Reciprocity – a symmetry of quantum and wave scattering processes

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L. Deák[1] and T. Fülöp[1], *Annals of Physics* **327** (2012) 1050

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What is reciprocity?

‘The path of a light ray can be reversed’

G. Stokes, *Cambridge Dublin Math. J.* **4** (1849) 89: introduced in geometrical optics, used for amplitudes t, r at $n_1|n_2$

H. von Helmholtz, *Handbuch der Physiologischen Optik* (1866); J.W. Strutt, Baron Rayleigh, *The theory of sound* (1877): for waves (electromagnetic, acoustic, elastic),

signal (source at A, measurement at B) =

= signal (source at B, measurement at A)

Signal: amplitude and phase

Reciprocity: invariance under the interchange of source and detector

Nonscalar signals (polarization, spin etc.): what to do with these degrees of freedom? Different suggestions

Landau-Lifshitz started to use the term ‘reciprocity’ with the meaning ‘invariance under time reversal’ (T)

Note: reciprocity may hold for absorption as well

Others: invariance under 180° rotation (R_{180°)

Even others: invariance under PT

‘microreversibility’, ‘principle of detailed balance’; ‘dichroism’

Here, now:

– the original meaning

– presented in a generally applicable formalism

– with illustrations and examples

Preparations:

Let's (re)write the equation of our linear wave propagation as $i\partial_t \psi = H\psi$ with a one- or multicomponent ψ

Note: Maxwell eqns.: $\psi = \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix}$, KG: $\psi = \begin{pmatrix} \phi \\ (\partial_t - ieA_0)\phi \end{pmatrix}$,

can also be some effective equation (e.g., to approximate multiple scattering with scattering on an effective medium)

$H = H_0 + V$, where

H_0: free wave propagation

V: scatterer

We need a scalar product (probability, energy functional etc.)

$V = V^\dagger$ not required: absorption of probability/energy

Scattering theory: Lippmann-Schwinger formalism (notations: Schiff)

$$G_E^{\pm} := (E - H \pm i\epsilon)^{-1}$$

$$\chi^{\pm} := u + G_E^{\pm} V u, \qquad \chi^{T\pm} := u + G_E^{\mp\,\dagger} V^{\dagger} u,$$

where u is an eigenstate of H_0 with real eigenvalue E (free incoming/outgoing part)

The transition amplitude of an elastic scattering process $u_\alpha \to u_\beta$ $(E_\alpha = E_\beta)$:

$$\langle \beta \,|\mathtt{T}|\, \alpha \rangle := \left(u_\beta, V \chi_\alpha^{+}\right) = \left(\chi_\beta^{T-}, V u_\alpha\right)$$

Antiunitary operators: (a.k.a. conjugate unitary)

isometric: $\|K\psi\| = \|\psi\|$

and conjugate linear: $K(\lambda_1\psi_1 + \lambda_2\psi_2) = \lambda_1^* K\psi_1 + \lambda_2^* K\psi_2$

Consequence: $(K\psi_1, K\psi_2) = (\psi_2, \psi_1)$ (physics: in $\leftrightarrow$ out)

Example from mathematics: complex conjugation: $J\psi = \psi^*$

Example from physics: time reversal T

They are 'the same in number' as unitary ones: with any fixed K_0, any K can be written as $K = UK_0$ with a unitary U

"A good definition should be the hypothesis of a theorem."

(J. Glimm)

K is a *reciprocity operator* of a system if

$$K H_0 K^{-1} = H_0 \quad \text{and} \quad K V K^{-1} = V^\dagger$$

$\Longrightarrow$ **Reciprocity theorem** for the Green's operators:

$$K G_E^\pm K^{-1} = G_E^{\pm\,\dagger}$$

The reciprocal partner of $u_\alpha \to u_\beta$: $K u_\beta \to K u_\alpha$

$\Longrightarrow$ **Reciprocity theorem** for the transition amplitude:

$$\langle \beta \,|\mathtt{T}|\, \alpha \rangle = \langle K\alpha \,|\mathtt{T}|\, K\beta \rangle$$

Discussion:

Condition on both the scatterer and the process
(vs. ‘magneto-optical materials are nonreciprocal’)

Not a rotation: *anti*unitary

T is just one possible K from the many
Particle physics: CP violated, CPT probably not $\Longrightarrow$
T violated [plus BABAR, PRL 109 (2012) 211801],
but some other K is possible

T may be a reciprocity operator even if no T invariance
($TVT^{-1} \neq V$ does not exclude $TVT^{-1} = V^\dagger$)
Nuclear physics: optical potentials

Reciprocity violation:

with $V = V_+ + V_-, \qquad V_\pm := \frac{1}{2}\left(V \pm K^{-1}V^\dagger K\right)$,

we can find

$$\left\langle\beta\middle|\mathtt{T}\middle|\alpha\right\rangle - \left\langle K\alpha\middle|\mathtt{T}\middle|K\beta\right\rangle = \left(\chi^{T-}_{+,\beta}, V_-\chi^+_\alpha\right) - \left(\chi^{T-}_{+,K\alpha}, V_-\chi^+_{K\beta}\right)$$

In the first Born approximation,

$$\approx 2\left(u_\beta, V_- u_\alpha\right)$$

n-component wave function:

$$\psi(\mathbf{r}) = \begin{pmatrix} \psi_1(\mathbf{r}) \\ \psi_2(\mathbf{r}) \\ \vdots \\ \psi_n(\mathbf{r}) \end{pmatrix}, \qquad V(\mathbf{r})\ n \times n \text{ matrix}$$

photon pol.
neutron spin
isospin
etc.

$$H_0 := -\Delta\ , \qquad u_\alpha(\mathbf{r}) = p_\alpha e^{i\mathbf{k}_\alpha \mathbf{r}}$$

$K := UJ$, where U constant $n \times n$, $J\psi = \psi^*$ (serving as K_0)

$$(Ku_\alpha)(\mathbf{r}) = (Up_\alpha^*)\, e^{-i\mathbf{k}_\alpha \mathbf{r}}\ : \qquad \mathbf{k}_\alpha \mapsto -\mathbf{k}_\alpha, \quad p_\alpha \mapsto Up_\alpha^*$$

$$KVK^{-1} = V^\dagger: \quad V = UV^\mathsf{T}U^{-1} \qquad \left(V^\mathsf{T}\right)_{ij} = V_{ji}$$

Mathematical results:

$n \leq 7$ dimensional complex matrices:

$$(\exists U) \;\; V = U V^{\mathsf{T}} U^{-1} \iff (\exists \tilde{U}) \;\; \left(\tilde{U} V \tilde{U}^{-1}\right)^{\mathsf{T}} = \tilde{U} V \tilde{U}^{-1}$$

(i.e., can be transformed to self-transpose)

And, for any finite n:

complete classification of Vs with $V = U V^{\mathsf{T}} U^{-1}$

S.R. Garcia, J.E. Tener, *J. Operator Theory* **68** (2012) 179

U acting on $\mathbf{r}$ as well: open question

Neutron or Mössbauer scattering: $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$,

$$V = v_0\sigma_0 + v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3 = v_0\sigma_0 + \mathbf{v}\boldsymbol{\sigma}$$

with $\sigma_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$

neutron: $\mathbf{v} = c\mathbf{B}$, photon Mössbauer: $\mathbf{v} = \mathbf{v}(\mathbf{B})$

In a good polarization basis: $\exists K \iff V_{12}(\mathbf{r}) = V_{21}(\mathbf{r})$

m homogeneous layers, V_1, V_2, $\ldots V_m$: criterion:

$\mathrm{Re}\,\mathbf{v}_1$, $\mathrm{Im}\,\mathbf{v}_1$, $\mathrm{Re}\,\mathbf{v}_2$, $\mathrm{Im}\,\mathbf{v}_2$, $\ldots$ $\mathrm{Re}\,\mathbf{v}_m$, $\mathrm{Im}\,\mathbf{v}_m$ in one plane

Magnitude reciprocity: $\left|\langle\beta\,|\mathtt{T}|\,\alpha\rangle\right|^2 = \left|\langle K\alpha\,|\mathtt{T}|\,K\beta\rangle\right|^2$

(most measurements see only intensities!)

$V_{12}(\mathbf{r}) = e^{i\delta} V_{21}(\mathbf{r})$ (with an $\mathbf{r}$ independent δ)

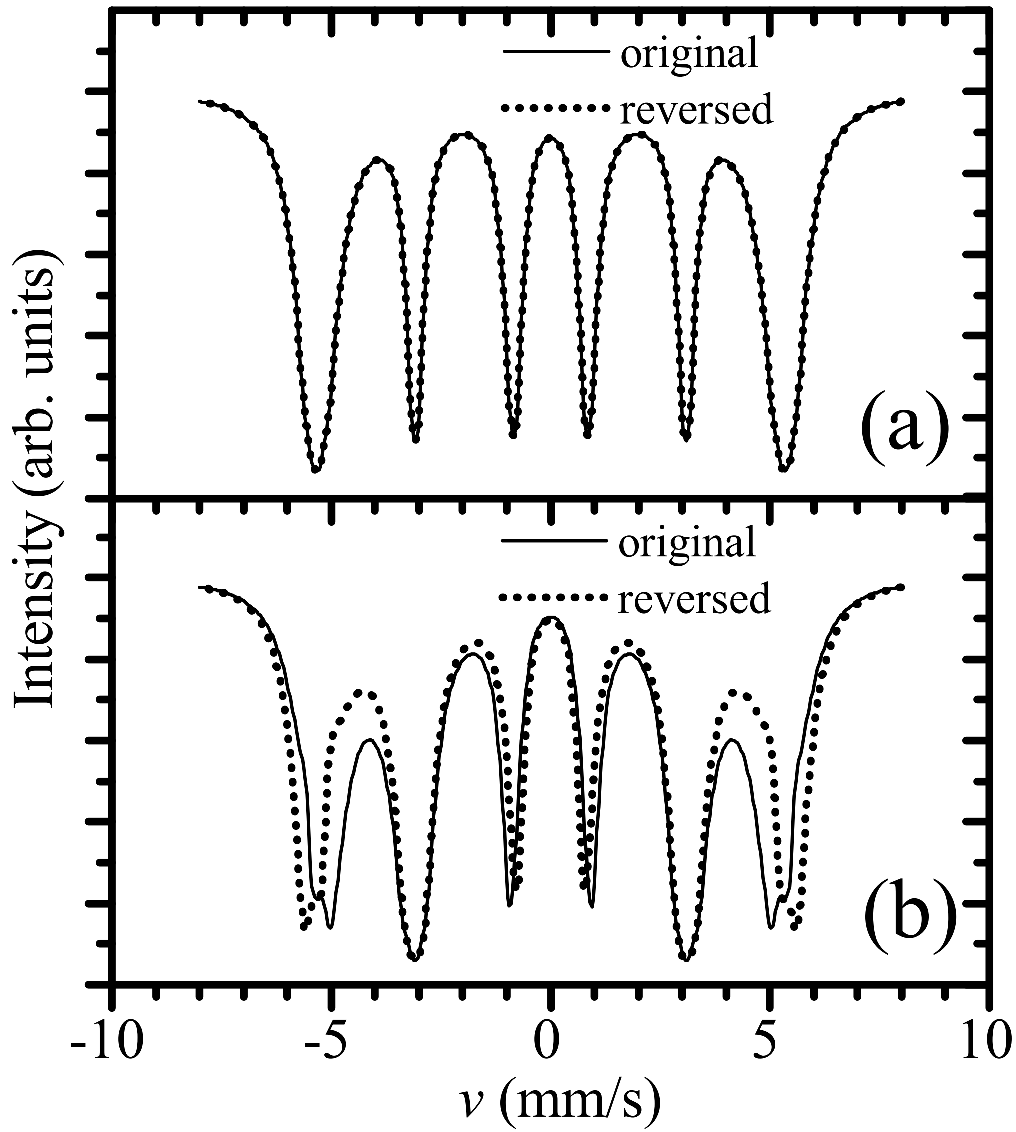
In experiments:

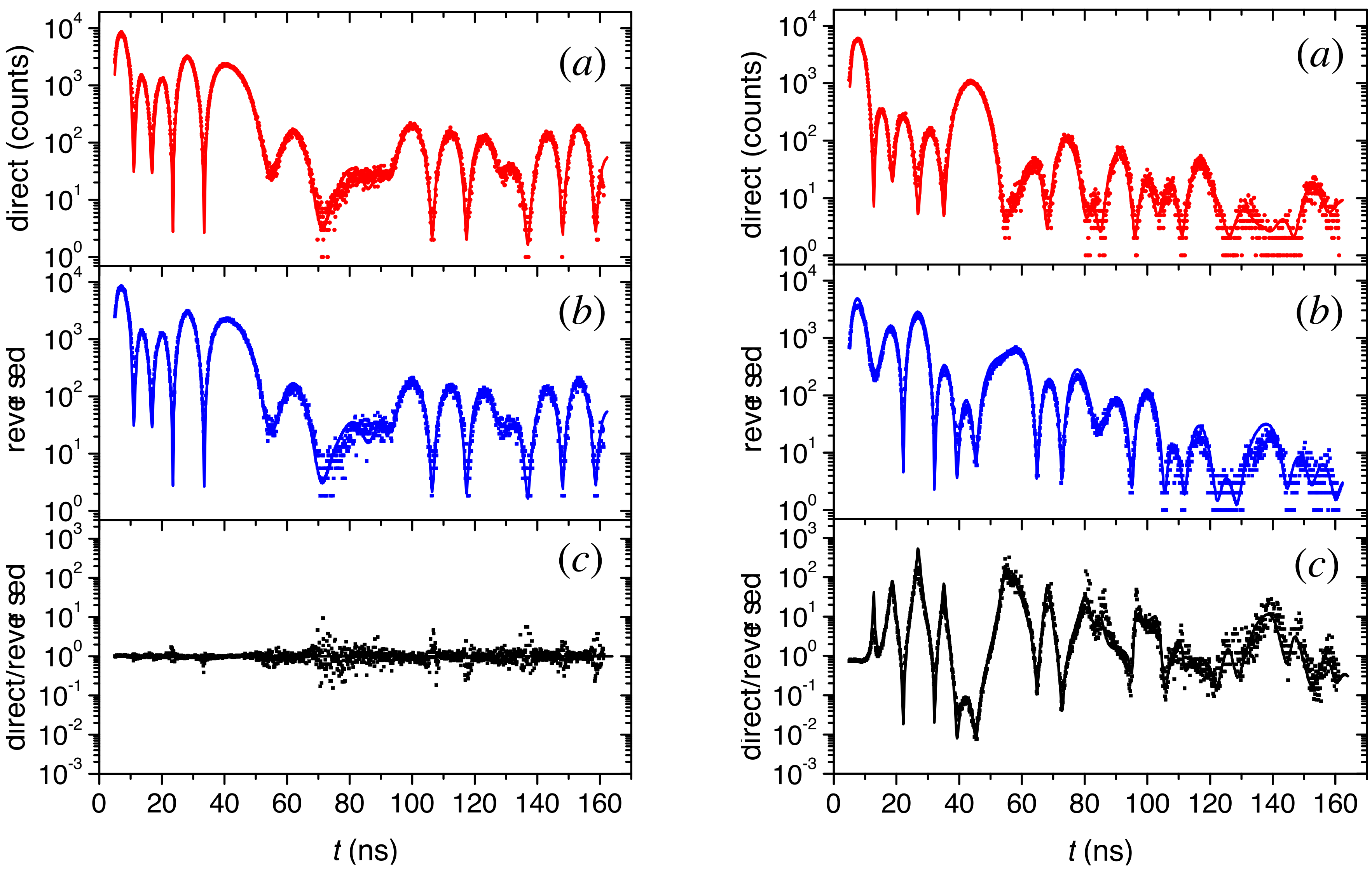
there is considerable absorption $\Longrightarrow$ no T invariance

choosing layers appropriately $\Longrightarrow$ no R_{180° invariance

Tuning the relative position/orientation of layers and the polarization conditions, reciprocity can be maintained or violated at wish

Example energy spectra and time spectra:


original
reversed
(a)
original
reversed
(b)
Intensity (arb. units)
-10
-5
0
5
10
v (mm/s)



(a)
(b)
(c)
direct (counts)
reversed
direct/reversed
t (ns)

Applications of reciprocity:

electric circuits

antenna theory

neutron and x-ray diagnostics

microelectronic devices (on-chip integration)

acoustics

seismology

nuclear physics

particle physics