

Self-Similar Solution of the three dimensional compressible Navier-Stokes Equations

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Outline

- **Solutions of PDEs** *self-similar, traveling wave*
- **non-compressible Navier-Stokes equation**
with my 3D Ansatz & geometry *my solution + other solutions, replay from last year*
- **compressible Navier-Stokes equation** *with the same Ansatz, some part of the solutions, traveling wave analysis*
- **Summary & Outlook** *more EOS & viscosity functions*

Physically important solutions of PDEs

- Travelling waves:
arbitrary wave fronts

$$u(x,t) \sim g(x-ct), g(x+ct)$$

- Self-similar

$$u(x,t) = t^{-\alpha} f(x/t^\beta)$$

Sedov, Barenblatt, Zeldovich

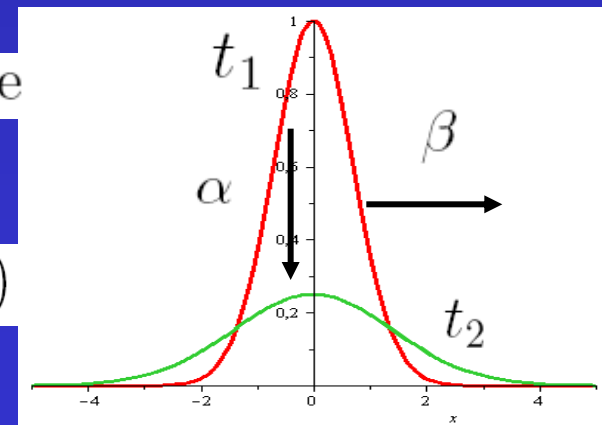
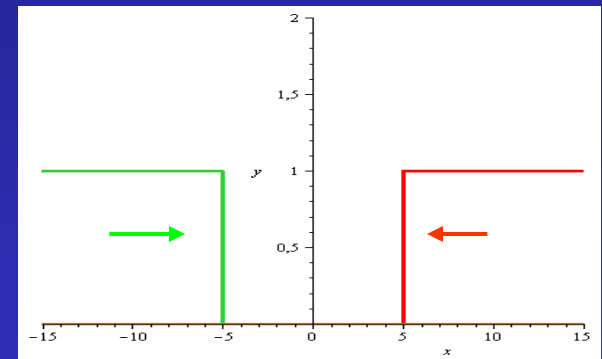
α and β are of primary physical importance

α represents the rate of decay

β is the rate of spread (or contraction if $\beta < 0$)

$$t_1 < t_2$$

in Fourier heat-conduction



The non-compressible Navier-Stokes equation

3 dimensional cartesian coordinates,
Euler description

\mathbf{v} velocity field, p pressure, \mathbf{a} external field

ν kinematic viscosity, ρ constant density
Newtonian fluid

$$\nabla \mathbf{v} = 0,$$

$$\mathbf{v}_t + (\mathbf{v} \nabla) \mathbf{v} = \nu \Delta \mathbf{v} - \frac{\nabla p}{\rho} + \mathbf{a}$$

$$\mathbf{v}(x, y, z, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)) \quad p(x, y, z, t)$$

Consider the most general case

$$u_x + v_y + w_z = 0$$

$$u_t + uu_x + vu_y + wu_z = \nu(u_{xx} + u_{yy} + u_{zz}) - \frac{p_x}{\rho}$$

$$v_t + uv_x + vv_y + wv_z = \nu(v_{xx} + v_{yy} + v_{zz}) - \frac{p_y}{\rho}$$

$$w_t + uw_x + vw_y + ww_z = \nu(w_{xx} + w_{yy} + w_{zz}) - \frac{p_z}{\rho} + a_z$$

just to write out all the coordinates

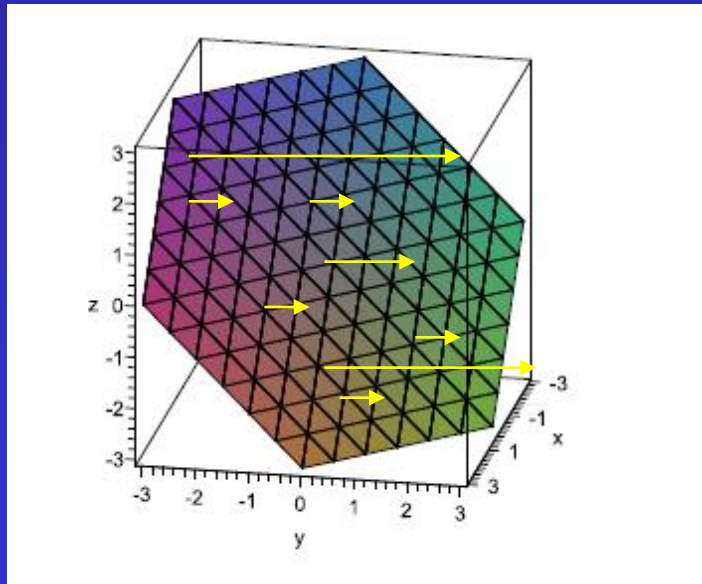
My 3 dimensional Ansatz

$$u(x, t) = t^{-\alpha} f(x/t^\beta)$$



$$u(x, y, z, t) = t^{-\alpha} f\left(\frac{F(x, y, z)}{t^\beta}\right) := t^{-\alpha} f\left(\frac{x + y + z}{t^\beta}\right) := t^{-\alpha} f(\omega)$$

$$F(x, y, z) = x + y + z = 0$$



The graph of the $x + y + z = 0$ plane.

A more general function does not work for N-S

~~$$u(x, y, z, t) = t^{-\alpha} f\left(\frac{\sqrt{x^2 + y^2 + z^2} - a}{t^\beta}\right)$$~~

Geometrical meaning:
all v components with
coordinate constrain $x+y+z=0$
lie in a plane = equivalent

The final applied forms

$$u(x, y, z, t) = t^{-\alpha} f\left(\frac{x + y + z}{t^\beta}\right), \quad v(x, y, z, t) = t^{-\gamma} g\left(\frac{x + y + z}{t^\delta}\right)$$

$$w(x, y, z, t) = t^{-\epsilon} h\left(\frac{x + y + z}{t^\zeta}\right), \quad p(x, y, z, t) = t^{-\eta} l\left(\frac{x + y + z}{t^\theta}\right)$$

The obtained ODE system

$$\begin{aligned}f'(\omega) + g'(\omega) + h'(\omega) &= 0 \\-\frac{1}{2}f(\omega) - \frac{1}{2}\omega f'(\omega) + [f(\omega) + g(\omega) + h(\omega)]f'(\omega) &= 3\nu f''(\omega) - \frac{l'(\omega)}{\rho} \\-\frac{1}{2}g(\omega) - \frac{1}{2}\omega g'(\omega) + [f(\omega) + g(\omega) + h(\omega)]g'(\omega) &= 3\nu g''(\omega) - \frac{l'(\omega)}{\rho} \\-\frac{1}{2}h(\omega) - \frac{1}{2}\omega h'(\omega) + [f(\omega) + g(\omega) + h(\omega)]h'(\omega) &= 3\nu h''(\omega) - \frac{l'(\omega)}{\rho} + a.\end{aligned}$$

as constraints we got for the exponents:

$$\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \theta = 1/2, \quad \eta = 1 \quad \text{universality relations}$$

$$\begin{aligned}u(x, y, z, t) = t^{-1/2} f\left(\frac{x + y + z}{t^{1/2}}\right) &= t^{-1/2} f(\omega), \quad v(x, y, z, t) = t^{-1/2} g(\omega), \\w(x, y, z, t) = t^{-1/2} h(\omega), \quad p(x, y, z, t) &= t^{-1} l(\omega),\end{aligned}$$

Continuity eq. helps us to get an additional constraint:

$$f(\omega) + g(\omega) + h(\omega) = c, \quad \text{and} \quad f''(\omega) + g''(\omega) + h''(\omega) = 0$$

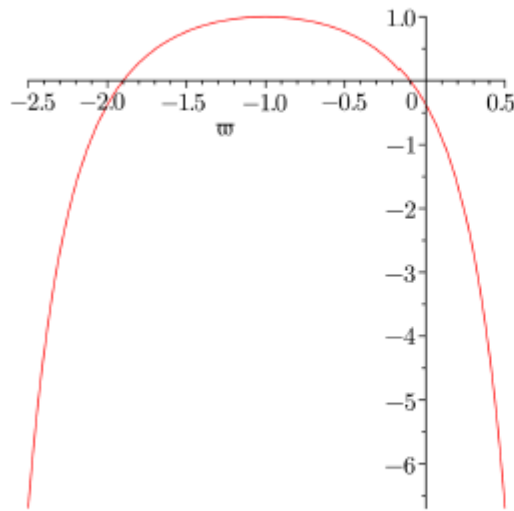
c is prop. to mass flow rate

Solutions of the ODE

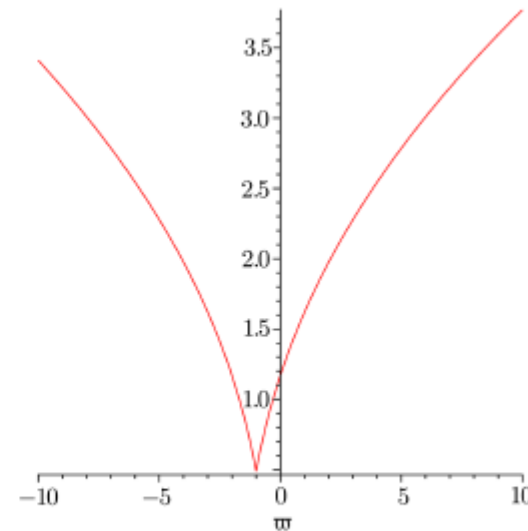
a single Eq. remains

$$9\nu f''(\omega) - 3(\omega + c)f'(\omega) + \frac{3}{2}f(\omega) - \frac{c}{2} + a = 0.$$

$$f(\omega) = c_1 \cdot \text{KummerU} \left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega + c)^2}{6\nu} \right) + c_2 \cdot \text{KummerM} \left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega + c)^2}{6\nu} \right) + \frac{c}{3} - \frac{2a}{3}$$



The $\text{KummerM}(-1/4, 1/2, (\omega + c)^2/6\nu)$ function for $c = 1$ and $\nu = 0.1$.



The $\text{KummerU}(-1/4, 1/2, (\omega + c)^2/6\nu)$ function for $c = 1$ and $\nu = 0.1$.

Kummer is spec.

$(a)_n$ is the Pochhammer symbol

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), (a)_0 = 1$$

$$M(a, b, z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \cdots + \frac{(a)_n z^n}{(b)_n n!},$$

$$U(a, b, z) = \frac{\pi}{\sin(\pi b)} \left[\frac{M(a, b, z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b} \frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)} \right]$$

Solutions of N-S

$$u(x, y, z, t) = t^{-1/2} f(\omega) = t^{-1/2} \left[c_1 \cdot \text{KummerU} \left(\frac{-1}{4}, \frac{1}{2}, \frac{((x+y+z)/t^{1/2} + c)^2}{6\nu} \right) \right] \\ + t^{-1/2} \left[c_2 \cdot \text{KummerM} \left(-\frac{1}{4}, \frac{1}{2}, \frac{((x+y+z)/t^{1/2} + c)^2}{6\nu} \right) + \frac{c}{3} - \frac{2a}{3} \right]$$

analytic only for one velocity component ☹️

Geometrical explanation:

all v components with coordinate constrain $x+y+z=0$ lie in a plane = equivalent

Naver-Stokes makes a dynamics of this plane

getting a multi-valued surface

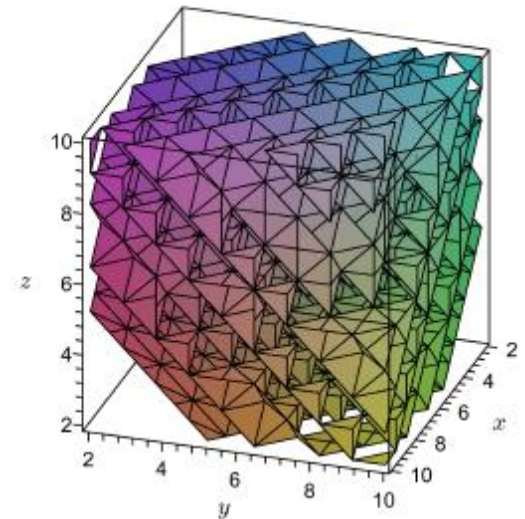
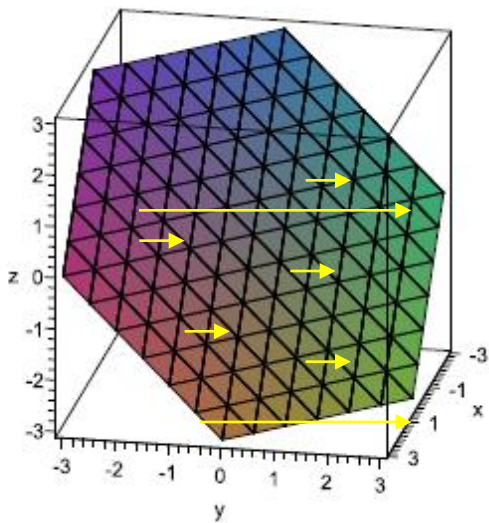


Fig. 5 The implicit plot of the self-similar solution Eq. (17). Only the KummerU function is presented for $t = 1$, $c_1 = 1$, $c_2 = 0$, $a = 0$, $c = 1$, and $\nu = 0.1$.

*I.F. Barna <http://arxiv.org/abs/1102.5504>
Commun. Theor. Phys. 56 (2011) 745-750*

for fixed space it decays in time $t^{-1/2}$ KummerT or U(1/t) ☹️

Other analytic solutions

Without completeness, usually from Lie algebra studies
all are for non-compressible N-S

W. I. Fushchich, W. M. Shtelen and S. L. Slavutsky J. Phys. A: Math. Gen. 24 (1990) 971.

$$\omega = z/\sqrt{t}$$

Presented 19 various solutions
one of them is:

$$u(z, t) = \frac{f(\omega)}{\sqrt{t}}, \quad v(y, z) = \frac{g(\omega)}{\sqrt{t}} + \frac{y}{t}, \quad w(z, t) = \frac{h(\omega)}{\sqrt{t}}, \quad p(t, z) = \frac{l(\omega)}{\sqrt{t}}$$

V. Grassi, R.A. Leo, G. Soliani and P. Tempesta, Physica 286 (2000) 79

Ansatz:

$$U_1 = Y(y)T(z)\Phi(t).$$

JOURNAL OF MATHEMATICAL PHYSICS 50, 083101 (2009)

Analytical solutions to the Navier–Stokes equations
with density-dependent viscosity and with pressure

Ling Hei Yeung^{1,a)} and Yuen Manwai^{2,b)}

Solutions are Kummer functions as well

“Only” Radial solution
for 2 or 3 D

Ansatz:

$$p(t, r) = \frac{f(r/a(t))}{a(t)^N}, \quad u(t, r) = \frac{\dot{a}(t)}{a(t)} r,$$

Nonlinear Instability of the Solutions of the Navier–Stokes
Equations: Formulas for Constructing Exact Solutions

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Received May 5, 2009

Ansatz:

$$V_n = f_n(z, t)x + g_n(z, t)y, \quad n = 1, 2; \quad V_3 = F(z, t).$$

Ukrainian Mathematical Journal, Vol. 49, No. 9, 1997

ON NAVIER–STOKES FIELDS WITH LINEAR VORTICITY

G. V. Popovich and R. O. Popovich

$$\text{rot } \vec{u} = H(t)\vec{x} + \vec{k}(t).$$

$$\vec{u} = \nabla\varphi + \frac{1}{3}(H\vec{x}) \times \vec{x} + \frac{1}{2}\vec{k} \times \vec{x},$$

Sedov, stationary N-S,
only the angular part

$$v_r = \frac{\nu}{r} f(\theta), \quad v_\theta = \frac{\nu}{r} \varphi(\theta), \quad v_\lambda = \frac{\nu}{r} \psi(\theta), \quad U - \frac{p}{\rho} = \frac{\nu^2}{r^2} F(\theta).$$

The compressible Navier-Stokes eq.

$$\rho_t + \text{div}(\rho \mathbf{v}) = 0$$

$$\rho[\mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v}] = \nu_1 \Delta \mathbf{v} + \frac{\nu_2}{3} \text{grad div } \mathbf{v} - \nabla p + \mathbf{a}; \quad \text{EOS } p = \kappa \rho^n$$

3 dimensional cartesian coordinates,
Euler description, Newtonian fluid, politropic EOS (these can be changed later)
 \mathbf{v} velocity field, p pressure, \mathbf{a} external field

ν_1, ν_2 viscosities, ρ density (No temperature at this point)

Consider the most
general case:

$$\mathbf{v}(x, y, z, t) = u(x, y, z, t), v(x, y, z, t), w(x, y, z, t) \quad \rho(x, y, z, t)$$

just write out
all the coordinates:

$$\begin{aligned} \rho_t + \rho_x u + \rho_y v + \rho_z w + \rho(u_x + v_y + w_z) &= 0 \\ \rho[u_t + uu_x + vu_y + wu_z] - \nu_1(u_{xx} + u_{yy} + u_{zz}) - \frac{\nu_2}{3}(u_{xx} + v_{xy} + w_{xz}) + \kappa n \rho^{n-1} \rho_x &= 0 \\ \rho[v_t + uv_x + vv_y + wv_z] - \nu_1(v_{xx} + v_{yy} + v_{zz}) - \frac{\nu_2}{3}(u_{xy} + v_{yy} + w_{yz}) + \kappa n \rho^{n-1} \rho_y &= 0 \\ \rho[w_t + uw_x + vw_y + ww_z] - \nu_1(w_{xx} + w_{yy} + w_{zz}) - \frac{\nu_2}{3}(u_{xz} + v_{yz} + w_{zz}) + \kappa n \rho^{n-1} \rho_z &= 0. \end{aligned}$$

The applied Ansatz & Universality Relations

A more general function does not work for N-S

$$\rho(x, y, z, t) = t^{-\alpha} f\left(\frac{x + y + z}{t^\beta}\right) = t^{-\alpha} f(\eta) \quad u(x, y, z, t) = t^{-\delta} g(\eta),$$
$$v(x, y, z, t) = t^{-\epsilon} h(\eta), \quad w(x, y, z, t) = t^{-\omega} i(\eta).$$

Where all the exponents $\alpha, \beta, \delta, \epsilon, \omega$ are real numbers.

as constraints we got for the exponents:
universality relations

$$\alpha = \beta = \frac{2}{n+1} \quad \& \quad \delta = \epsilon = \omega = 2 - \frac{4}{n+1}.$$

Note, that n remains free, presenting some physics in the system, polytropic EOS

$$p = \kappa \rho^n$$

The obtained ODE system

The most general case, n is free

Continuity can be integrated

$$\alpha[f + f'\eta] = f'[g + h + i] + f[g' + h' + i'] \longrightarrow$$

$$\alpha f \eta = f[g + h + i] + c_0$$

$$f[-\delta g - \alpha \eta g' + gg' + hg' + ig'] = -\kappa n f^{n-1} f' + 3\nu_1 g'' + \frac{\nu_2}{3}[g'' + h'' + i'']$$

$$f[-\delta h - \alpha \eta h' + gh' + hh' + ih'] = -\kappa n f^{n-1} f' + 3\nu_1 h'' + \frac{\nu_2}{3}[g'' + h'' + i'']$$

$$f[-\delta i - \alpha \eta i' + gi' + hi' + ii'] = -\kappa n f^{n-1} f' + 3\nu_1 i'' + \frac{\nu_2}{3}[g'' + h'' + i'']$$

if $\delta = 0$

N-S can be intergated once, after some algebra getting an ODE of:

$$4\nu c_0 f' + 3\kappa f^3 + f^2[-c_0 \eta - 4\nu + c_4] + c_0^2 f = 0$$

This is for the density

where $\nu = \nu_1 = \nu_2$ and c_4 is a new constant

No analytic solutions exist , but the direction field can be investigated for reasonable parameters

The properties of the solutions

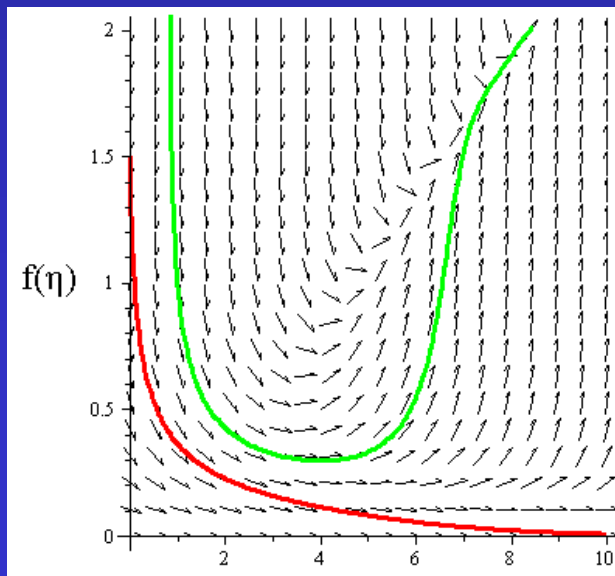
From the universality relations the global properties of the solutions are known

$$\delta = 0 \rightarrow \alpha = \beta = 1, n = 1, \rho = t^{-1} f\left(\frac{x+y+z}{t}\right), u = g\left(\frac{x+y+z}{t}\right), p = \kappa\rho$$

has decay & spreading

just spreading in time

for fixed space



$\delta \neq 0$ is under recent investigation

$v := 0.4; c_0 := 1; c_4 := 1; k := 1.4;$

General properties of the solutions for other exponents

$$\rho(x, y, z, t) = t^{-\alpha} f\left(\frac{x+y+z}{t^\beta}\right) = t^{-\alpha} f(\eta) \quad u(x, y, z, t) = t^{-\delta} g(\eta)$$

$$\alpha = \beta = \frac{2}{n+1} \quad \& \quad \delta = \epsilon = \omega = 2 - \frac{4}{n+1}$$

There are different regimes for different n s

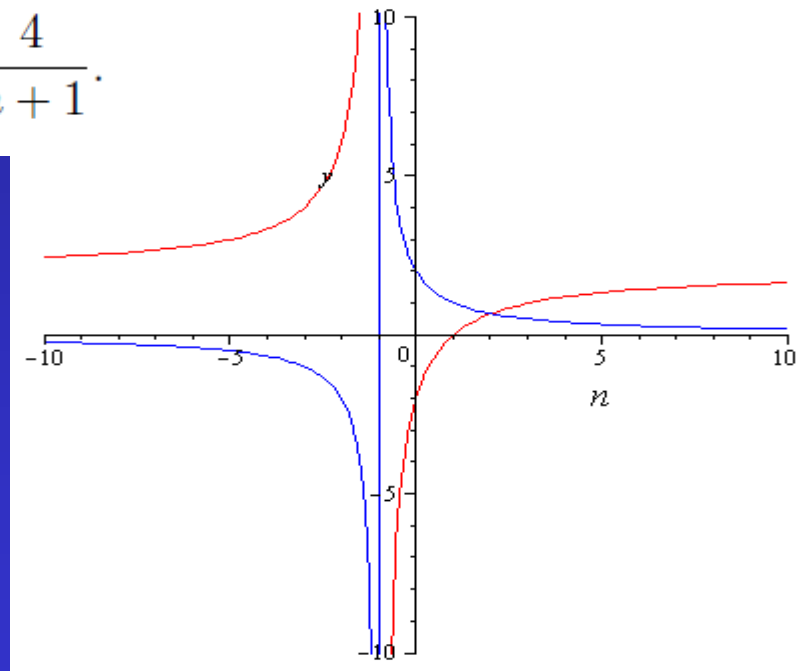
$n > 1$ all exponents are positive decaying, spreading solutions for speed and density

$n = 1$ see above

$-1 \leq n \leq +1$ decaying and spreading density & enhancing velocity in time

$n \neq -1$

$n \leq -1$ sharpening and enhancing density & decaying and sharpening velocity

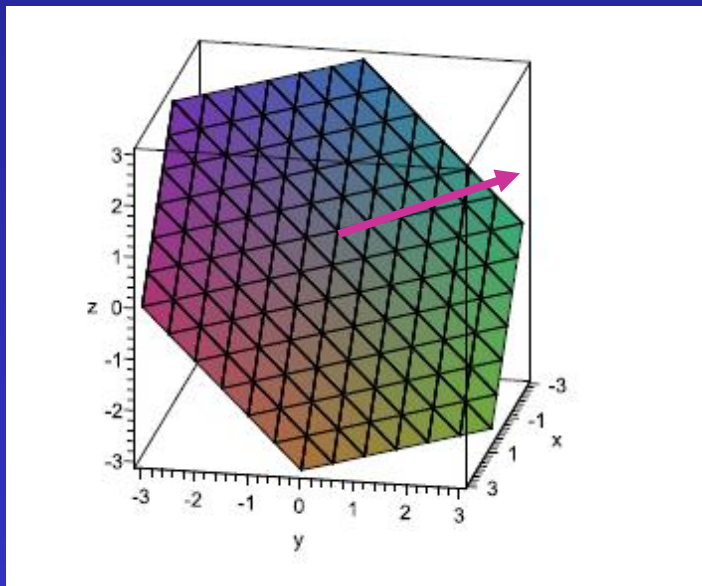


Relevant physics is for $n > 1$ the analysis is in progress to see the shape functions

Traveling wave solutions

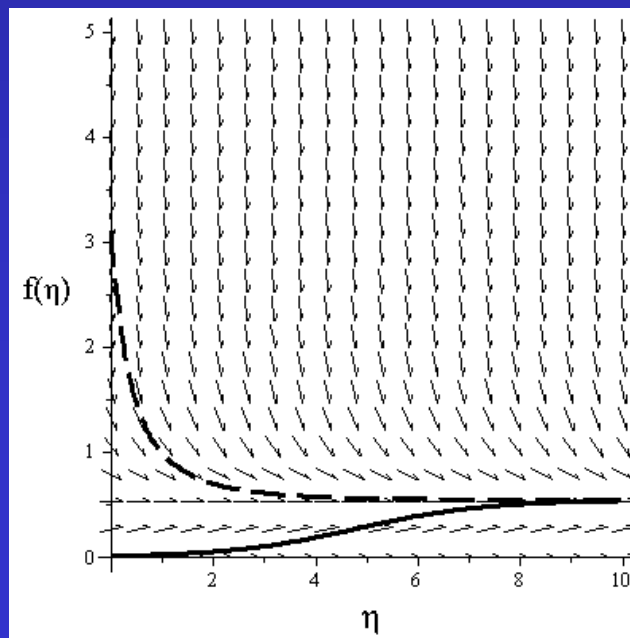
As a second Ansatz we may try to find traveling wave solutions of the NS system.

$$\rho(x, y, z, t) = f(x + y + z - Ct) = f(\eta), \quad u = g(\eta), \quad v = h(\eta), \quad w = i(\eta) \quad \text{where } C \text{ is the wave velocity}$$



After some algebra the next ODE can be obtained: (for $n = 1$)

$$-4\nu c_0 f' + 3\kappa f^3 + f^2[c_4 - c_0 c] + c_0^2 f = 0$$



Detailed analysis is in progress

Summary & Outlook

- *The self-similar Ansatz is presented as a tool for non-linear PDA*
- *The non-compressible & compressible N-S eq. is investigated and the results are discussed*
- *An in-depth analysis is in progress for further EOS, more general viscosity functions could be analysed like the Ostwald-de Waele power law*

$$\tau_{yx} = K \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y}$$

K : consistency index for non-Newtonian viscosity

$$\left(\gamma = \frac{K}{\rho} \right)$$

$n > 0$: power-law index $0 < n < 1$ pseudoplastic fluids or shear-thinning fluids
 $n = 1$ Newtonian fluid
 $n > 1$ dilatant or shear-thickening fluids-

- *To investigate some relativistic cases, which may attracts the interest of the recent community*

Thank you for



your attention!

Questions, Remarks, Comments?...