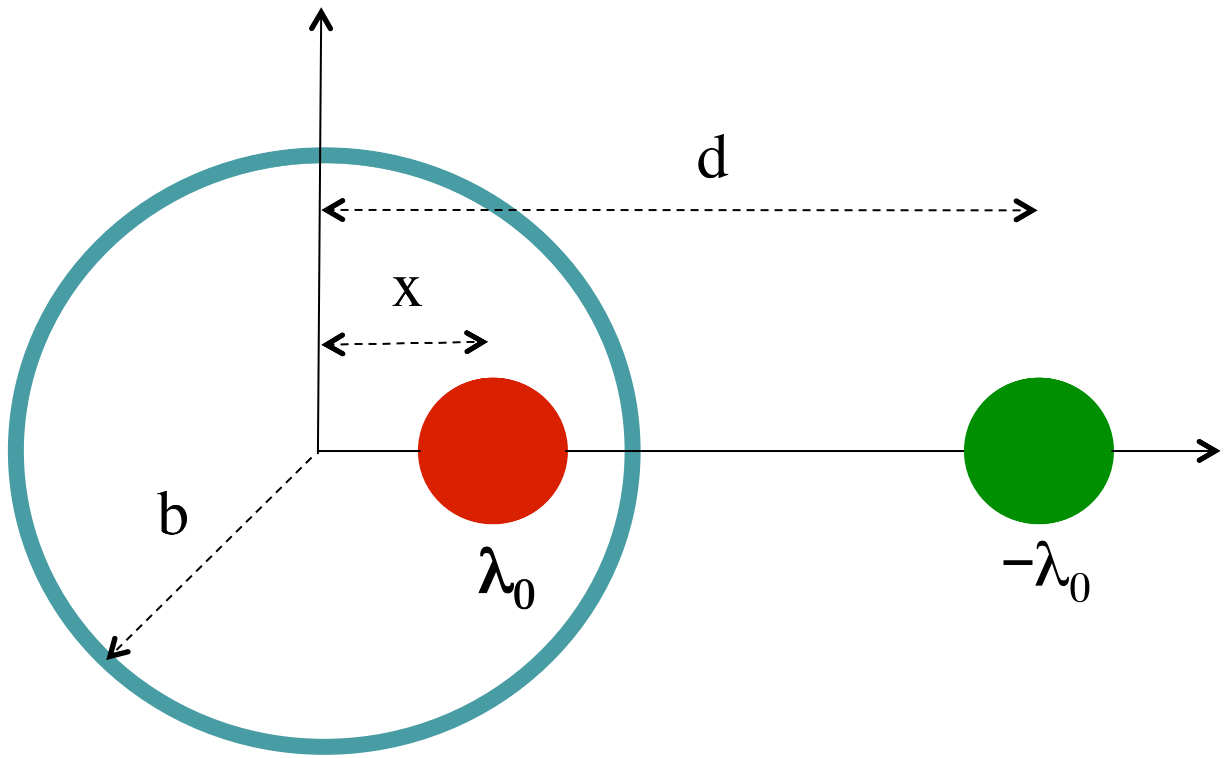
CERN PS Booster accelerates proton bunches from 50 to 800 MeV in about 0.6 s. The tunes occupied by the particles are indicated in the diagram by the shaded area. As time goes on, the energy increases and the space charge tune spread gets smaller covering at t=100 ms the tune area shown by the darker area. The point of highest tune corresponds to the particles which are least affected by the space charge. This point moves in the Q diagram since the external focusing is adjusted such that the reduced tune spread lies in a region free of harmful resonances.

The small red area shows the situation at t=600 ms when the beam has reached the energy of 800 MeV. The tune spread reduction is lower than expected with the energy increase ($1/\gamma^3$) dependence since the bunch dimensions also decrease during the acceleration.

Transverse Coherent Effects

If the beam experiences a transverse deflection kick, it starts to perform betatron oscillations as a whole. The beam, source of the space charge fields moves transversely inside the pipe, while individual particles still continue their incoherent motion around the common coherent trajectory.

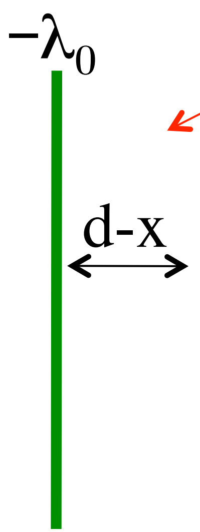




$$\boxed{d = \frac{b^2}{x}}$$

The image charge is at a distance "d" such that the pipe surface is at constant voltage, and pulls the beam away from the center of the pipe.

The effect is defocusing: the horizontal electric image field E and the horizontal force F are:



$$E_{xc}(\boldsymbol{x}) = \frac{\lambda_0}{2\pi\varepsilon_0}\frac{1}{d-x} \approx \frac{\lambda_0}{2\pi\varepsilon_0}\frac{1}{d} = \frac{\lambda_0}{2\pi\varepsilon_0}\frac{x}{b^2}$$

$$F_{xc}(r) \approx \frac{e\lambda_0}{2\pi\varepsilon_0}\frac{x}{b^2}$$

$$\Delta Q_{xc} = -\frac{\rho_x^2}{2\beta^2 E_o Q_{x0}}\left(\frac{\partial F_{xc}}{\partial x}\right) = -\frac{\rho_x^2}{2\beta^2 E_o Q_{x0}}\frac{e\lambda_0}{2\pi\ \varepsilon_o b^2}$$

$$\Delta Q_{xc} = -\frac{r_e \rho_x^2}{\beta^2 \gamma Q_{x0}}\frac{N}{b^2 l_0}$$

$$r_{e,p} = \frac{e^2}{4\pi\varepsilon_0 m_0 c^2}$$

Example 4: coherent betatron tune shift for a uniform electron beam of length l_o=100μm, inside a circular perfectly conducting pipe of radius $b=2$cm, (energy E_0=1GeV, N=10^{10}, ϱ_x=20m, Q_{xo}=4.15)

$$r_e = \frac{e^2}{4\pi\varepsilon_o m_o c^2} = 2.82\times 10^{-15} m$$

$$\boxed{\Delta Q_{xc} = -\frac{r_e \rho_x^2}{\beta^2 \gamma Q_{x0}} \frac{N}{b^2 l_0} \approx -0.7}$$

Self Fields and synchrotron motion

The longitudinal motion is governed by the RF voltage

$$V(z) = V_0 \cos\left(\frac{h\omega_0}{c} z - \varphi_s\right)$$

h=harmonic number, ω_o revolution frequency, φ_s synchronous phase

$$z'' + \left(\frac{Q_s}{\rho_x}\right)^2 z = 0; \qquad Q_s = \frac{\omega_s}{\omega_0} = \sqrt{\frac{e\eta h V_0 \sin\varphi_s}{2\pi\beta E_0}}$$

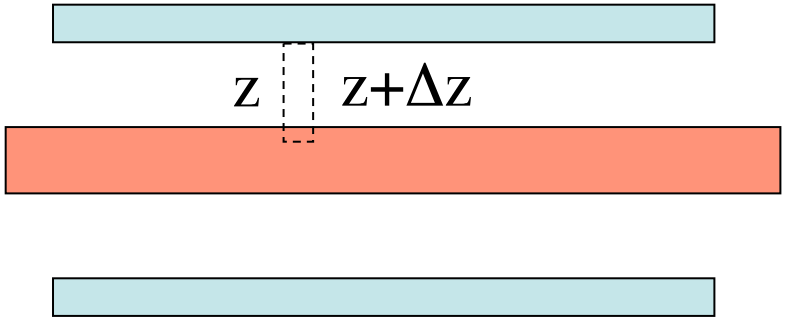
With longitudinal space charge forces the equation becomes:

$$z'' + \left(\frac{Q_s}{\rho_x}\right)^2 z = \frac{\eta F_z}{\beta^2 E_0}; \qquad \eta = \frac{1}{\gamma^2} - \alpha_c$$

LONGITUDINAL FORCES

In order to derive the relationship between the longitudinal and transverse forces inside a beam, let us consider the case of cylindrical symmetry and ultra-relativistic bunches. We know from Faraday's law of induction that a varying magnetic field produces a rotational electric field:

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{n} dS$$



We choose as path a rectangle going through the beam pipe and the beam, parallel to the axis.

$$E_z(r,z)\Delta z + \int_r^b E_r(r,z+\Delta z)dr - E_z(b,z)\Delta z - \int_r^b E_r(r,z)dr =$$

$$= -\Delta z \frac{\partial}{\partial t}\int_r^b B_\theta(r,z)dr \qquad\qquad E_r(r,z+\Delta z) - E_r(r,z) = \frac{\partial E_r(r,z)}{\partial z}\Delta z$$

$$E_z(r,z) = E_z(b,z) - \int_r^b \left[\frac{\partial E_r(r,z)}{\partial z} + \frac{\partial B_\theta(r,z)}{\partial t}\right]dr$$

$$\boxed{dz = -vdt}$$

$$E_z(r,z) = E_z(b,z) - \frac{\partial}{\partial z}\int_r^b \left[E_r(r,z) - vB_\theta(r,z)\right]dr$$

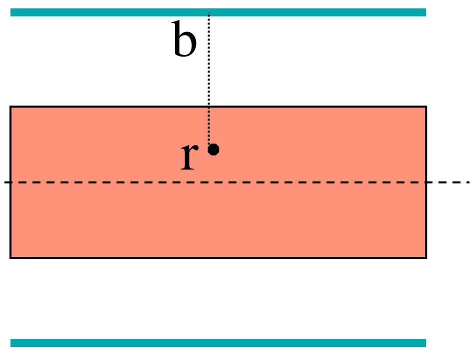
$$E_z(r,z) = E_z(b,z) - \frac{\partial}{\partial z}\int_r^b \left[E_r(r,z) - \beta^2 E_r(r,z)\right]dr$$

$$\boxed{E_z(r,z) = E_z(b,z) - (1-\beta^2)\frac{\partial}{\partial z}\int_r^b E_r(r,z)dr}$$

where $(1-\beta^2)=1/\gamma^2$. For perfectly conducting walls $E_z=0$.

$$\boxed{E_z(r,z) = -\frac{1}{\gamma^2}\frac{\partial}{\partial z}\int_r^b E_r(r,z)dr}$$

Transverse uniform beam in a circular p.c. pipe.



$$\boxed{F_z(r,z) = -\frac{e}{4\pi\varepsilon_0\gamma^2}(1-\frac{r^2}{a^2}+2\ln\frac{b}{a})\frac{\partial\lambda(z)}{\partial z}}$$