Thermodynamics of the $O(N)$ model using Φ-derivable approximations

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- Motivation
- The effective potential of the model at 2-loop level: gap and field equations
- Renormalization of the model and the emergence of a “hybrid” approximation
- Numerical method, Fourier techniques for convolution integrals
- Parametrization of the $O(N)$ model
- Results: description of the phase transition, thermodynamic quantities ...
- Conclusions and outlook

Collaborators: Gergely Markó (PhD student at Eötvös University) and Urko Reinosa (École Polytechnique)

Motivation

Not all the resummation schemes can capture the order of the thermal phase transition of a scalar model.

How good is a systematically improvable resummation scheme, such as the Φ-derivable ($\equiv$ two-particle-irreducible (2PI) $\equiv$ Cornwall-Jackiw-Tomboulis (CJT)) formalism, for the description of equilibrium properties ? Up to which order of the approximation do we have to go to obtain acceptable critical exponents?

1. Study the phase transition beyond Hartree level. [Done already for $N = 1$ in Minkowski space Arrizabalaga & Reinosa NPA785 (2007) 234, proceedings of SEWM2006.]

2. Work out explicitly at finite temperature the renormalization of the self-consistent equation propagator, field equation and effective potential. [Done, appeared on the poster of SEWM2006, but not published]

3. Using Fourier techniques as in Borsányi & Reinosa PRD 80 (2009) 125029 obtain (if possible) a more accurate Euclidean solution of the equations than those in NPA785 (2007) 234.

Tasks 1, 2, 3 completed for $N = 1$ in G. Markó, U. Reinosa, Zs. Szép, PRD 86, 085031

What about the physically more interesting $N = 4$ case?

2PI formalism

A bilocal source is introduced in the generating functional

$$e^{W[J,K]} = \int \mathcal{D}\varphi \exp\left[-S_0(\varphi) - S_{\text{int}}(\varphi) + \int_x \varphi(x)J(x) + \frac{1}{2}\int_x\int_y \varphi(x)K(x,y)\varphi(y)\right]$$

The 2PI effective action defined through a double Legendre transform

$$\Gamma[\phi, G] = W[J,K] - \int d^4x \underbrace{\frac{\delta W[J,K]}{\delta J(x)}}_{\phi(x)} J(x) - \int d^4x \int d^4y \underbrace{\frac{\delta W[J,K]}{\delta K(x,y)}}_{[\phi(x)\phi(y)+G(x,y)]/2} K(x,y)$$

At vanishing sources ($J, K \to 0$) the physical $\bar{\phi}(x)$ and $\bar{G}(x,y)$ are determined from stationarity conditions

$$\left.\frac{\delta\Gamma[\phi,G]}{\delta\phi(x)}\right|_{\bar{\phi}(x)} = 0, \qquad \left.\frac{\delta\Gamma[\phi,G]}{\delta G(x,y)}\right|_{\bar{G}(x,y)} = 0$$

$\Gamma[\phi, G]$ can be rewritten as (see J. Cornwall et al., PRD 10, 2428)

$$\Gamma[\phi, G] = S_0(\phi) + \frac{1}{2}\mathrm{Tr}\log G^{-1} + \frac{1}{2}\mathrm{Tr}\left[G_0^{-1}G - 1\right] + \Gamma_{\mathrm{int}}[\phi, G]$$

S_0 is the free action
G_0 is the free propagator,
$\Gamma_{\mathrm{int}}[\phi, G]$ contains all the two-particle-irreducible graphs (graphs which do not split apart when two propagators are cut) constructed with vertices taken from $S_{\mathrm{int}}(\phi + \varphi)$

The 1PI effective action is recovered: $\Gamma_{1\mathrm{PI}}[\phi] = \Gamma[\phi, \bar{G}]$.

The $O(N)$ model

Effective potential in the two-loop approximation: (homogeneous ϕ_a)

$$\gamma[\phi, G] = \frac{1}{2} m_2^2 \, \phi_a \phi_a + \frac{1}{4!} \hat{\lambda}_{abcd} \, \phi_a \phi_b \phi_c \phi_d + \frac{1}{2} \int_Q^T \left[\ln G^{-1}(Q) + (Q^2 + m_0^2) G(Q) - 1 \right]_{aa}$$

$$+ \frac{1}{4} \lambda_{ab,cd} \, \phi_a \phi_b \int_Q^T G_{cd}(Q) + \frac{1}{8} \bar{\lambda}_{ab,cd} \left(\int_P^T G_{ab}(P) \right) \left(\int_Q^T G_{cd}(Q) \right)$$

$$- \frac{\lambda_\star^2}{36 N^2} \phi_a \phi_b \int_Q^T \int_K^T G_{ab}(Q) G_{cd}(K) G_{cd}(-K-Q)$$

$$- \frac{\lambda_\star^2}{18 N^2} \phi_a \phi_b \int_Q^T \int_K^T G_{ad}(Q) G_{cd}(K) G_{cb}(-K-Q)$$

where $m_0^2 = m_\star^2 + \delta m_0^2, \quad \cdots \quad \lambda_2^{(A)} = \lambda_\star + \delta \lambda_2^{(A)}, \quad \cdots \quad \lambda_4 = \lambda_\star + \delta \lambda_4$

$$\hat{\lambda}_{abcd} \equiv \frac{\lambda_4}{3N} t_{123}^{abcd}, \quad \lambda_{ab,cd} \equiv \frac{1}{3N} \left(\lambda_2^{(A)} t_1^{abcd} + \lambda_2^{(B)} t_{23}^{abcd} \right), \quad \bar{\lambda}_{ab,cd} \equiv \frac{1}{3N} \left(\lambda_0^{(A)} t_1^{abcd} + \lambda_0^{(B)} t_{23}^{abcd} \right)$$

$$t_1^{abcd} = \delta_{ab} \delta_{cd}, \quad t_{23}^{abcd} = \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}, \quad t_{123}^{abcd} = \delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}$$

Gap equations from $\left.\frac{\delta\gamma}{\delta G_{ab}}\right|_{\bar{G}} = 0$ upon projection: $\quad (\bar{G}(K) = [K^2 + \bar{M}^2(K)]^{-1})$

$$\bar{M}_L^2(K) = m_0^2 + \frac{\lambda_2^{(A)} + 2\lambda_2^{(B)}}{6N}\phi^2 + \frac{\lambda_0^{(A)} + 2\lambda_0^{(B)}}{6N}\mathcal{T}[\bar{G}_L] + \frac{(N-1)\lambda_0^{(A)}}{6N}\mathcal{T}[\bar{G}_T]$$
$$- \frac{\lambda_\star^2}{18N^2}\phi^2\left[9\mathcal{B}[\bar{G}_L](K) + (N-1)\mathcal{B}[\bar{G}_T](K)\right]$$

$$\bar{M}_T^2(K) = m_0^2 + \frac{\lambda_2^{(A)}}{6N}\phi^2 + \frac{\lambda_0^{(A)}}{6N}\mathcal{T}[\bar{G}_L] + \frac{(N-1)\lambda_0^{(A)} + 2\lambda_0^{(B)}}{6N}\mathcal{T}[\bar{G}_T]$$
$$- \frac{\lambda_\star^2}{9N^2}\phi^2\mathcal{B}[\bar{G}_L, G_T](K)$$

Field equation $\left.\frac{\delta\gamma}{\delta\phi}\right|_{\bar{\phi}} = 0$

$$0 = \phi_a\Bigg(m_2^2 + \frac{\lambda_4}{6N}\phi^2 + \frac{\lambda_2^{(A)} + 2\lambda_2^{(B)}}{6N}\mathcal{T}[\bar{G}_L] + (N-1)\frac{\lambda_2^{(A)}}{6N}\mathcal{T}[\bar{G}_T]$$
$$-\frac{\lambda_\star^2}{18N^2}\left[3\mathcal{S}[\bar{G}_L] + (N-1)\mathcal{S}[\bar{G}_L, \bar{G}_T]\right]\Bigg)$$

Curvature tensor

Defining $\gamma(\phi^2) := \gamma[\phi, \bar{G}_L, \bar{G}_T]$ (valid for $h = 0$) one derives

$$\begin{aligned}
\hat{M}_{ij} &= \left.\frac{d^2\gamma(\phi)}{d\phi_i d\phi_j}\right|_{\phi=\bar{\phi}} = \frac{d}{d\phi_j}\left(2\gamma'(\phi^2)\phi_i\right)\Big|_{\phi=\bar{\phi}} \\
&= 4\gamma''(\bar{\phi}^2)\bar{\phi}_i\bar{\phi}_j + 2\gamma'(\bar{\phi}^2)\delta_{ij}
\end{aligned}$$

Using $P^L_{ij} = \phi_i\phi_j/\phi^2$ and $P^T_{ij} = \delta_{ij} - P^L_{ij}$, one obtains the two eigenmodes:

$$\hat{M}^2_L = 4\bar{\phi}^2\gamma''(\bar{\phi}^2) + 2\gamma'(\bar{\phi}^2), \qquad \hat{M}^2_T = 2\gamma'(\bar{\phi}^2)$$

Corroborated with the field equation $\bar{\phi}_i\gamma'(\bar{\phi}^2) = 0$, one sees:

1. in the symmetric phase ($\bar{\phi}_i = 0$) $\hat{M}^2_L = \hat{M}^2_T$

2. in the broken phase ($\bar{\phi}_i \neq 0$) $\hat{M}^2_T = 0 \Longrightarrow$ Goldstone theorem fulfilled

For $h \neq 0$ $\hat{M}^2_L$ and $\hat{M}^2_T$ have the same expressions and show the restoration of chiral symmetry: $\hat{M}^2_L \to \hat{M}^2_T$ for $\bar{\phi}(T \to \infty) \to 0$.

Renormalization method

Berges et al., Annals Phys. **320** (2005) 344

Mass renormalization

In the 2PI formalism there are two expressions for the two point functions, which are equivalent in the exact theory, but could be inequivalent in a given truncation, e.g. when $\Sigma_{ab}\big|_{\phi=0} := \frac{\delta^2\Gamma_{\rm int}[\phi,\Gamma]}{\delta\phi_a\delta\phi_b}\Big|_{\phi=0} \neq 2\frac{\delta\Gamma_{\rm int}[\phi,\Gamma]}{\delta G_{ab}}\Big|_{\phi=0} =: \bar\Sigma_{ab}\big|_{\phi=0}$, which influences also the relation between the possible definitions of the three-point functions (all equivalent in the exact theory).

In the two-loop truncation: $\bar M_{\phi=0} = m_0^2 + \frac{1}{6N}\big(N\lambda_0^{(A)} + 2\lambda_0^{(B)}\big)\mathcal{T}[\bar G_{\phi=0}]$

$$\hat M_{\phi=0} = m_2^2 + \frac{1}{6N}\big(N\lambda_2^{(A)} + 2\lambda_2^{(B)}\big)\mathcal{T}[\bar G_{\phi=0}] - \frac{N+2}{18N^2}\lambda_\star^2\mathcal{S}[\bar G_{\phi=0}]$$

$\Longrightarrow m_0^2 \neq m_2^2$, two conditions needed:

1.) $\bar M_{\phi=0,T_\star} = m_\star^2$ (renormalization condition)
2.) $\hat M_{\phi=0,T_\star} = \bar M_{\phi=0,T_\star}$ (consistency condition, which needed to be imposed for the approximation scheme to converge to the exact theory as the order of the approximation is increased, while keeping the same conditions.)

$$\Longrightarrow m_0^2 = m_\star^2 - \frac{N\lambda_0^{(A)}+2\lambda_0^B}{6N}\mathcal{T}_\star[G_\star], \quad m_2^2 = m_\star^2 - \frac{N\lambda_2^{(A)}+2\lambda_2^B}{6N}\mathcal{T}_\star[G_\star] + \frac{N+2}{18N^2}\lambda_\star^2\mathcal{S}_\star[G_\star]$$

The quantities on which the coupling renormalization conditions are imposed appear by taking functional derivatives of the effective potential w.r.t ϕ.

- 1st derivative   simple due to the stationarity condition $\left.\frac{\delta\Gamma[\phi,G]}{\delta G_{ij}}\right|_{\bar{G}} = 0$

$$\frac{\delta\Gamma(\phi)}{\delta\phi_a} = \left.\frac{\delta\Gamma(\phi)}{\delta\phi_a}\right|_{\bar{G}} + \int_Q^T \left.\frac{\delta\Gamma[\phi,G]}{\delta G_{ij}(Q)}\right|_{\bar{G}} \frac{\delta\bar{G}_{ij}(Q)}{\delta\phi_a} = \left.\frac{\delta\Gamma(\phi)}{\delta\phi_a}\right|_{\bar{G}}$$

- 2nd derivative   use $(\bar{G}^{-1})_{ab}(Q) = (Q^2 + m_0^2)\delta_{ab} + \bar{\Sigma}_{ab}(Q)$ where $\bar{\Sigma}_{ab} = 2\frac{\delta\Gamma_{\rm int}}{\delta\bar{G}_{ba}}$

$\frac{\delta\bar{\Sigma}}{\delta\phi_n}$ appears in the equation for $\frac{\delta^2\Gamma(\phi)}{\delta\phi_n\delta\phi_m}$

self-consistent equation for $\frac{\delta\bar{\Sigma}}{\delta\phi_n}$: $\bar{\Lambda}_{ab,cd}(K,Q) = 4\left.\frac{\delta^2\Gamma_{\rm int}[\phi,G]}{\delta G_{ab}(K)\delta G_{cd}(Q)}\right|_{\bar{G}}$

$$\frac{\delta\Sigma_{ab}(K)}{\delta\phi_n} = 2\left.\frac{\delta^2\Gamma_{\rm int}[\phi,G]}{\delta G_{ba}(K)\delta\phi_n}\right|_{\bar{G}} - \frac{1}{2}\int_Q^T \bar{\Lambda}_{ab,cd}(K,Q)\bar{G}_{ce}(Q)\bar{G}_{fd}(Q)\frac{\delta\Sigma_{ef}(Q)}{\delta\phi_n}$$

The formal solution for $\frac{\delta \bar{\Sigma}_{ab}}{\delta \phi_n}$ can be given in terms of $\bar{V}(K,P)$ satisfying a Bethe-Salpeter-type equation

$$\bar{V}_{ab,cd}(K,P) = \bar{\Lambda}_{ab,cd}(K,P) - \frac{1}{2}\int_Q^T \bar{\Lambda}_{ab,mn}(K,Q)\bar{G}_{me}(Q)\bar{G}_{fn}(Q)\bar{V}_{ef,cd}(Q,P)$$

- in the 3rd derivative $V_{ab,cd}(K,0) := \frac{\delta^2 \bar{\Sigma}_{ab}(K)}{\delta \phi_c \delta \phi_d}$ appears. The equation for $V(K,0)$ is simple if taken at $\phi = 0$ ($\bar{G}^{\phi=0}_{ab} = \delta_{ab}\bar{G}_{\phi=0}$) $\bar{\Lambda}^{\phi=0}_{ab,cd}(K,0) = \left.\frac{2\delta^3 \Gamma_{\text{int}}[\phi, G]}{\delta G_{ab}(K)\delta \phi_c \delta \phi_d}\right|_{\bar{G},\phi=0}$

$$V^{\phi=0}_{ab,cd}(K,0) = \Lambda^{\phi=0}_{ab,cd}(K,0) - \frac{1}{2}\int_Q^T \bar{\Lambda}_{ab,ef}(K,Q)\bar{G}^2_{\phi=0}(Q)V^{\phi=0}_{fe,cd}(Q,0)$$

the formal solution of $V^{\phi=0}$ can be given in terms of $\bar{V}^{\phi=0}$

$$V^{\phi=0}_{ab,cd}(K,0) = \Lambda^{\phi=0}_{ab,cd}(K,0) - \frac{1}{2}\int_Q^T \bar{V}^{\phi=0}_{ab,ef}(K,Q)\bar{G}^2_{\phi=0}(Q)\Lambda^{\phi=0}_{fe,cd}(Q,0)$$

- 4th derivative $\hat{V}_{abcd} := \frac{\delta^4\Gamma(\phi)}{\delta\phi_a\delta\phi_b\delta\phi_c\delta\phi_d}$ simplifies at $\phi = 0$ $\quad \hat{\Lambda}^{\phi=0}_{abcd} = \frac{\delta^4\Gamma_{\rm int}[\phi, G]}{\delta\phi_a\delta\phi_b\delta\phi_c\delta\phi_d}\Big|_{\phi=0}$

$$\hat{V}^{\phi=0}_{abcd} = \hat{\Lambda}^{\phi=0}_{abcd} - \frac{1}{2}\int_Q^T \Lambda^{\phi=0}_{ab,mn}(0,Q)\bar{G}^2_{\phi=0}(Q)V^{\phi=0}_{mn,cd}(Q,0)$$
$$- \frac{1}{2}\int_Q^T \Lambda^{\phi=0}_{bc,mn}(0,Q)\bar{G}^2_{\phi=0}(Q)V^{\phi=0}_{mn,ad}(Q,0) - \frac{1}{2}\int_Q^T \Lambda^{\phi=0}_{bd,mn}(0,Q)\bar{G}^2_{\phi=0}(Q)V^{\phi=0}_{mn,ac}(Q,0)$$

These expressions can be worked out for a concrete truncation. Assuming in our case the same tensor decomposition for $\bar{V}$, V, and $\hat{V}$ as for $\bar{\Lambda}$, Λ, and $\hat{\Lambda}$

$$\bar{\Lambda}^{\phi=0}_{ab,cd} = \bar{\Lambda}^{(A)}_{\phi=0}t_1^{abcd} + \bar{\Lambda}^{(B)}_{\phi=0}t_{23}^{abcd}, \quad \Lambda^{\phi=0}_{ab,cd} = \Lambda^{(A)}_{\phi=0}t_1^{abcd} + \Lambda^{(B)}_{\phi=0}t_{23}^{abcd}, \quad \hat{\Lambda}^{\phi=0}_{abcd} = \frac{\lambda_4}{3N}t_{123}^{abcd}$$
$$\bar{V}^{\phi=0}_{ab,cd} = \bar{V}^{(A)}_{\phi=0}t_1^{abcd} + \bar{V}^{(B)}_{\phi=0}t_{23}^{abcd}, \quad V^{\phi=0}_{ab,cd} = V^{(A)}_{\phi=0}t_1^{abcd} + V^{(B)}_{\phi=0}t_{23}^{abcd}, \quad \hat{V}^{\phi=0}_{abcd} = \frac{\lambda_4}{3N}t_{123}^{abcd}$$

one imposes (one) renormalization and (five) consistency conditions which determine the bare parameters (counterterms)

$$\hat{V}_{\phi=0,T_\star} = V^{(A)}_{\phi=0,T_\star}(K_\star = 0) = V^{(B)}_{\phi=0,T_\star}(K_\star = 0) = \bar{V}^{(A)}_{\phi=0,T_\star} = \bar{V}^{(B)}_{\phi=0,T_\star} = \frac{\lambda_\star}{3N}\,.$$

The simplest example:

$\bar{\Lambda}^{(A)} = \frac{\lambda_0^{(A)}}{3N}$, $\bar{\Lambda}^{(B)} = \frac{\lambda_0^{(B)}}{3N}$, but the equations for $\bar{V}^{(A)}, \bar{V}^{(B)}$ are coupled

Introduce $\bar{\Lambda}^{(C)}_{\phi=0} \equiv N\bar{\Lambda}^{(A)}_{\phi=0} + 2\bar{\Lambda}^{(B)}_{\phi=0} = \frac{N\lambda_0^{(A)}+2\lambda_0^{(B)}}{3N}$, $\quad \bar{V}^{(C)}_{\phi=0} \equiv N\bar{V}^{(A)}_{\phi=0} + 2\bar{V}^{(B)}_{\phi=0}$

$$\Longrightarrow \bar{V}^{(B)}_{\phi=0} = \bar{\Lambda}^{(B)}_{\phi=0} - \int_Q^T \bar{\Lambda}^{(B)}_{\phi=0}\bar{G}^2_{\phi=0}(Q)\bar{V}^{(B)}_{\phi=0}, \qquad \bar{V}^{(C)}_{\phi=0} = \bar{\Lambda}^{(C)}_{\phi=0} - \frac{1}{2}\int_Q^T \bar{\Lambda}^{(C)}_{\phi=0}\bar{G}^2_{\phi=0}(Q)\bar{V}^{(C)}_{\phi=0}$$

Rearrange

$$\Longrightarrow \frac{1}{\bar{V}^{(B)}_{\phi=0}} = \frac{3N}{\lambda_0^{(B)}} + \mathcal{B}[\bar{G}_{\phi=0}](0), \qquad \frac{1}{\bar{V}^{(C)}_{\phi=0}} = \frac{3N}{N\lambda_0^{(A)} + 2\lambda_0^{(B)}} + \frac{1}{2}\mathcal{B}[\bar{G}_{\phi=0}](0)$$

Impose $\bar{V}^{(A)}_{\phi=0,T_\star} = \bar{V}^{(B)}_{\phi=0,T_\star} = \frac{\lambda_\star}{3N}$ that is $\bar{V}^{(B)}_{\phi=0,T_\star} = \frac{\lambda_\star}{3N}$ and $\bar{V}^{(C)}{}_{\phi=0,T_\star} = \frac{N+2}{3N}\lambda_\star$

$$\Longrightarrow \frac{1}{\lambda_0^{(B)}} = \frac{1}{\lambda_\star}\left[1 - \frac{2\lambda_\star}{6N}\mathcal{B}_{\star,\Lambda}[G_\star](0)\right], \qquad \frac{1}{\lambda_0^{(A)}} = \frac{1}{\lambda_0^{(B)}}\left[1 - \frac{(N+2)\lambda_\star}{6N}\mathcal{B}_{\star,\Lambda}[G_\star](0)\right]$$

$\mathcal{B}_{\star,\Lambda}[G_\star](0)$ increases with increasing Λ
$\Longrightarrow$ for $N > 0$ $1/\lambda_0^{(A)}$ vanishes before $1/\lambda_0^{(B)}$ at the Landau pole $\Lambda_{\rm p}$.

Hybrid approximation

In the 2-loop approximation it is possible to derive explicitly finite equations, i.e. do not contain counterterms (for $N = 1$ details in Markó et al., PRD 86, 085031). They involve the spectral density, which is not easily accessible when working in the imaginary time formalism of the finte-T QFT.

$\Longrightarrow$ we constructed also a hybrid approximation in which the propagators do not satisfy the stationarity conditions, they are kept at the Hartree level:

$$\begin{aligned}\bar{M}_L^2 &= m_\star^2 + \frac{\lambda_\star}{2N}\big(\phi^2 + \mathcal{T}_{\rm F}[\bar{G}_L]\big) + \frac{(N-1)\lambda_\star}{6N}\mathcal{T}_{\rm F}[\bar{G}_T],\\ \bar{M}_T^2 &= m_\star^2 + \frac{\lambda_\star}{6N}\big(\phi^2 + \mathcal{T}_{\rm F}[\bar{G}_L]\big) + \frac{(N+1)\lambda_\star}{6N}\mathcal{T}_{\rm F}[\bar{G}_T],\end{aligned}$$

where $G(Q) = 1/(Q^2 + M^2)$ and

$$\mathcal{T}_{\rm F}[G] = \mathcal{T}[G] - \mathcal{T}_\star[G_\star] + (M^2 - m_\star^2)\mathcal{B}_\star[G_\star](0)$$

is the finite tadpole integral.

The effective potential goes beyond the Hartree level ($\gamma_H[\phi, \bar{G}_L, \bar{G}_T]$) as the setting-suns are included

$$\gamma(\phi) \equiv \gamma[\phi, \bar{G}_L, \bar{G}_T] = \gamma_H[\phi, \bar{G}_L, \bar{G}_T] + \frac{\lambda_\star \phi^2}{36N^2}\mathcal{C}_N[\bar{G}_L, \bar{G}_T, G_\star],$$

where

$$\mathcal{C}_N[\bar{G}_L, \bar{G}_T, G_\star] = (N+8)\mathcal{C}[\bar{G}_L, G_\star] + (N-1)\tilde{\mathcal{C}}[\bar{G}_L, \bar{G}_T, G_\star],$$

with

$$\begin{aligned}
\mathcal{C}[\bar{G}_L, G_\star] &= \Big[\mathcal{T}[\bar{G}_L] - \mathcal{T}_\star[G_\star] + (\bar{M}_L^2 - m_\star^2)\mathcal{B}_\star[G_\star](0)\Big]\mathcal{B}_\star[G_\star](0) \\
&- \frac{1}{3}\left[\mathcal{S}[\bar{G}_L] - S_\star[G_\star] - (\bar{M}_L^2 - m_\star^2)\frac{d\mathcal{S}_\star[G_\star]}{dm_\star^2}\right], \\
\tilde{\mathcal{C}}[\bar{G}_L, \bar{G}_T, G_\star] &= 2\Big[\mathcal{T}[\bar{G}_T] - \mathcal{T}_\star[G_\star] + (\bar{M}_T^2 - m_\star^2)\mathcal{B}_\star[G_\star](0)\Big]\mathcal{B}_\star[G_\star](0) \\
&- \frac{1}{3}\left[3\mathcal{S}[\bar{G}_L, \bar{G}_T, \bar{G}_T] - \mathcal{S}[\bar{G}_T] - 2S_\star[G_\star] - 2(\bar{M}_L^2 - m_\star^2)\frac{d\mathcal{S}_\star[G_\star]}{dm_\star^2}\right]
\end{aligned}$$

The hybrid model can be solved using adaptive integral routines.

Fourier techniques for the 2-loop approximation

Aim: Calculate using FFT the convolution in momentum space of two rotation symmetric functions:

$$C[\tilde{f},\tilde{g}](|\vec{p}|) = \int \frac{d^3k}{(2\pi)^3} \tilde{f}(|\vec{k}|)\tilde{g}(|\vec{p}-\vec{k}|)$$

The 3d FT and IFT: $\tilde{f}(|\vec{p}|) = \int d^3x\, f(|\vec{x}|) e^{-i\vec{p}\vec{x}}$ $\qquad f(|\vec{x}|) = \int \frac{d^3p}{(2\pi)^3} \tilde{f}(|\vec{p}|) e^{i\vec{p}\vec{x}}$

Exploiting rotational invariance: $F(x) := x f(x) = \frac{1}{4\pi^2}\left[\mathcal{F}^S_{1d,p}\right]^{-1}[p\tilde{f}(p)](x)$

$$\tilde{F}(p) := p\tilde{f}(p) = 2\pi \mathcal{F}^S_{1d,x}[x f(x)](p)$$

with the 1d Sine and Inverse Sine transforms

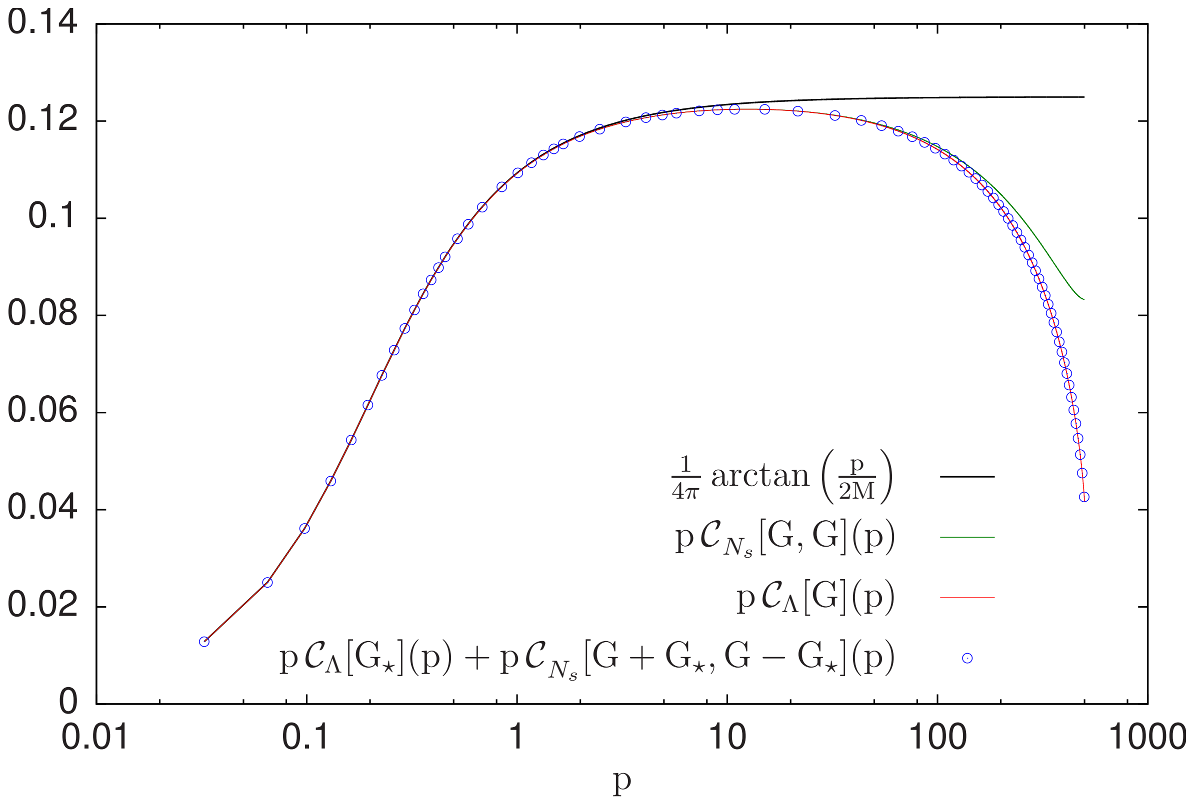
$$\mathcal{F}^S_{1d,x}[\alpha(x)](p) = 2\int_0^\infty dx\, \alpha(x)\sin(xp) \qquad \left[\mathcal{F}^S_{1d,p}\right]^{-1}[\tilde{\alpha}](x) = 2\int_0^\infty dp\, \tilde{\alpha}(p)\sin(px)$$

$$\Longrightarrow C[\tilde{f},\tilde{g}](|\vec{p}|) = \mathcal{F}^{(S)}_{1d,x}\left[\frac{F(x)G(x)}{x}\right]$$

Cutoff and discretization effects in a 3d convolution integral

For $G(p) = \frac{1}{p^2 + M^2}$ the result is known exactly: $C[G,G](p) = \frac{1}{4\pi p} \arctan \frac{p}{2M}$

With cutoff regularization $C_\Lambda[G](p) = \frac{1}{8\pi^2 p} \int_0^\Lambda dk \frac{k}{k^2+M^2} \log\left(\frac{\min^2(k+p,\Lambda)+M^2}{(p-k)^2+M^2}\right)$



Questions:

1. How the continuum limit is approached?

2. Does the Fourier method reproduce the result from the cutoff-dependent formula?

Method to compute accurately the 3d convolution using FFT:

$$C_\Lambda[G](p) = \int_{\substack{|q|<\Lambda \\ |q-p|<\Lambda}} G(q)G(q-p) = \int_{\substack{|q|<\Lambda \\ |q-p|<\Lambda}} G_\star(q)G_\star(q-p) + \int_{\substack{|q|<\Lambda \\ |q-p|<\Lambda}} [G(q)+G_\star(q)][G(q-p)-G_\star(q-p)]$$

Method for the use of FFT

Longitudinal gap equation after mass renormalization:

$$\bar{M}_L^2(K) = m_\star^2 + \frac{\lambda_{2l}^{(A)} + 2\lambda_{2l}^{(B)}}{6N}\phi^2$$

$$+\frac{\lambda_0^{(A)} + 2\lambda_0^{(B)}}{6N}\big(\mathcal{T}[\bar{G}_L] - \mathcal{T}_\star[G_\star]\big) + (N-1)\frac{\lambda_0^{(A)}}{6N}\big(\mathcal{T}[\bar{G}_T] - \mathcal{T}_\star[G_\star]\big)$$

$$-\frac{\lambda_\star^2}{18N^2}\phi^2\left[9\big(\mathcal{B}[\bar{G}_L](K) - \mathcal{B}_\star[G_\star](0)\big) + (N-1)\big(\mathcal{B}[\bar{G}_T](K) - \mathcal{B}_\star[G_\star](0)\big)\right]$$

where $K = (\omega_n, \mathbf{k})$, $\bar{G}_{L/T}(K) = [K^2 + \bar{M}_{L/T}^2(K)]^{-1}$ and $G_\star(Q) = [Q^2 + m_\star^2]^{-1}$

$$\frac{\lambda_{2l}^{(B)}}{\lambda_0^{(B)}} = 1 - \frac{N+6}{18N^2}\lambda_\star^2 \int_{Q_\star}^{T_\star} G_\star^2(Q_\star)\Big[\mathcal{B}_\star[G_\star](Q_\star) - \mathcal{B}_\star[G_\star](0)\Big]$$

$$\frac{N\lambda_{2l}^{(A)} + 2\lambda_{2l}^{(B)}}{N\lambda_0^{(A)} + 2\lambda_0^{(B)}} = 1 - \frac{N+2}{6N^2}\lambda_\star^2 \int_{Q_\star}^{T_\star} G_\star^2(Q_\star)\Big[\mathcal{B}_\star[G_\star](Q_\star) - \mathcal{B}_\star[G_\star](0)\Big]$$

A. Difference of Tadpoles

$$\int_{Q<\Lambda}^{T} \bar{G}(Q) - \int_{Q_\star<\Lambda}^{T_\star} G_\star(Q_\star) = \int_{Q<\Lambda}^{T} \left[\bar{G}(Q) - G_\star(Q)\right] + \int_{Q<\Lambda}^{T} G_\star(Q) - \int_{Q_\star<\Lambda}^{T_\star} G_\star(Q_\star)$$

$$\int_{Q<\Lambda}^{T} G_\star(Q) - \int_{Q_\star<\Lambda}^{T_\star} G_\star(Q_\star) \quad = \quad \frac{1}{2\pi^2}\int_0^\Lambda dq \frac{q^2}{\epsilon_q^\star}\left[\frac{1}{e^{\epsilon_q^\star/T}-1} - \frac{1}{e^{\epsilon_q^\star/T_\star}-1}\right]$$

B. Difference of Bubbles

$$\int_{\substack{Q<\Lambda \\ Q-K<\Lambda}}^{T} \bar{G}(Q)\bar{G}(Q-K) - \int_{Q_\star<\Lambda}^{T_\star} G_\star^2(Q_\star) = \int_{\substack{Q<\Lambda \\ Q-K<\Lambda}}^{T} \bar{G}(Q)\bar{G}(Q-K) - \int_{\substack{Q<\Lambda \\ Q-K<\Lambda}}^{T} G_\star(Q)G_\star(Q-K)$$

$$+ \int_{\substack{Q<\Lambda \\ Q-K<\Lambda}}^{T} G_\star(Q)G_\star(Q-K) - \int_{Q_\star<\Lambda}^{T_\star} G_\star^2(Q_\star)$$

$K_\star = (\omega_\star, \mathbf{k})$, $n_T(\epsilon_q^\star) = 1/(\exp(\epsilon_q^\star/T) - 1)$ and $n_{T_\star}(\epsilon_q^\star) = 1/(\exp(\epsilon_q^\star/T_\star) - 1)$

numerical integration:

$$\int_{\substack{Q<\Lambda\\Q-K<\Lambda}}^{T} G_\star(Q)G_\star(Q-K) - \int_{Q_\star<\Lambda}^{T_\star} G_\star^2(Q_\star) =$$

$$= \quad \frac{\theta(\Lambda-k)}{8\pi^2 k}\left[\int_0^{\Lambda-k} dq\frac{q}{\epsilon_q^\star}\left(\frac{1}{2}+n_T(\epsilon_q^\star)\right)\ln\frac{(k^2+2kq+\omega^2)^2+4\omega^2(\epsilon_q^\star)^2}{(k^2-2kq+\omega^2)^2+4\omega^2(\epsilon_q^\star)^2}\right.$$

$$\left.+\int_{\Lambda-k}^{\Lambda} dq\frac{q}{\epsilon_q^\star}\left(\frac{1}{2}+n_T(\epsilon_q^\star)\right)\ln\frac{(\Lambda^2-q^2+\omega^2)^2+4\omega^2(\epsilon_q^\star)^2}{(k^2-2kq+\omega^2)^2+4\omega^2(\epsilon_q^\star)^2}\right]$$

$$-\frac{1}{4\pi^2}\left[-\frac{\Lambda}{\epsilon_\Lambda^\star}\left(\frac{1}{2}+n_{T_\star}(\epsilon_\Lambda^\star)\right)+\int_0^{\Lambda} dq\frac{1}{\epsilon_q^\star}\left(\frac{1}{2}+n_{T_\star}(\epsilon_q^\star)\right)\right]$$

FFT:

$$\int_{\substack{Q<\Lambda\\Q-K<\Lambda}}^{T} \bar{G}(Q)\bar{G}(Q-K) - \int_{\substack{Q<\Lambda\\Q-K<\Lambda}}^{T} G_\star(Q)G_\star(Q-K) =$$

$$= \int_{\substack{Q<\Lambda\\Q-K<\Lambda}}^{T} \left\{\bar{G}(Q)\left[\bar{G}(Q-K)-G_\star(Q-K)\right]+\left[\bar{G}(Q)-G_\star(Q)\right]G_\star(Q-K)\right\}$$

$$= \int_{\substack{Q<\Lambda\\Q-K<\Lambda}}^{T} \left[\bar{G}(Q)+G_\star(Q)\right]\left[\bar{G}(Q-K)-G_\star(Q-K)\right]$$

. . . Similar procedure for the difference of TrLogs and setting-suns.

The final form of the longitudinal gap equation ($\delta\bar{G}(Q) = \bar{G}(Q) - G_\star(Q)$)

$$\bar{M}_L^2(K) = m_\star^2 + \frac{\lambda_{2,l}^{(A)} + 2\lambda_{2,l}^{(B)}}{6N}\phi^2 + \frac{\lambda_0^{(A)} + 2\lambda_0^{(B)}}{6N}\int_{Q<\Lambda}^T \delta\bar{G}_L(Q) + (N-1)\frac{\lambda_0^{(A)}}{6N}\int_{Q<\Lambda}^T \delta\bar{G}_T(Q)$$
$$+ \frac{N\lambda_0^{(A)} + 2\lambda_0^{(B)}}{6N}\left(\int_{Q<\Lambda}^T G_\star(Q) - \int_{Q_\star<\Lambda}^{T_\star} G_\star(Q_\star)\right)$$
$$- \frac{\lambda_\star^2}{18N^2}\phi^2\left[9\int_{\substack{Q<\Lambda \\ Q-K<\Lambda}}^T [\bar{G}_L(Q) + G_\star(Q)]\,\delta\bar{G}_L(Q-K) + (N-1)\int_{\substack{Q<\Lambda \\ Q-K<\Lambda}}^T [\bar{G}_T(Q) + G_\star(Q)]\,\delta\bar{G}_T(Q-K)\right.$$
$$\left. + (N+8)\left(\int_{\substack{Q<\Lambda \\ Q-K<\Lambda}}^T G_\star(Q)G_\star(Q-K) - \int_{Q_\star<\Lambda}^{T_\star} G_\star^2(Q)\right)\right]$$

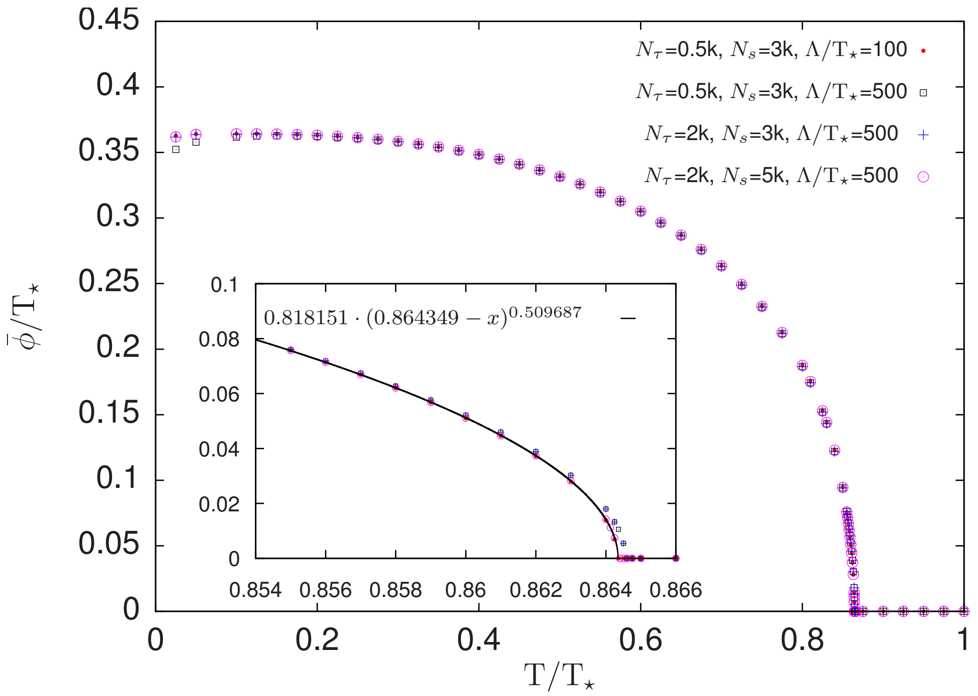
Some integrals evaluated using adaptive numerical routines, some using FFT.

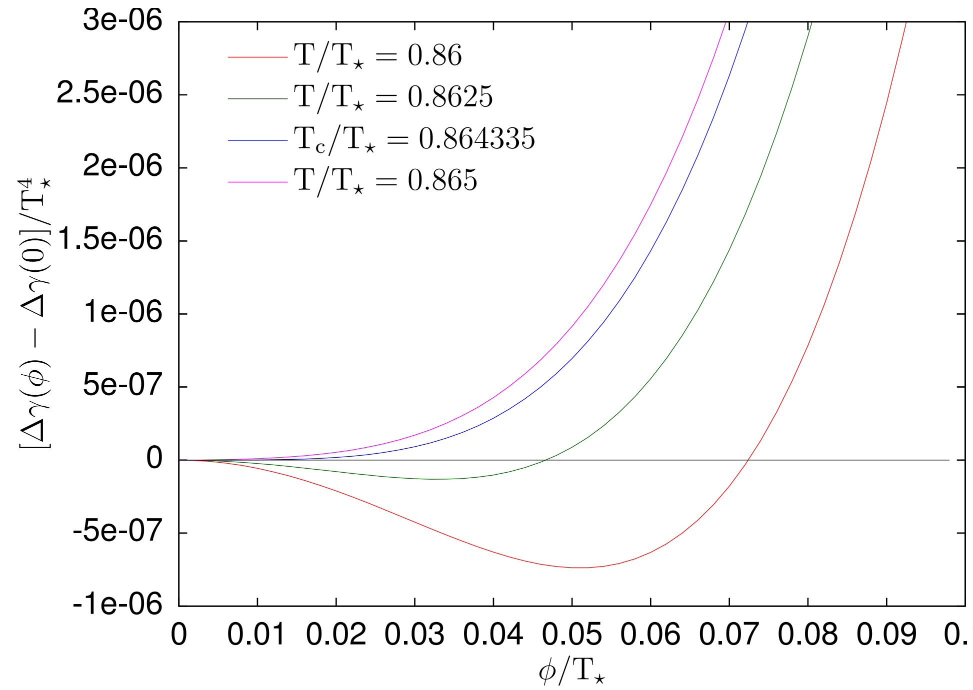
The same method is applied for the transverse gap equation, field equation and the effective potential.

Do we see any improvement over the Hartree approximation?

Yes, we do.

Order of the phase transition in the 2-loop approximation (N=1)

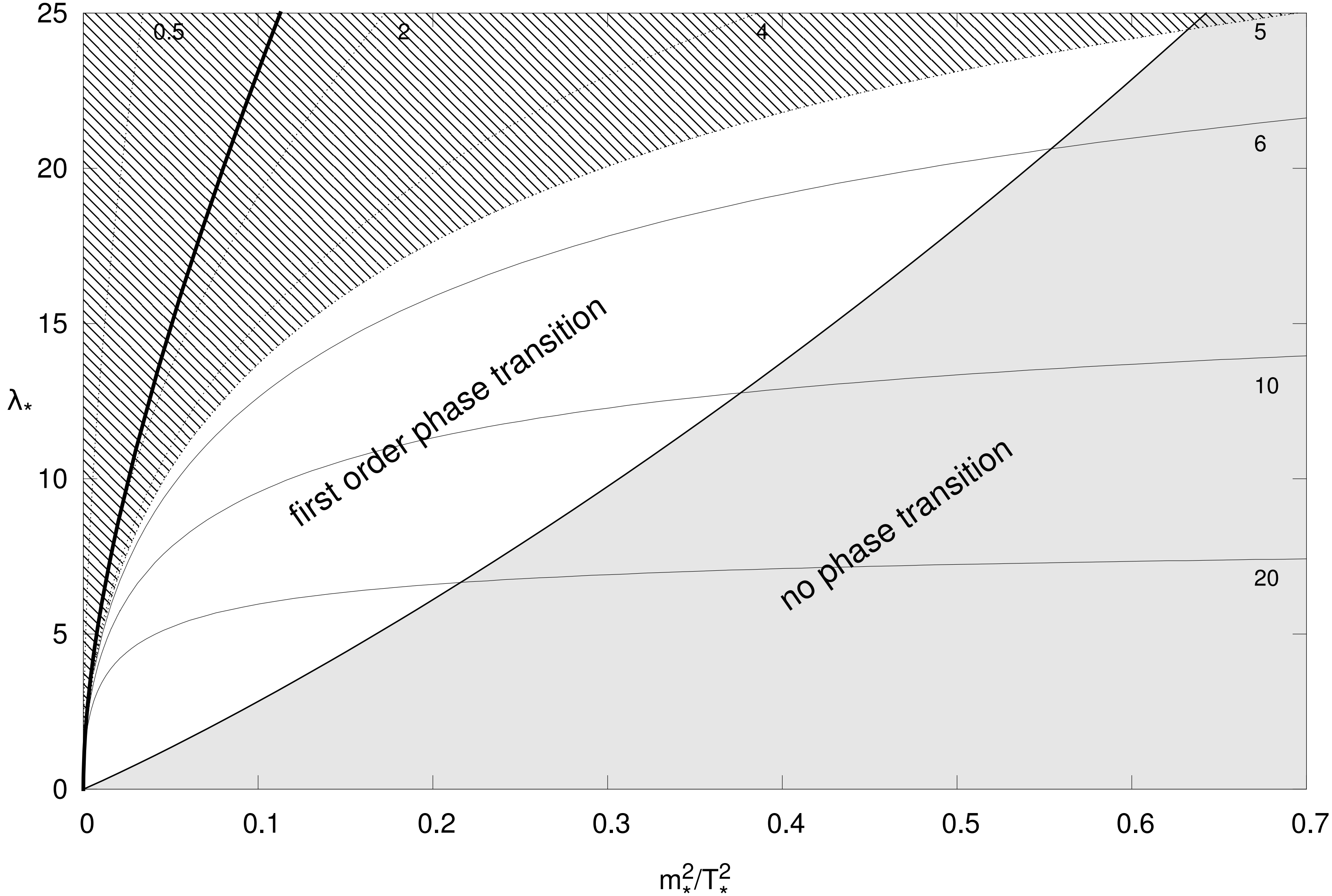




The expected 2nd order nature of the phase transition is confirmed by the temperature evolution of the effective potential.

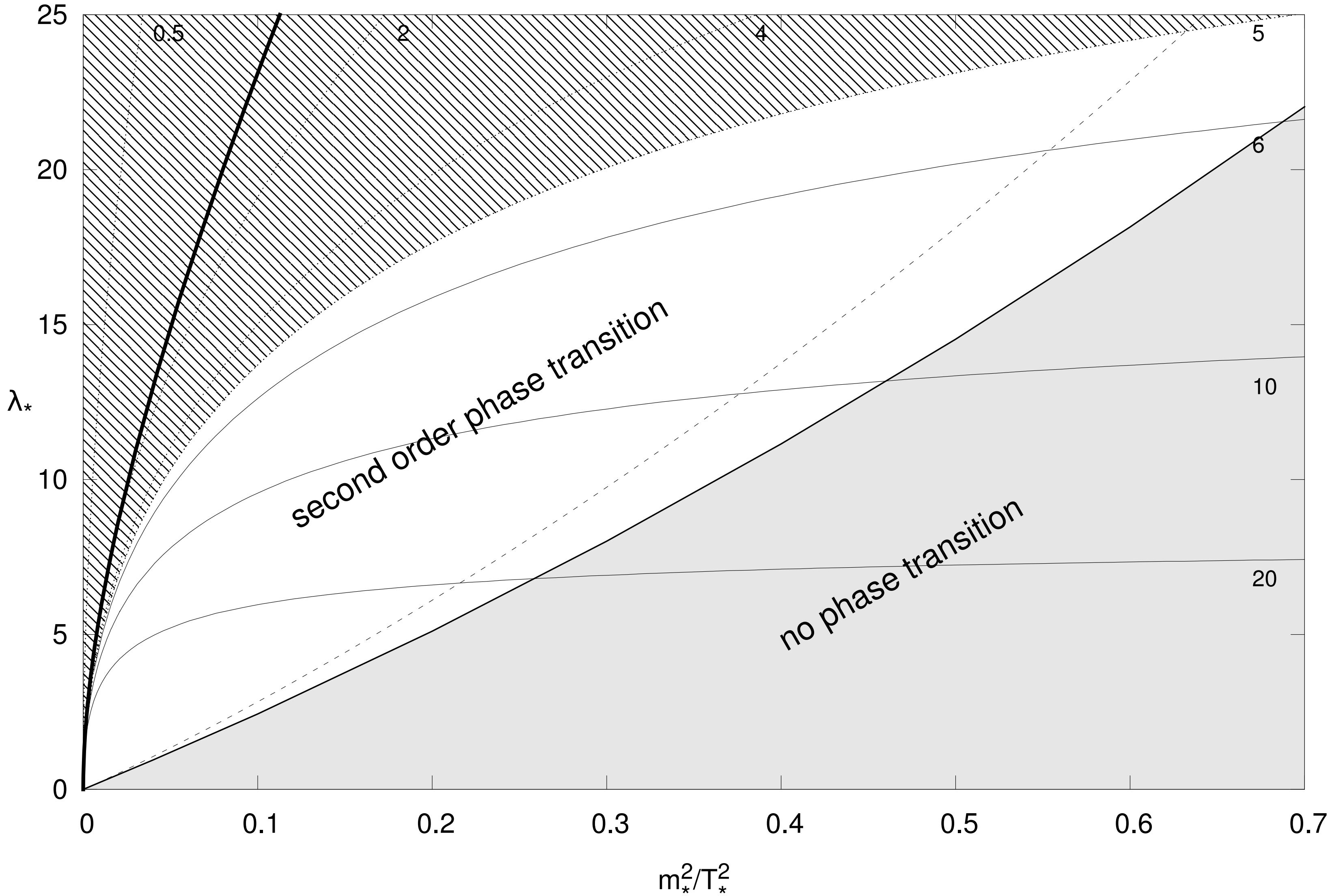
The hybrid approximation also gives a 2nd order phase transition $\Longrightarrow$ what is important compared to the Hartree approximation is to include the setting-sun in the field equation

Hartree vs. 2-loop 2PI ($N = 1$)



PRD 83, 125026

Hartree vs. 2-loop 2PI ($N = 1$)



PRD 86, 085031

Parametrization of the $O(N=4)$ model

Without approximation (truncation) the equation are $T_\star$-invariant.

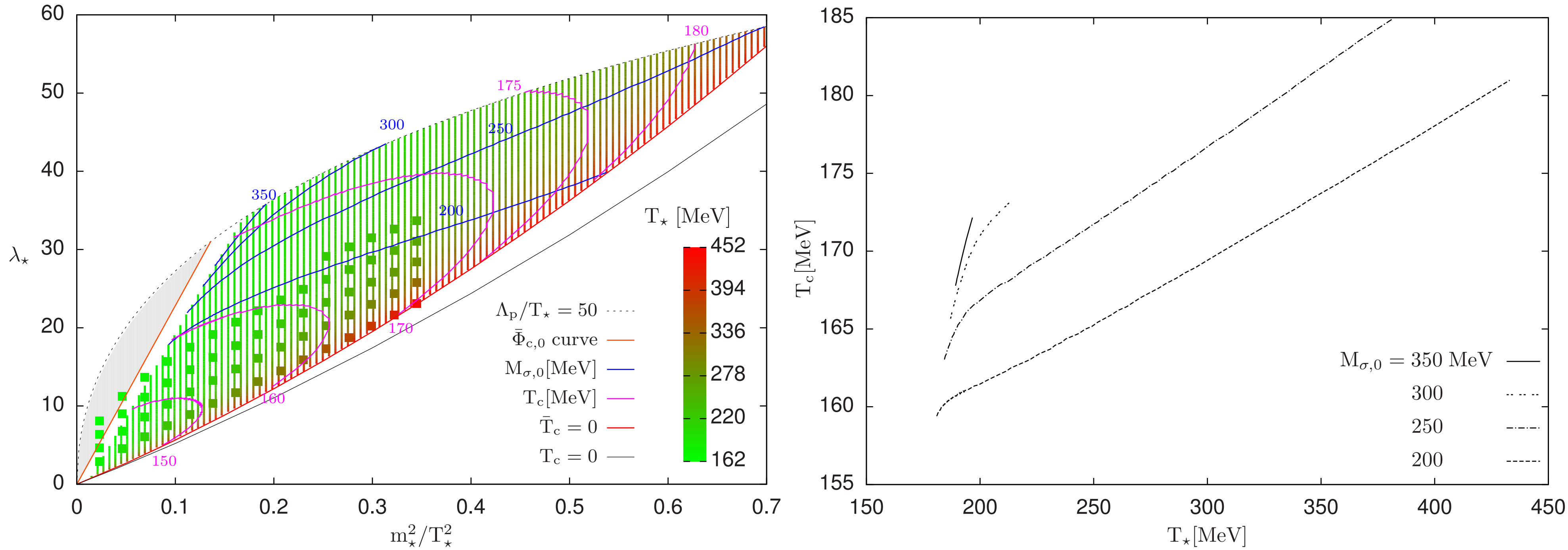
When an approximation is used the $T_\star$-invariance is lost and practical calculation requires to fix the renormalization scale $T_\star$ too.

In the chiral limit we have to determine $m_\star^2, \lambda_\star, T_\star$ and additionally h in the physical case.

The problem is that only f_π and m_π are known accurately, m_σ and T_c are not.

We choose to scan the parameter space $(m_\star^2/T_\star^2, \lambda_\star, h/T_\star^3)$, measure everything in units of $T_\star$ ($T_\star$ set to 1 in the code) and determine $T_\star$ in physical units (MeV) from the requirement that the value of $\bar\phi_0 \equiv \bar\phi(T=0)$ in physical units is f_π.

Parametrization in the chiral limit $h = 0$

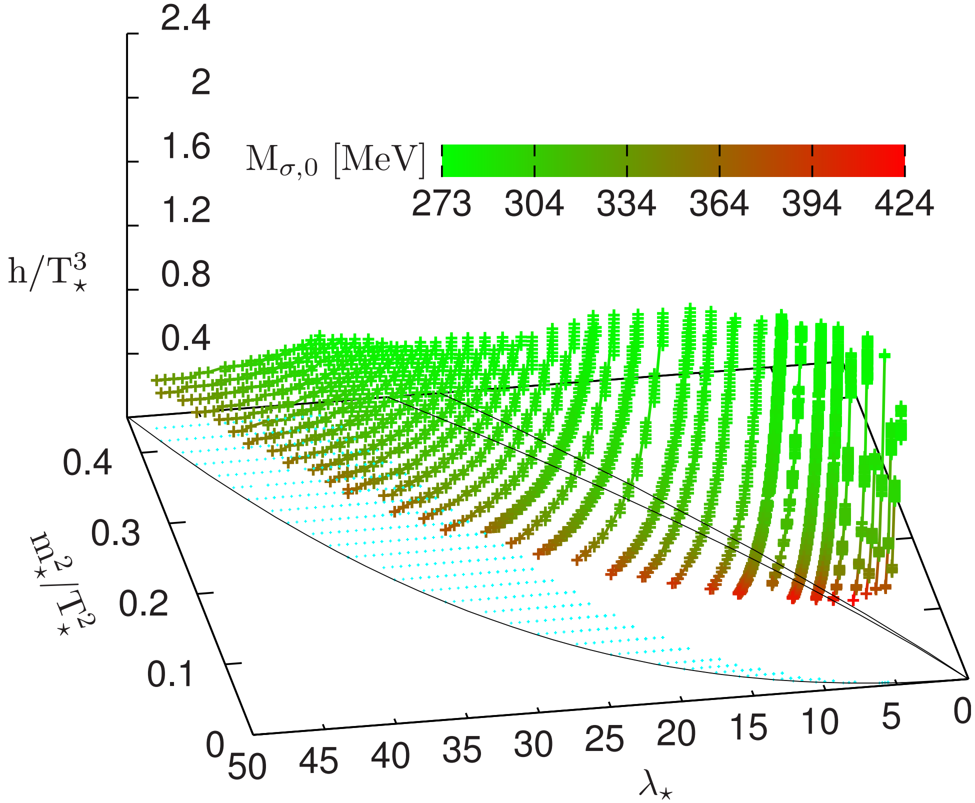


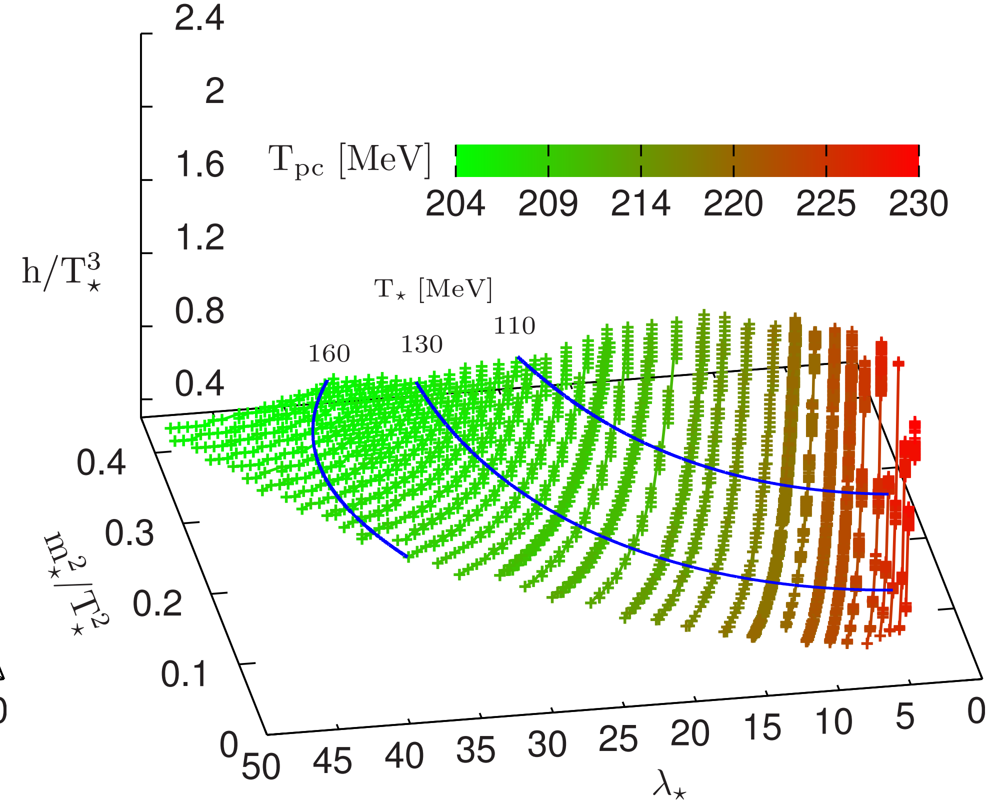
along an iso-$M_{\sigma,0}$ line $\bar{\phi}_0 = f_\pi$ (line of constant physics) and although $T_\star$[MeV] varies along the line, in the exact theory physical quantities should not depend on $T_\star$[MeV]. In our two-loop approximate theory the dependence of T_c on $T_\star$[MeV] is around 10%.

σ is light, as it is also in the large-N limit, where $M_{\sigma,0}^{\max} \simeq 333$ MeV was observed

Patkós et al., PLB 537, 77; Andersen et al., PRD 70, 116007

Parametrization in the hybrid approximation at $h \neq 0$

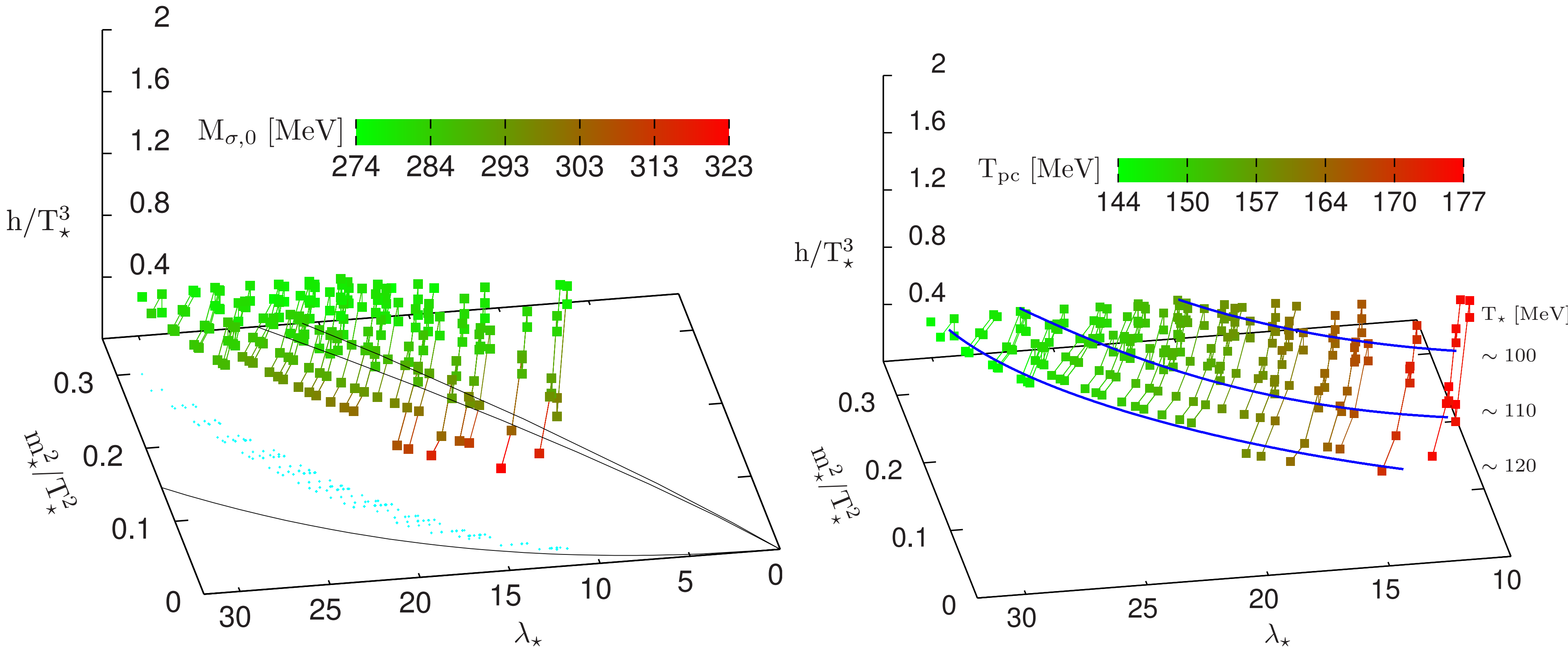




only points satisfying $M_{\sigma,0} > 2M_{\pi,0}$ and $M_{\pi,0} = 138 \pm 14$ MeV are considered where $M_{\sigma/\pi,0} = \hat{M}_{L/T}(T=0)$[MeV]

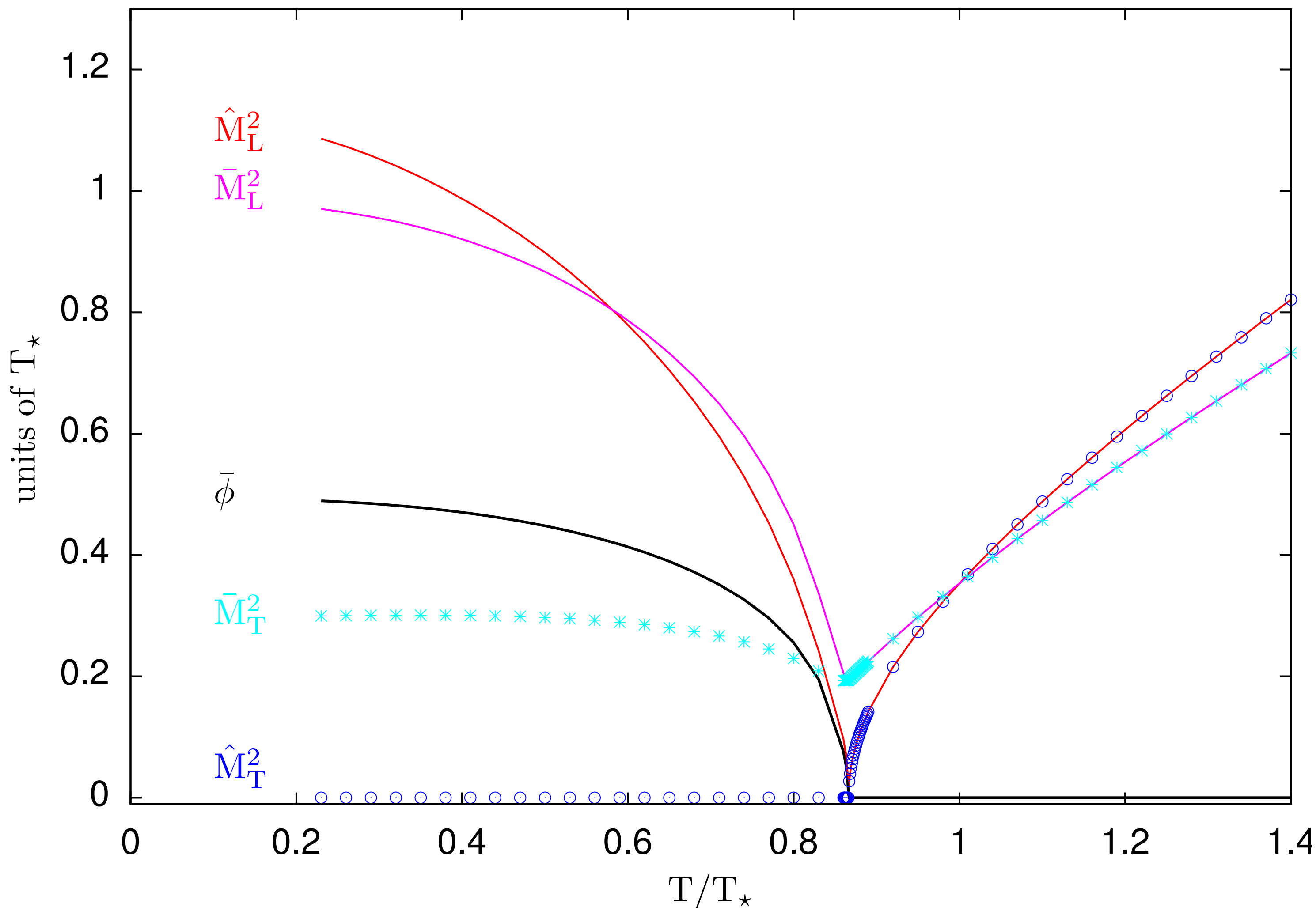
$M_{\sigma,0}$ and $T_{\rm pc}$ increase compared to the chiral limit.

Parametrization in the two-loop approximation at $h \neq 0$



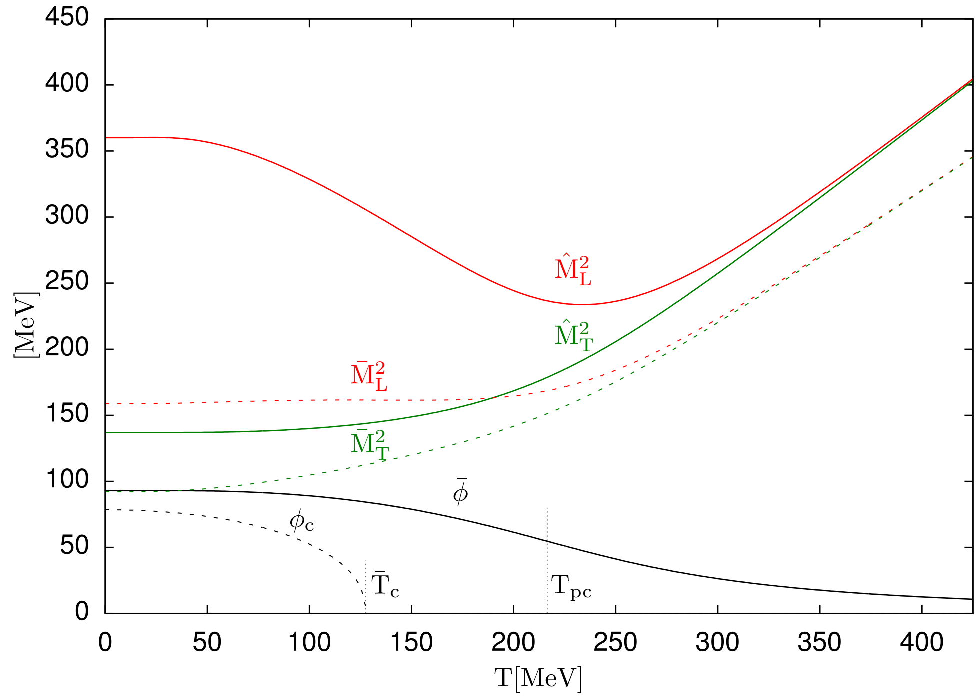
$M_{\sigma,0}$ and $T_{\rm pc}$ decrease compared to the hybrid approximation.

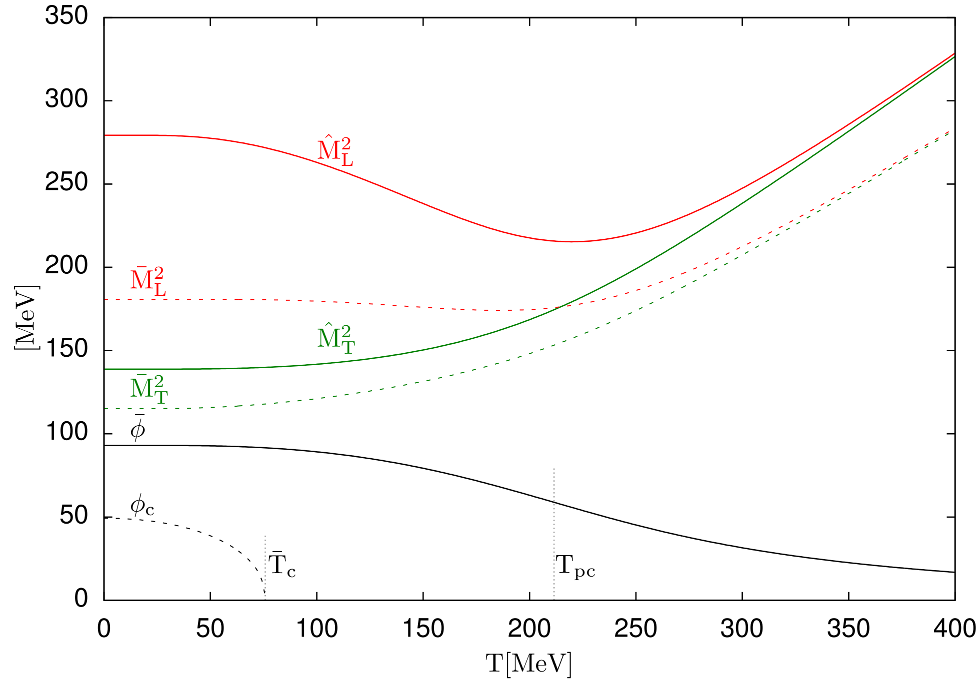
Temperature evolution in the chiral limit (2-loop approximation)

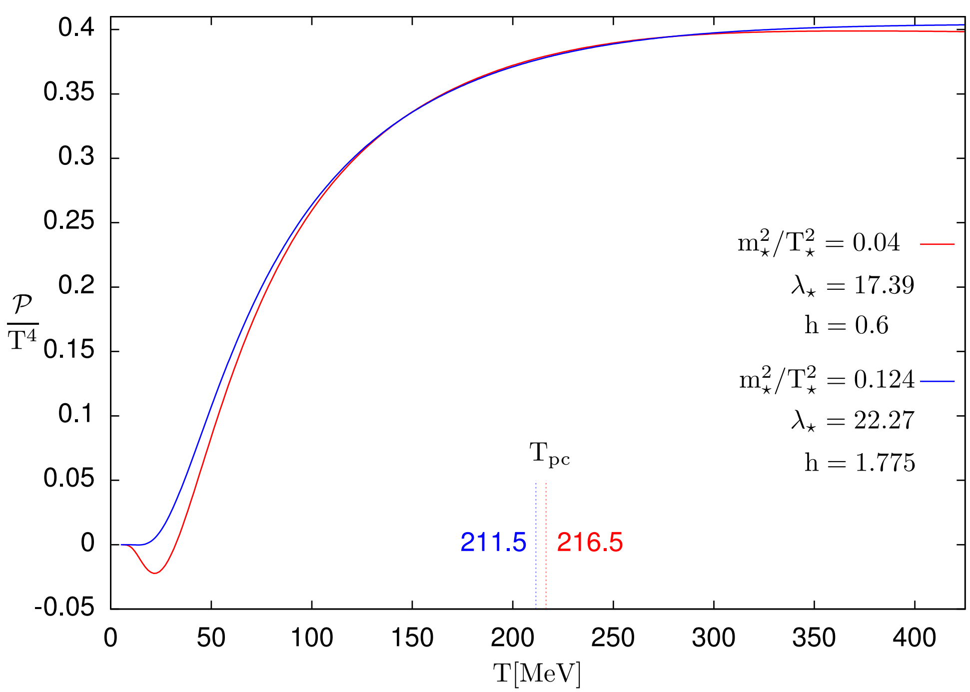


the phase transition is of second order

Temperature evolution in the hybrid case $h \neq 0$







The negative pressure is related to the fact that at low T $\phi(T)$ increases a bit and only after decreases.

This seems to be an inconsistency of the hybrid approximation.

Conclusions & Outlook

• A method was constructed to accurately compute at finite-T convolution integrals of propagators using Fourier techniques in cutoff regularized theory.

There are practical limitation (huge memory requirement) for a direct use of the method at small values of the temperature and in case of vanishing masses, but some sort of extrapolation can be applied.

• The inclusion of the setting-sun diagram in the Φ-derivable (2PI) effective action renders the phase transition second order $\Longrightarrow$ the method can be use to study thermodynamic quantities.

• The statical critical exponents have mean-field values $\Longrightarrow$ higher order skeleton diagrams need to be included in the 2PI effective action.

• apply the Fourier method to the $O(N)$ model at NLO of the 2PI-$1/N$ expansion. Huge memory requirement expected because some of the procedures used to accelerate the convergence of Matsubara integrals with increasing N_τ will not work.