

QCD Thermodynamics With Continuum Extrapolated Wilson Fermions

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Outline

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 - Fermion Fields
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Quantum Chromodynamics

Theory of Strong Nuclear Interaction

- SU(3) gauge theory
- *Asymptotic Freedom*
 - Coupling becomes small in short distance or large momentum transfer processes.
 - Perturbation theory is successful.
- *Confinement*
 - Coupling becomes large in the low energy regime.
 - **Perturbation theory does not work.**
- Non-perturbative methods are necessary:
 - Effective model calculations
 - Lattice field theory

Path Integral in Euclidian Space-time

- Partition function:

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S[A_\mu, \bar{\psi}, \psi]}$$

- Action:

$$S = \int d^4x \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi} M \psi \right)$$

- An observable:

$$\langle \mathcal{O}(A_\mu, \bar{\psi}, \psi) \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{O}(A_\mu, \bar{\psi}, \psi) e^{-S[A_\mu, \bar{\psi}, \psi]}$$

- We have to regularize these integrals.

Discretization of Space-time

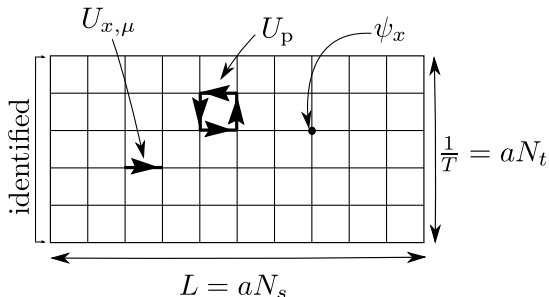
- 4D isotropic hypercubic grid:

$$\Lambda = a\mathbb{Z}^4 = \{x \mid x_\mu/a \in \mathbb{Z}\}$$

- lattice spacing: a
- lattice size: $N_s^3 \times N_t$
- $N_t \gg N_s$: zero temperature, $N_t < N_s$: finite temperature

Physical
Temperature

$$T = \frac{1}{aN_t}$$



Gauge Fields on a Lattice

SU(3) Wilson Gauge Action

The *plaquette* variable is the only relevant operator:

$$U_p = U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu}^\dagger U_{x,\nu}^\dagger$$

The action:

$$\begin{aligned} S_g^W[U] &= -\beta \sum_p \left\{ \frac{2}{2\text{Tr}\mathbf{1}} \text{Tr} \left(U_p + U_p^\dagger \right) - 1 \right\} \\ &= \beta \sum_p \left(1 - \frac{1}{3} \text{Re Tr } U_p \right), \quad \beta = \frac{6}{g^2} \\ \lim_{a \rightarrow 0} S_g^W &= \int d^4x F_{\mu\nu} F_{\mu\nu} + \mathcal{O}(a^5) \end{aligned}$$

Fermion Fields on a Lattice

Naive Discretization

- $S_f^{\text{cont}} = \int d^4x \bar{\psi} (\not{D} + m) \psi$
- $\bar{\psi}, \psi$ Grassmann variables
- $\partial_\mu \longrightarrow \frac{1}{2} (\Delta_\mu^f + \Delta_\mu^b)$ finite differences
- Naive fermion action:

$$S_f^N = \sum_{x,y \in \Lambda} \bar{\psi}_x \underbrace{(D_{xy} + am\delta_{xy})}_{M_{xy}} \psi_y$$

$$D_{xy} = \frac{1}{2} \sum_{\mu=1}^4 \gamma_\mu \left(U_{x,\mu} \delta_{x+\hat{\mu},y} - U_{x-\hat{\mu},\mu}^\dagger \delta_{x-\hat{\mu},y} \right)$$

The Doubling Problem

Free Propagator ($U = 1$)

$$\tilde{G}^{-1}(p) = m + \frac{i}{a} \sum_{\mu} \gamma_{\mu} \sin p_{\mu} a$$

It has 16 poles in the Brillouin cell in the limit $a \rightarrow 0$.

Nielsen–Ninomiya no-go theorem

There exists no D lattice Dirac operator which has

- ① **correct continuum limit** ($\tilde{D}(p) = i\not{p} + \mathcal{O}(ap^2)$, $p \ll \pi/2$)
- ② **locality** ($\|D_{xy}\| \leq c e^{-\lambda|x-y|}$, $c, \lambda > 0$)
- ③ **chiral symmetry** ($\{D, \gamma_5\} = 0$)
- ④ **no fermion doublers** ($\nexists \tilde{D}^{-1}(p)$, $p \neq 0$)

simultaneously.

Solutions for the Doubling Problem

Drop one condition of the Nielsen–Ninomiya theorem

- **Wilson discretization:** It breaks chiral symmetry which restores in the continuum limit.
- **staggered discretization:** Possibly breaks locality.
- **chiral fermions:**

$$\gamma_5 D + D \gamma_5 = a D \gamma_5 D$$

(Ginsparg–Wilson relation)

Computational Costs

staggered fermions $\ll_{\times 10}$ Wilson fermions $\ll_{\times 100}$ chiral fermions

Wilson Fermions

- idea: $\bar{\psi} (\not{D} + m) \psi \longrightarrow \bar{\psi} (\not{D} + a\nabla^2 + m) \psi$
- Wilson action:

$$S_f^W = \sum_{x \in \Lambda} \left\{ \bar{\psi}_x \psi_x - \kappa \sum_{\mu=\pm 1}^{\pm 4} [\bar{\psi}_{x+\hat{\mu}} (1 + \gamma_\mu) U_{x,\mu} \psi_x] \right\},$$

$$\kappa = 1/(2am + 8)$$

Free Wilson Propagator ($U = 1$)

$$\tilde{G}^{-1}(p) = \frac{1}{a} \left(1 - 2\kappa \sum_{\mu} \cos p_{\mu} a + 2i\kappa \sum_{\mu} \gamma_{\mu} \sin p_{\mu} a \right)$$

15 doubler masses $\rightarrow \infty$ in the limit $a \rightarrow 0$.

Staggered Fermions

- idea: “Half the Brillouin cell and double the lattice spacing.”
- staggered action:

$$S_f^{\text{stagg}} = \sum_{x \in \Lambda} \left\{ am \bar{\psi}_x \psi_x + \frac{1}{2} \sum_{\mu=1}^4 \alpha_{x,\mu} \left[\bar{\psi}_x U_{x,\mu}^\dagger \psi_{x+\hat{\mu}} - \bar{\psi}_{x+\hat{\mu}} U_{x,\mu} \psi_x \right] \right\}$$

- We still have 4 doublers.
- Partition function of N_f flavour, **locality!**

$$Z = \int \prod_{x,\mu} dU_{x,\mu} (\det M)^{N_f/4} e^{-S_g}$$

- Taste symmetry violation at finite a .

Comparison of staggered and Wilson discretization

	Wilson	staggered
discretization effects	$\mathcal{O}(a)$	$\mathcal{O}(a^2)$
chiral symmetry	\emptyset	U(1)
$m \rightarrow -m$	not a symmetry	symmetry
quark mass renormalization	both additive and multiplicative	multiplicative
number of fermions	1	4

Simulation Points and Techniques

The action

- **Gauge sector:** Symanzik tree-level improved gauge action ^a
- **Fermion sector:** 6 step stout smeared $\mathcal{O}(a)$ clover improved Wilson fermion action ^b

^aK. Symanzik, Nucl. Phys. B **226** (1983) 187

M. Lüscher and P. Weisz, Commun. Math. Phys. **97**, 59 (1985) [Erratum-ibid. **98**, 433 (1985)]

^bB. Sheikholeslami and R. Wohlert, Nucl. Phys. B **259** (1985) 572

C. Morningstar and M. J. Peardon, Phys. Rev. D **69** (2004) 054501 [[arXiv:hep-lat/0311018](https://arxiv.org/abs/hep-lat/0311018)]

R. Hoffmann, A. Hasenfratz and S. Schaefer, PoS **LAT2007** (2007) 104 [[arXiv:0710.0471](https://arxiv.org/abs/0710.0471)] [[hep-lat](#)]]

Dürr *et al.*, Science **322** (2008) 1224-1227, [arXiv:0906.3599](https://arxiv.org/abs/0906.3599)

Algorithms for $N_f = 2 + 1$ flavour simulations

- $m_u = m_d \equiv m_{ud}$ **light quarks:** Hybrid Monte Carlo ^a
- m_s **heavy quark:** Rational Hybrid Monte Carlo ^b

^aS. Duane, A. D. Kennedy, B. J. Pendleton and D. Roweth, Phys. Lett. B **195** (1987) 216

^bM. A. Clark and A. D. Kennedy, Phys. Rev. Lett. **98** (2007) 051601 [[arXiv:hep-lat/0608015](https://arxiv.org/abs/hep-lat/0608015)]

Simulation Points

- 4 different lattice spacings
- Bare masses were tuned to $m_\pi/m_\Omega = 0.326(4)$,
 $m_K/m_\Omega = 0.366(4)$
- $m_\pi \approx 545$ MeV and $m_K \approx 614$ MeV
- Lattice sizes: $m_\pi L \gtrsim 8$
- The temperature was controlled with N_t .
- Comparison with staggered results at same m_π and m_K .

Simulation Parameters and Lattice Sizes

a [fm]	β	am_{ud}	am_s	N_s	N_t
0.139(1)	3.30	-0.0985	-0.0710	32	4 - 16, 32
0.093(1)	3.57	-0.0260	-0.0115	32	4 - 16, 64
0.070(1)	3.70	-0.0111	-0.0000	48	8 - 28, 48
0.057(1)	3.85	-0.00336	-0.0050	64	12 - 28, 64

Chiral Condensate of the Light Quarks

Definition

$$\langle \bar{\psi}\psi \rangle = \frac{T}{V} \frac{\partial \ln Z}{\partial m_{ud}}$$

Properties

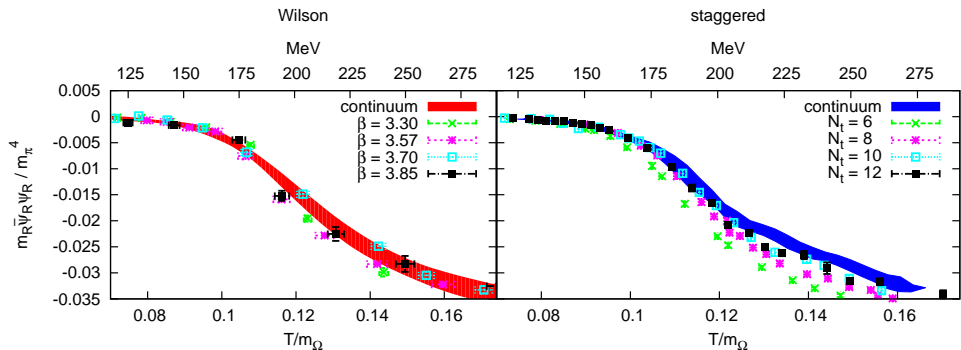
- This is the order parameter of the chiral phase transition in the chiral limit.
- There are both additive (power-like) and multiplicative (logarithmic) divergences need to be removed.

Renormalization Group Invariant Quantities

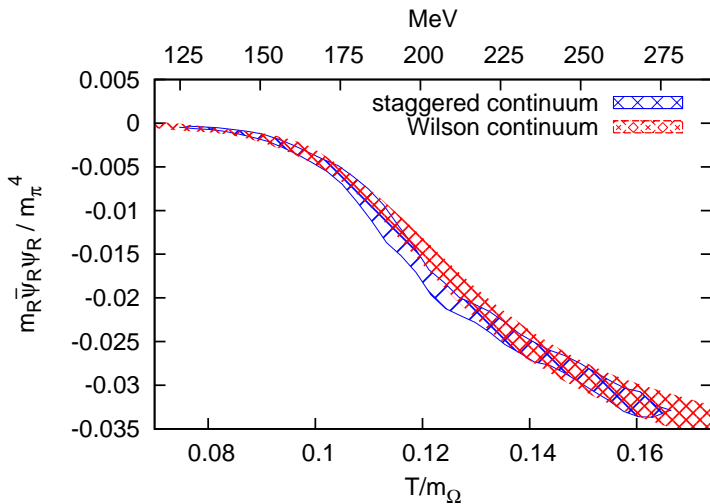
$$m_R \langle \bar{\psi}\psi \rangle_R(T) = 2N_f m_{\text{PCAC}}^2 Z_A^2 \Delta_{PP}(T)$$

$$m_R \langle \bar{\psi}\psi \rangle_R(T) = m_{\text{PCAC}} Z_A \Delta_{\bar{\psi}\psi}(T)$$

Renormalized Chiral Condensate



Continuum Limit of the Chiral Condensate



Strange Quark Number Susceptibility

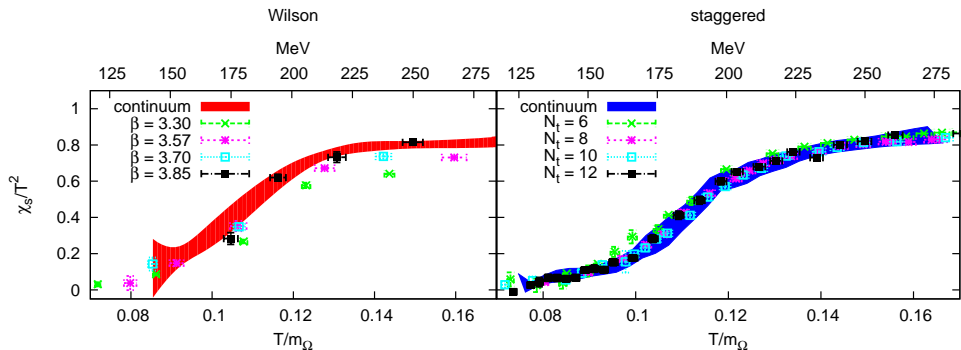
Definition

$$\frac{\chi_s}{T^2} \equiv \frac{1}{T^2} \frac{\partial n_s}{\partial \mu_s} = \frac{1}{TV} \frac{\partial^2 \ln Z}{\partial \mu_s^2} \Big|_{\mu_s=0}$$

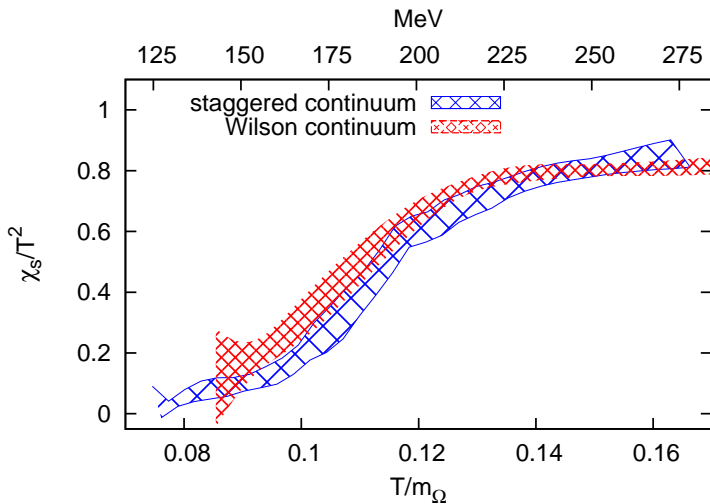
Properties

- It measures strangeness fluctuations.
- There is no need for renormalization, the continuum limit is straightforward.

Strange Quark Number Susceptibility



Continuum Limit of the Quark Number Susceptibility



Polyakov Loop

Definition

$$L = \frac{1}{N_s^3} \sum_{x|x_4=0} \text{Tr} \left(U_{x,4} U_{x+\hat{4},4} \cdots U_{x+(N_t-1)\hat{4},4} \right)$$

Properties

- Pure gauge theory has a Z_3 symmetry which spontaneously breaks at high temperature. The Polyakov loop is the order parameter of the corresponding phase transition.
- Connection with the free energy of a quark-antiquark pair and with confinement:

$$|\langle L \rangle|^2 \propto e^{-\Delta F_{q\bar{q}}(r \rightarrow \infty)/T}$$

Polyakov Loop

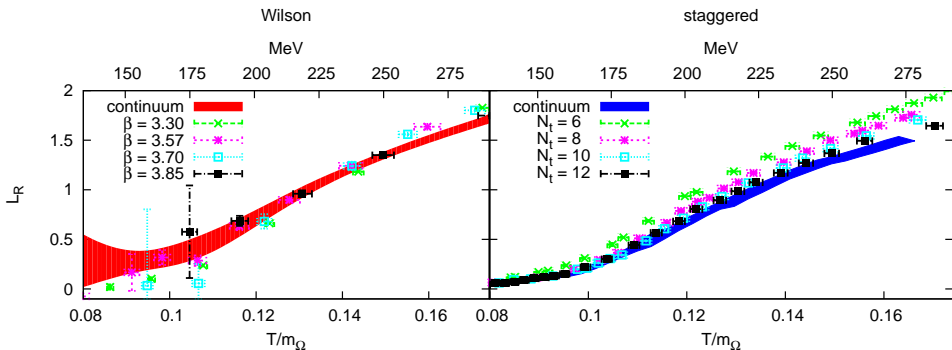
Renormalization

- Additive divergences in $\Delta F_{q\bar{q}} \implies$ multiplicative divergences in L .
- Demand a fix L_* value for fixed $T_* > T_c$:

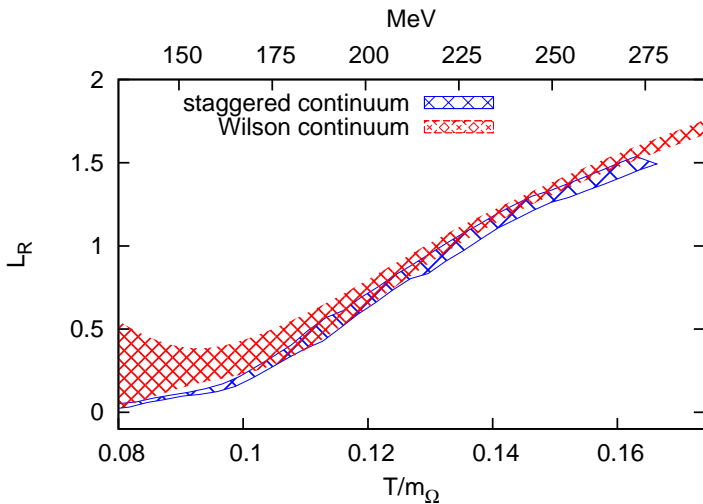
$$L_R(T) = \left(\frac{L_*}{L_0(T_*)} \right)^{\frac{T_*}{T}} L_0(T)$$

We chose $T_* = 0.143m_\Omega$ and $L_* = 1.2$.

Renormalized Polyakov Loop



Continuum Limit of the Polyakov Loop



Summary

- Continuum extrapolation of results with Wilson fermions are possible.
- There is an agreement with the corresponding staggered results ($\approx 1\sigma$).
- Staggered and Wilson results were also tested with chiral fermions ².

- Outlook
 - Lowering the light quark mass towards their physical value.

²Phys. Lett. **B713** (2012) 342-346 arXiv:1204.4089 [hep-lat]