# Introduction to Transverse Beam Dynamics

## Lecture 4: Dispersion / Errors in fields and gradient

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# Dispersion function and orbit

We need to study the motion for particles with $\Delta p = p - p_0 \neq 0$ :

$$x''(s) + K(s)\,x(s) = \frac{1}{\rho}\frac{\Delta p}{p_0}$$

The general solution of this equation is:

$$x(s) = x_\beta(s) + x_D(s) \qquad \begin{cases} x''_\beta(s) + K(s)\,x_\beta(s) = 0 \\ D''(s) + K(s)\,D(s) = \dfrac{1}{\rho} \end{cases}$$

with $x_D(s) = D(s)\,\frac{\Delta p}{p_0}$.

Remarks

- $x_D(s)$ describes the deviation from the closed orbit for off-momentum particles with a fixed $\Delta p$
- $D(s)$ is that special orbit that a particle would have for $\Delta p/p = 1$
- the orbit of a generic particle is the sum of the well known $x_\beta(s)$ and $x_D(s)$

# Dispersion function and orbit

$$\begin{cases} x(s) = x_\beta(s) + x_D(s) \\ x(s) = C(s)\,x_0 + S(s)\,x_0' + D(s)\,\dfrac{\Delta p}{p} \end{cases}$$

In matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}_0$$

We can rewrite the solution in matrix form:

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$

Inside a magnet, the dispersion trajectory is solution of $D''(s) + K(s)\,D(s) = \frac{1}{\rho}$ :

$$D(s) = S(s) \int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s) \int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

Exercise: show that $D(s)$ is a solution for the equation of motion, with the initial conditions $D_0 = D_0' = 0$.

# Dispersion function examples

Let's study, for different magnetic elements, the solution of:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

at the exit of the element: that is, in $D(L_{\text{magnet}})$

- Drift space:

$$M_{\text{Drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$C(t) = 1,\ S(t) = L \quad \Rightarrow$ the integrals cancel

$$M_{\text{Drift}} = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Dispersion function: sector dipole

- Sector dipole:
  $K = \frac{1}{\rho^2}$:

$$M_{\text{Dipole}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\sin\frac{L}{\rho} \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} \end{pmatrix}$$

which gives

$$D(s) = \rho\left(1 - \cos\frac{L}{\rho}\right)$$
$$D'(s) = \sin\frac{L}{\rho}$$

therefore

$$M_{\text{Dipole}} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\sin\frac{L}{\rho} & \rho\left(1-\cos\frac{L}{\rho}\right) \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} & \sin\frac{L}{\rho} \\ 0 & 0 & 1 \end{pmatrix}$$

# Dispersion function: quadrupole

- Focusing quadrupole:

$$M_{\mathrm{QF}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) & 0 \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

- Defocusing quadrupole:

$$M_{\mathrm{QD}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}L\right) & \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}L\right) & 0 \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}L\right) & \cosh\left(\sqrt{|K|}L\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Dispersion propagation through the machine

- The equation:

$$D(s) = S(s)\int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s)\int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

  shows that the dispersion inside a magnet does not depend on the dispersion that might have been generated by the upstreams magnets.

- At the exit of a magnet of length $L_\mathrm{m}$ the dispersion reaches the value $D(L_\mathrm{m})$, then it propagates from there on through the rest of the machine, just like any other particle:

$$\begin{pmatrix} D \\ D' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} D \\ D' \end{pmatrix}_0$$

# Closed orbit for an off-momentum particle

In a circular accelerator, even the trajectory of an off-energy particle must be periodic.

That is, for $\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix}$ we want:

$$\begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta\prime \\ 1 \end{pmatrix}$$

Let's rewrite this in $2 \times 2$ form:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} + \begin{pmatrix} D \\ D' \end{pmatrix}$$

$$\begin{pmatrix} 1 - C & -S \\ -C' & 1 - S' \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} D \\ D' \end{pmatrix}$$

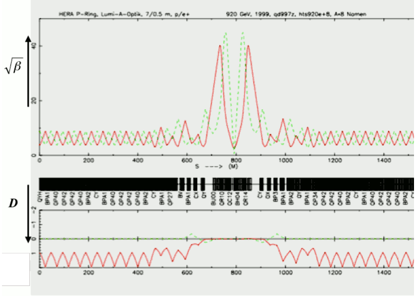
The solution is:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \frac{1}{(1 - C)(1 - S') - C'S} \begin{pmatrix} 1 - S' & S \\ C' & 1 - C \end{pmatrix} \begin{pmatrix} D \\ D' \end{pmatrix}$$

# Dispersion function: example

In this example from the HERA storage ring (DESY) we see the twiss parameters and the dispersion near the interaction point. In the periodic region,

$$x_\beta(s) = 1 \ldots 2 \text{ mm}$$
$$D(s) = 1 \ldots 2 \text{ m}$$
$$\Delta p/p \approx 1 \cdot 10^{-3}$$




Remember:

$$x(s) = x_\beta(s) + D(s)\,\frac{\Delta p}{p}$$

Beware: the dispersion contributes to the beam size:

$$\sigma_x = \sqrt{\sigma_{x_\beta}^2 + \left(D \cdot \frac{\Delta p}{p}\right)^2} = \sqrt{\epsilon\beta + \left(D \cdot \frac{\Delta p}{p}\right)^2}$$

- We need to suppress the dispersion at the IP !
- We need a special insertion section: a *dispersion suppressor*

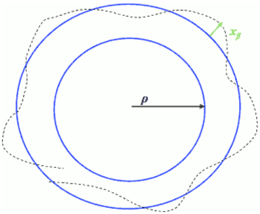
# The momentum compaction factor

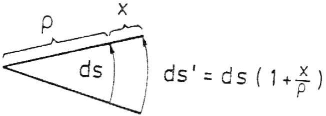
The dispersion function relates the momentum error of a particle to the horizontal orbit coordinate

The general solution of the equation of motion is

$$x\left(s\right) = x_{\beta}\left(s\right) + D\left(s\right)\frac{\Delta p}{p}$$

The dispersion changes also the length of the off-energy orbit.




particle with offset $x$ w.r.t. the design orbit:

$$\frac{\mathrm{d}s'}{\mathrm{d}s} = \frac{\rho + x}{\rho} \qquad \rightarrow \qquad \mathrm{d}s' = \left(1 + \frac{x}{\rho}\right)\mathrm{d}s$$

The circumference change is $\Delta C$, that is $C' = \oint\left(1 + \frac{x}{\rho}\right)\mathrm{d}s = C + \Delta C$

We define the "momentum compaction factor", $\alpha$, such that:

$$\frac{\Delta C}{C} = \alpha\frac{\Delta p}{p} \qquad \rightarrow \text{a rough estimate is } \alpha = \frac{1}{Q_x^2}$$

# The beam matrix

To track a beam of particles, let's assume with Gaussian distribution, the beam ellipse can be characterised by a "beam matrix" $\Sigma$

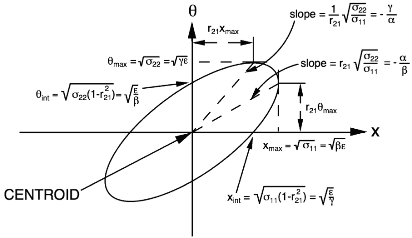
The equation of an ellipse can be written in matrix form:

$$X^T \Sigma X = 1$$

with $X = \begin{pmatrix} x \\ x' \end{pmatrix}$,

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

$\Sigma$ is the covariance matrix of the particles distribution




- The area of the ellipse is

$$A = \pi\sqrt{\det \Sigma} = \pi\epsilon$$

  with slope $r_{21} = \sigma_{21}/\sqrt{\sigma_{11}\sigma_{22}}$.

- The transformation that transports the beam ellipse from a position 0 to a position $s$ is:

$$\Sigma_s = M \, \Sigma \, M^T$$

# Beam matrix and Twiss parameters

The beam matrix is the covariance matrix of the particle distribution

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

this matrix can be also expressed in terms of Twiss parameters $\alpha$, $\beta$, $\gamma$ and $\epsilon$:

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix} = \epsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

We can transport the beam matrix, or the twiss parameters, from 0 to $s$ by two equivalent ways:

- Twiss $3 \times 3$ transport matrix

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

- Using the transfer matrix $M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}_{0 \to s}$:

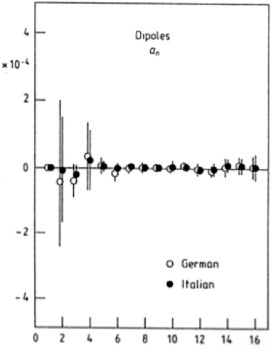
$$\Sigma_s = M \, \Sigma_0 \, M^T$$

# Magnetic imperfections

## HIgh order multipolar components

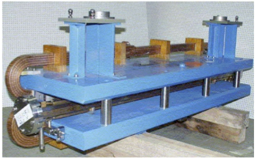
Taylor expansion of the $B$ field:

$$B_y(x) = \underbrace{B_{y0}}_{\text{dipole}} + \underbrace{\frac{\partial B_y}{\partial x}}_{\text{quad}} x + \frac{1}{2} \underbrace{\frac{\partial^2 B_y}{\partial x^2}}_{\text{sextupole}} x^2 + \frac{1}{3!} \underbrace{\frac{\partial^3 B_y}{\partial x^3}}_{\text{octupole}} x^3 + \ldots \qquad \text{divide by } B_{y0}$$




*There can be undesired multipolar components, due to small fabrication defects*

*Or also errors in the windings, in the gap h, ... remember:* $B = \frac{\mu_0 n I}{h}$



# Dipole magnet errors

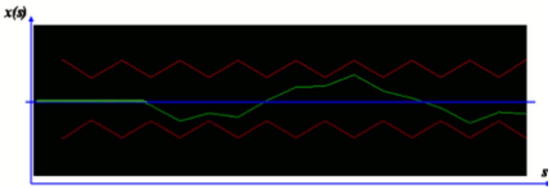
Let's imagine to have a magnet with $B_x = B_0 + \Delta B$. This will give an additional kick to each particle, and will distort the ideal design orbit

$$F_x = ev\,(B_0 + \Delta B)\,; \qquad \Delta x' = \Delta B ds / B\rho$$

A dipole error will cause a distortion of the closed orbit, that will „run around" the storage ring, being observable everywhere. If the distortion is small enough, it will still lead to a closed orbit.

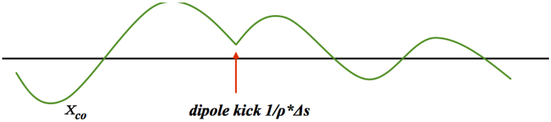
Example: 1 single dipole error

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M_{\text{lattice}} \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix}_0$$




In order to have bounded motion the tune $Q$ must be non-integer, $Q \neq 1$. We see that even for particles with reference momentum $p_0$ an integer $Q$ value is forbidden, since small field errors are always present.

# Orbit distortion for a single dipole field error




We consider a single thin dipole field error at the location $s = s_0$, with a kick angle $\Delta x'$.

$$X_- = \begin{pmatrix} x_0 \\ x_0' + \Delta x' \end{pmatrix}, \quad X_+ = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

are the phase space coordinates before and after the kick located at $s_0$. The closed-orbit condition becomes

$$M_{\mathrm{Lattice}} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' + \Delta x' \end{pmatrix}$$

The resulting closed orbit at $s_0$ is

$$x_0 = \frac{\beta_0 \Delta x'}{2 \sin \pi Q} \cos \pi Q; \quad x_0' = \frac{\Delta x'}{2 \sin \pi Q} (\sin \pi Q - \alpha_0 \cos \pi Q)$$

where $Q$ is the tune. The orbit at any other location $s$ is

$$x(s) = \frac{\sqrt{\beta_s \beta_0}}{2 \sin \pi Q} \cos \left( \pi Q - |\mu_s - \mu_0| \right) \Delta x'$$

(see the references for a demonstration)

# Orbit distortion for a distributed dipole field errors

One single dipole field error

$$x(s) = \frac{\sqrt{\beta(s)\,\beta(0)}}{2\sin\pi Q}\cos\left(\pi Q - |\mu(s) - \mu(0)|\right)\Delta x'$$

Distributed dipole field errors

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q}\oint\sqrt{\beta(t)}\cos\left(\pi Q - |\mu(s) - \mu(t)|\right)\Delta x'\mathrm{d}t$$

- orbit distortion is visible at any position $s$ in the ring, even if the dipole error is located at one single point $s_0$
- the $\beta$ function describes the sensitivity of the beam to external fields
- the $\beta$ function acts as amplification factor for the orbit amplitude at the given observation point
- there is a resonance denominator ($Q$ integer)

# The resonances

Closed orbit distortion due to dipole field errors:

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q} \oint \sqrt{\beta(t)} \cos\left(\pi Q - |\mu(s) - \mu(t)|\right) \Delta x' \mathrm{d}t$$

Remember the definition of tune:

$$Q = \frac{\mu_L}{2\pi}$$

is the phase advance for a revolution $\mu_L$ in units of $2\pi$.

Extremely important:

- In case of imperfections the orbit becomes unstable for $Q$ integer
- Integer tunes lead to a resonant increase of the closed orbit amplitude in presence of the smallest dipole field error!

# Tunes and resonances

The particles – oscillating under the influence of the external magnetic fields – can be excited in case of resonant tunes to infinite high amplitudes.

There is particle loss within a short number of turns.

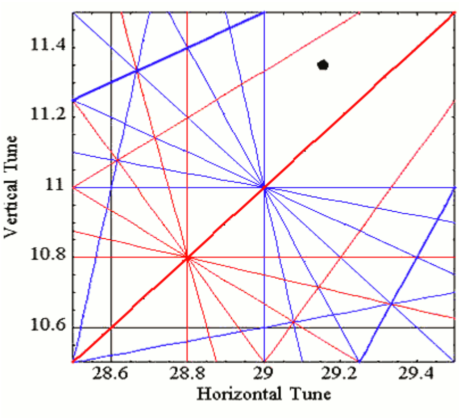
The cure:

1. **avoid** large magnet errors
2. **avoid forbidden tune values** in both planes

$$\mathrm{m} \cdot Q_x + \mathrm{n} \cdot Q_y \neq \mathrm{p}$$

with $m$, $n$, $p$ integer numbers

# Resonance diagram




A resonance diagram for the Diamond light source. The lines shown are the resonances and the black dot shows a suitable place where the machine could be operated.

# Quadrupole errors: tune shift

Orbit perturbation described by a thin lens quadrupole:

$$M_{\mathrm{Perturbed}} = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta k \mathrm{d}s & 1 \end{pmatrix}}_{\text{perturbation}} \underbrace{\begin{pmatrix} \cos\mu_0 + \alpha\sin\mu_0 & \beta\sin\mu_0 \\ -\gamma\sin\mu_0 & \cos\mu_0 - \alpha\sin\mu_0 \end{pmatrix}}_{\text{ideal ring}}$$

Let's see how the tunes changes:

$$M_{\mathrm{Perturbed}} = \begin{pmatrix} \cos\mu_0 + \alpha\sin\mu_0 & \beta\sin\mu_0 \\ \Delta k \mathrm{d}s\left(\cos\mu_0 + \alpha\sin\mu_0\right) - \gamma\sin\mu_0 & \Delta k \mathrm{d}s \beta\sin\mu_0 + \cos\mu_0 - \alpha\sin\mu_0 \end{pmatrix}$$

Remember the rule for computing the tune:

$$2\cos\mu = \mathrm{trace}\left(M\right) = 2\cos\mu_0 + \Delta k \mathrm{d}s \beta\sin\mu_0$$

# Quadrupole errors: tune shift (cont.)

We rewrite $\cos\mu = \cos(\mu_0 + \Delta\mu)$

$$\cos(\mu_0 + \Delta\mu) = \cos\mu_0 + \frac{1}{2}\Delta k \mathrm{d}s \beta \sin\mu_0$$

from which we can compute that

$$\Delta\mu = \frac{\Delta k \mathrm{d}s \beta}{2} \quad \text{shift in the phase advance}$$

$$\Delta Q = \int_{s_0}^{s_0+L} \frac{\Delta k(s)\beta(s)\mathrm{d}s}{4\pi} \quad \text{tune shift}$$

Important remarks:

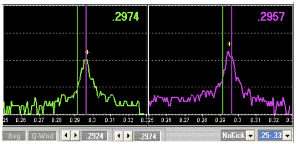
- the tune shift if proportional to the $\beta$-function at the quadrupole
  - field quality, power supply tolerances etc. are much tighter at places where $\beta$ is large
- $\beta$ is a measurement of the sensitivity of the beam

# Quadrupole errors: tune shift example

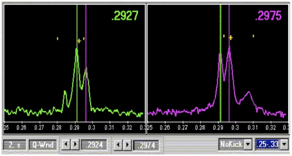
Deliberate change of a quadrupole strength in a synchrotron:

$$\Delta Q = \int_{s_0}^{s_0+L} \frac{\Delta K(s)\,\beta(s)\,\mathrm{d}s}{4\pi} \approx \frac{\Delta K(s)\; L_{\mathrm{quad}}\,\overline{\beta}}{4\pi}$$

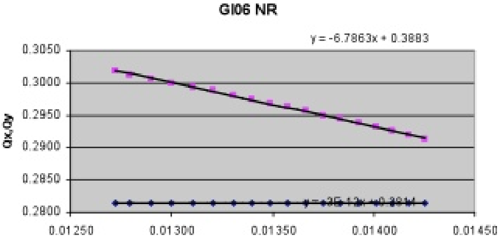
$\Rightarrow$




*the tune is measured permanently*




*After changing the strength of a quad:*
*we get a second peak*

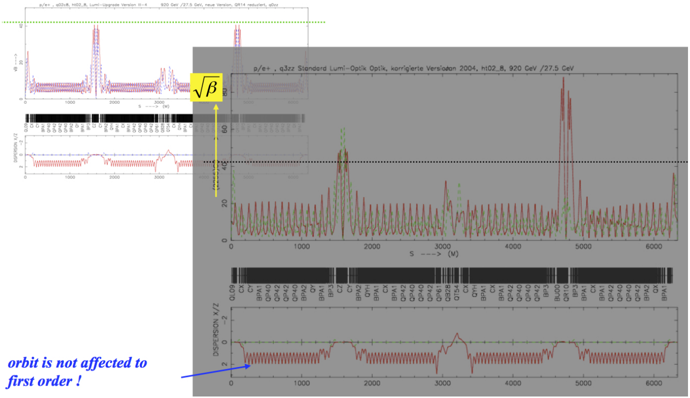


# Quadrupole errors: beta beat

A quadrupole error at $s_0$ causes distortion of $\beta$-function at $s$: $\Delta\beta(s)$ due to the errors of all quadrupoles:

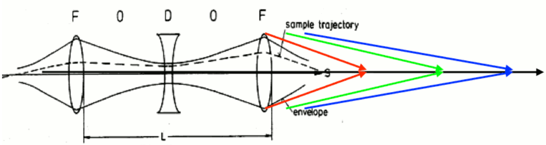
$$\frac{\Delta\beta(s)}{\beta(s)} = \frac{1}{2\sin 2\pi Q} \oint \beta(t)\,\Delta k(t) \cos\left(2\pi Q - 2\left(\mu(t) - \mu(s)\right)\right) \mathrm{d}t$$

affects the element $M_{12}$ of the $M$ matrix.



# Quadrupole errors: chromaticity, $\xi$

Is an error (optical aberration) that happens in quadrupoles when $\Delta p/p \neq 0$:




The chromaticity $\xi$ is the variation of tune $\Delta Q$ with the relative momentum error:

$$\Delta Q = \xi \frac{\Delta p}{p_0} \quad \Rightarrow \quad \xi = \frac{\mathrm{d}\Delta Q}{\mathrm{d}\Delta p/p}$$

Remember the quadrupole strength:

$$k = \frac{g}{p/e} \qquad \text{with } p = p_0 + \Delta p$$

then

$$k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0}\left(1 - \frac{\Delta p}{p_0}\right) g = k_0 + \Delta k$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

# Quadrupole errors: chromaticity (cont.)

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

$\Rightarrow$ Chromaticity acts like a quadrupole error and leads to a *tune spread:*

$$\Delta Q_{\text{one quad}} = -\frac{1}{4\pi}\frac{\Delta p}{p_0} k_0 \beta(s)\, \mathrm{d}s \qquad \Rightarrow \Delta Q_{\text{all quads}} = -\frac{1}{4\pi}\frac{\Delta p}{p_0}\oint k(s)\,\beta(s)\,\mathrm{d}s$$

Therefore the definition of chromaticity $\xi$ is

$$\xi = -\frac{1}{4\pi}\oint_{\text{quads}} k(s)\,\beta(s)\,\mathrm{d}s$$

The peculiarity of chromaticity is that it isn't due to external agents, it is generated by the lattice itself!

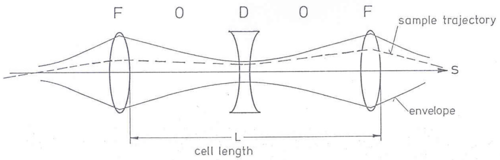
Remarks:

- $\xi$ is a number indicating the size of the tune spot in the working diagram
- $\xi$ is always created by the focusing strength $k$ of **all** quadrupoles

In other words, because of chromaticity the tune is not a point, but it is pancake

# Example: Chromaticity of the FODO cell

Consider a ring composed by $N_{\text{cell}}$ FODO cells like in figure, with two thin quads separated by length $L/2$,



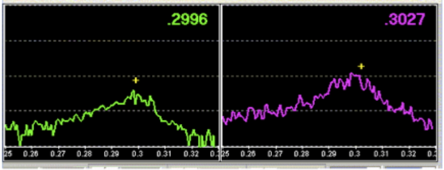

The natural chromaticity $\xi_N$ for the $N_{\text{cell}}$ cells is:

$$\begin{aligned}
\xi_N &= -\frac{1}{4\pi}\oint \beta(s)k(s)ds \\
&= -\frac{1}{4\pi}N_{\text{cell}}\int_{\text{cell}} \beta(s)\underbrace{k(s)d}_{\frac{1}{f}} \\
&= -\frac{1}{4\pi}N_{\text{cell}}\left[\beta^+\left(\frac{1}{f_F}\right) - \beta^-\left(\frac{1}{f_D}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{4\pi\sin\mu}N_{\text{cell}}\left[\left(L+\frac{L^2}{4f_D}\right)\frac{1}{f_F} - \left(L-\frac{L^2}{4f_F}\right)\frac{1}{f_D}\right] \\
&= -\frac{1}{4\pi\sin\mu}N_{\text{cell}}\left[\frac{L}{f_F} - \frac{L}{f_D} + \frac{L^2}{2f_F f_D}\right] \\
&\simeq -\frac{1}{8\pi\sin\mu}N_{\text{cell}}\frac{L^2}{f_F f_D}
\end{aligned}$$

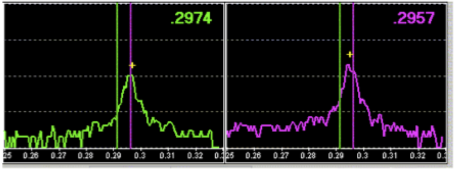
The chromaticity of the ring is the same as the FODO cell.

# Quadrupole errors: chromaticity




*Tune signal for a nearly uncompensated cromaticity ( Q' ≈ 20 )*

*Ideal situation: cromaticity well corrected, ( Q' ≈ 1 )*



# Summary

orbit for an off-momentum particle $x\left(s\right)=x_{\beta}\left(s\right)+D\left(s\right)\frac{\Delta p}{p}$

dispersion trajectory $D\left(s\right)=S\left(s\right)\int_0^s\frac{1}{\rho(t)}C\left(t\right)\mathrm{d}t-C\left(s\right)\int_0^s\frac{1}{\rho(t)}S\left(t\right)\mathrm{d}t$

equations of motion with dispersion

$$\left(\begin{array}{c}x\\x'\\\Delta p/p\end{array}\right)_s=\left(\begin{array}{ccc}C&S&D\\C'&S'&D'\\0&0&1\end{array}\right)\left(\begin{array}{c}x\\x'\\\Delta p/p\end{array}\right)_0$$

definition of momentum compaction $\frac{\Delta C}{C}=\alpha\frac{\Delta p}{p}$

stability condition $m\cdot Q_x+n\cdot Q_y\neq p$ with $n,m,p$ integers

tune shift $\Delta Q=\frac{1}{4\pi}\oint_{\mathrm{quads}}\Delta k\left(s\right)\beta\left(s\right)\mathrm{d}s$

beta beat

$$\frac{\Delta\beta\left(s\right)}{\beta\left(s\right)}=\frac{1}{2\sin 2\pi Q}\cdot\oint\beta\left(t\right)\Delta k\left(t\right)\cos\left(2\pi Q-2\left(\mu\left(t\right)-\mu\left(s\right)\right)\right)\mathrm{d}t$$

chromaticity $\xi=\frac{\mathrm{d}\Delta Q}{\mathrm{d}\Delta p/p}=-\frac{1}{4\pi}\oint_{\mathrm{quads}}k\left(s\right)\beta\left(s\right)\mathrm{d}s$

# References

Derivation of the equation of the orbit distortion for a dipole field errors:

1. Shyh-Yuan Lee: Accelerator Physics, World Scientific, 2004
2. The CERN Accelerator School (CAS) Proceedings