Non-extensive Boltzmann equation & thermalization

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Content

- Non-extensivity & kinetic equations
- Motivations
- Toy-model (colliding point-particles)
  - Details of kinetics
  - Detailed balance no-go
  - Numerical results about long-time evolution

Goal:

Long-time behaviour of two-component modified kinetic equations:

NESS (Non-equilibrium steady state)

Non-Extensivity

- Why can NOT we just sum up micro-/mesoscopic quantities?

⇒ **INTERACTION**

- Pair-interaction in general:

$$\frac{E}{N} = \frac{1}{N}\sum_{i} E_i + \frac{1}{N}\sum_{ij} V_{ij}$$

- **in TDL:**

Non-Extensivity

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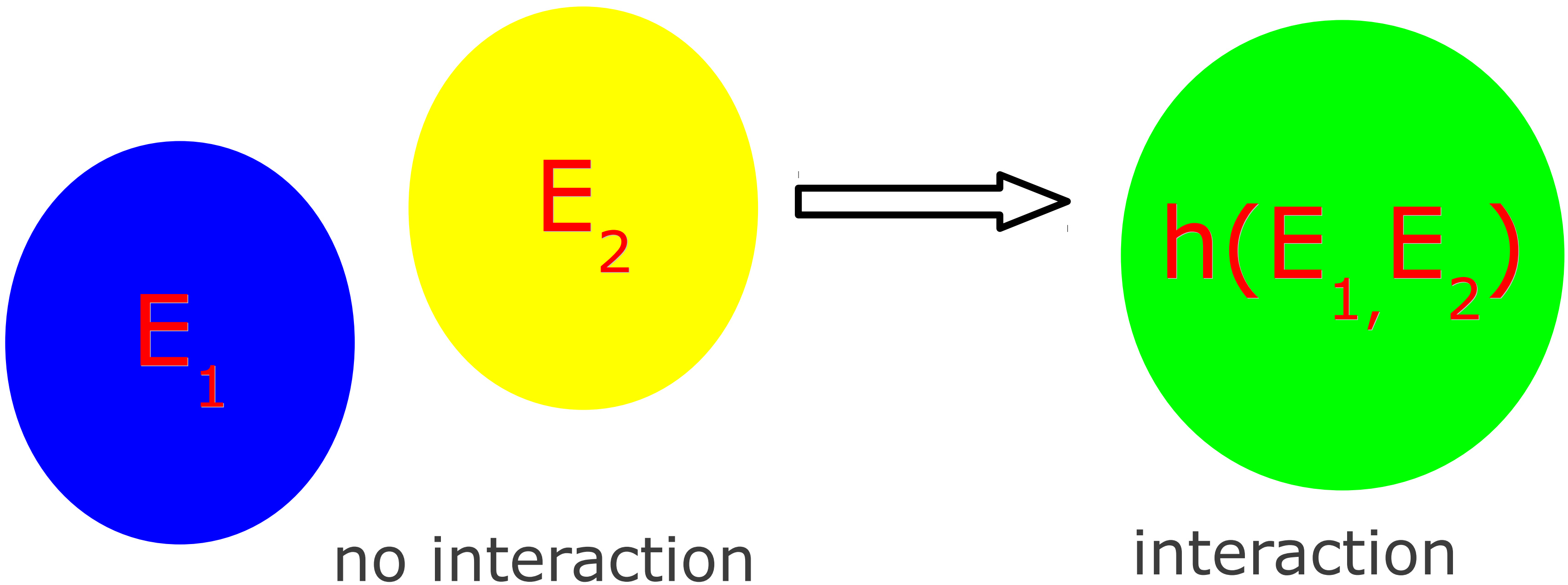
small: weak interaction (few pairs)

$\sim N^{\delta-1}$ big: strong interaction (a lot of pairs)

Non-Extensivity



Trick to use additive quantities

(if h is associative):

$$h(E,E') = K \Rightarrow L(E) + L(E') = L(K) = const.$$

$$L(E) = \int_0^E \frac{\mathrm{d}\epsilon}{\partial_2 h(\epsilon,0)}$$

Kinetic equations

$$\dot{f}_1^\alpha = \sum_\beta \iiint_{234} \mathcal{W}_{1234}^{\alpha\beta}(f_3^\alpha f_4^\beta - f_1^\alpha f_2^\beta)$$

- symmetries: $1234 \leftrightarrow 3412, \; 1234 \leftrightarrow 2134$

$$\alpha\beta \leftrightarrow \beta\alpha$$

- constraints: 2-p. (quasi-)energy, 2-p. total momentum

$$\mathcal{W}_{1234}^{\alpha\beta} = \omega_{1234}^{\alpha\beta}\delta(h(E_1, E_2) - h(E_3, E_4))\delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$

- conserving quantities:

  total momentum, number of particles

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Kinetic equations & entropy growth

$$S = -\sum_{\alpha} \int_1 f_1^{\alpha} \ln f_1^{\alpha}$$

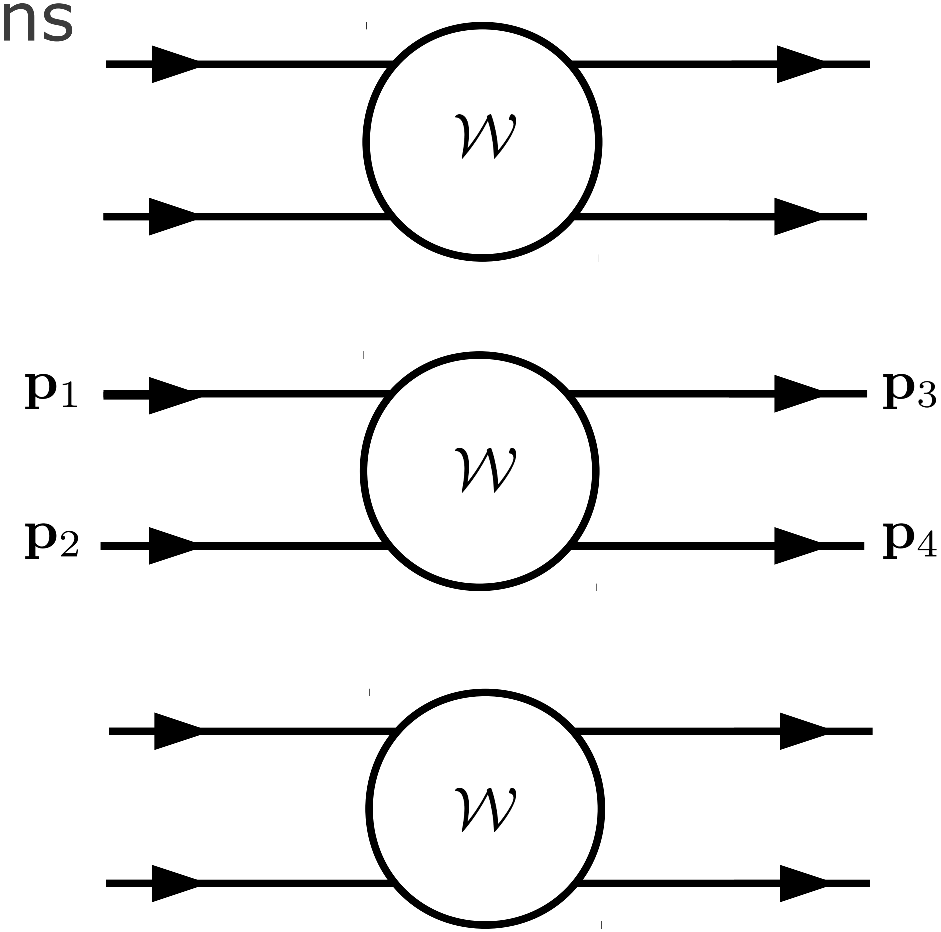
- H-theorem: S is increasing monotonically
- approaches its maximum in detailed balance (if exists):

$$f^{\alpha}(E_1) f^{\beta}(E_2) = f^{\alpha}(E_3) f^{\beta}(E_4)$$

$$L^{\alpha\beta}(E_1) + L^{\alpha\beta}(E_2) = L^{\alpha\beta}(E_3) + L^{\alpha\beta}(E_4)$$

Motivations

- power law tails in spectra
- not independent collisions in dense plasma
- modified dispersion relation



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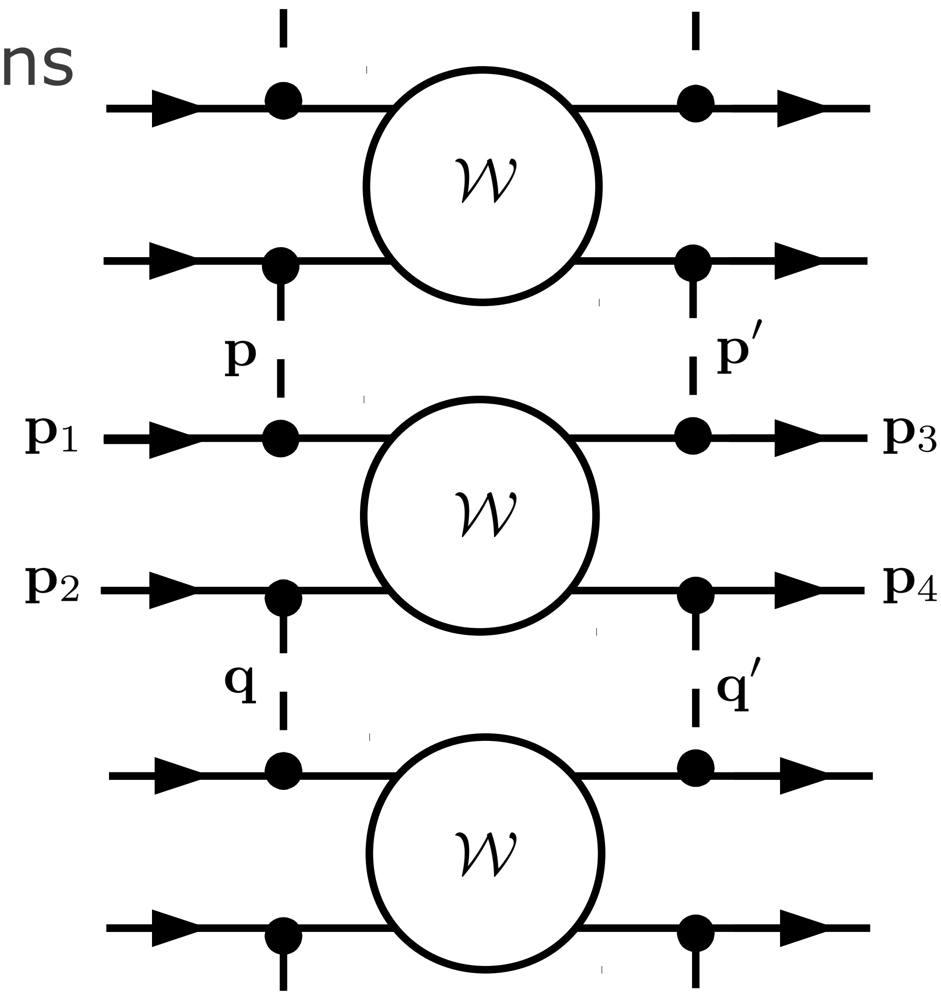
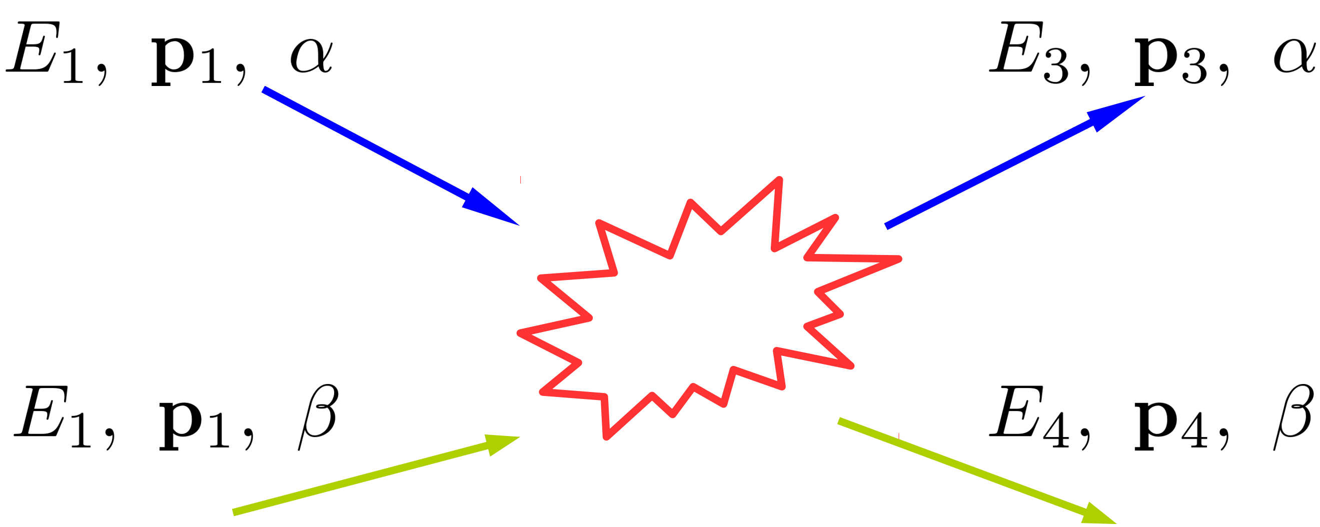


A toy-model – construction

- particles $\{E_i, \mathbf{p}_i, \alpha_i\}$
- interaction via binary collisions
- dispersion relation: $E = c|\mathbf{p}|^n$
- collisional invariants: $$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_3 + \mathbf{p}_4$$ $$h(E_1, E_2) = h(E_3, E_4)$$



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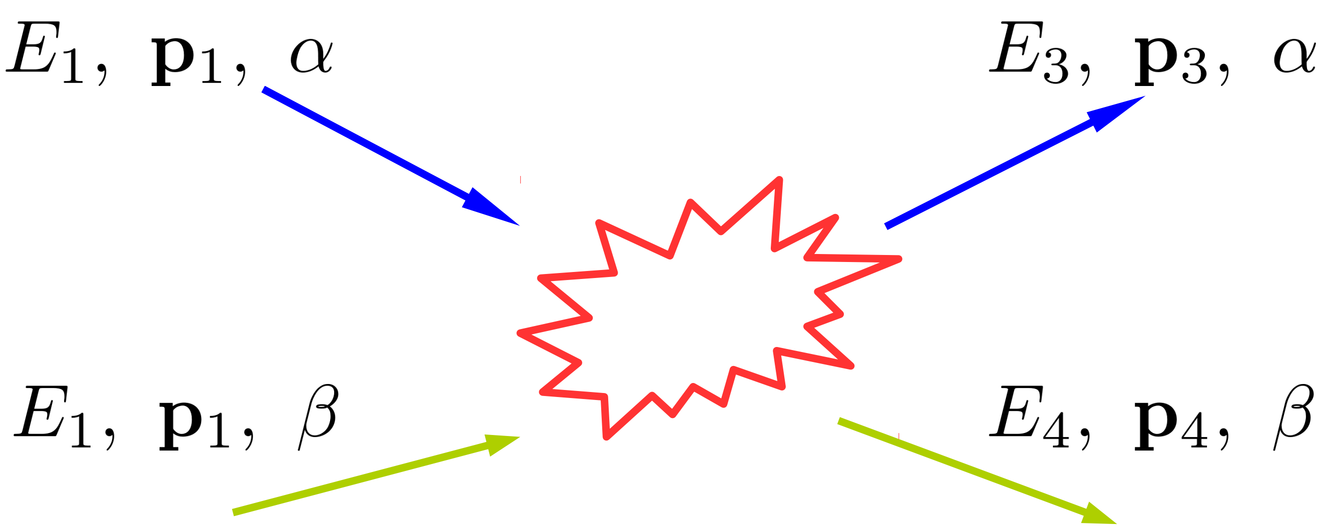

A toy-model – kinetic equation

continuum limit: $(N \to \infty, \;\; \Delta(E, E', \epsilon) \to 0)$

$$\dot{f}_1^{\alpha} = \sum_{\beta} \iiint_{234} \mathcal{W}_{1234}^{\alpha\beta} (f_3^{\alpha} f_4^{\beta} - f_1^{\alpha} f_2^{\beta})$$

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$$\mathcal{W}_{1234}^{\alpha\beta} = \rho^{\alpha\beta}(E_3, E_4, K)\delta(\underbrace{h^{\alpha\beta}(E_1, E_2)}_{K} - h^{\alpha\beta}(E_3, E_4))\delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$

$$\dot{f}^\alpha(E) = \sum_\beta \int_0^\Lambda \mathrm{d}E' \int_0^K \mathrm{d}\epsilon \left(f^\alpha(\epsilon) f^\beta(K \ominus \epsilon) - f^\alpha(E) f^\beta(E') \right) \rho^{\alpha\beta}$$

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modified

$$\dot{f}^{\alpha}(E) = \sum_{\beta} \int_0^{\Lambda} \mathrm{d}E' \int_0^{K} \mathrm{d}\epsilon \left(f^{\alpha}(\epsilon) f^{\beta}(K \ominus \epsilon) - f^{\alpha}(E) f^{\beta}(E') \right) \rho^{\alpha\beta}$$

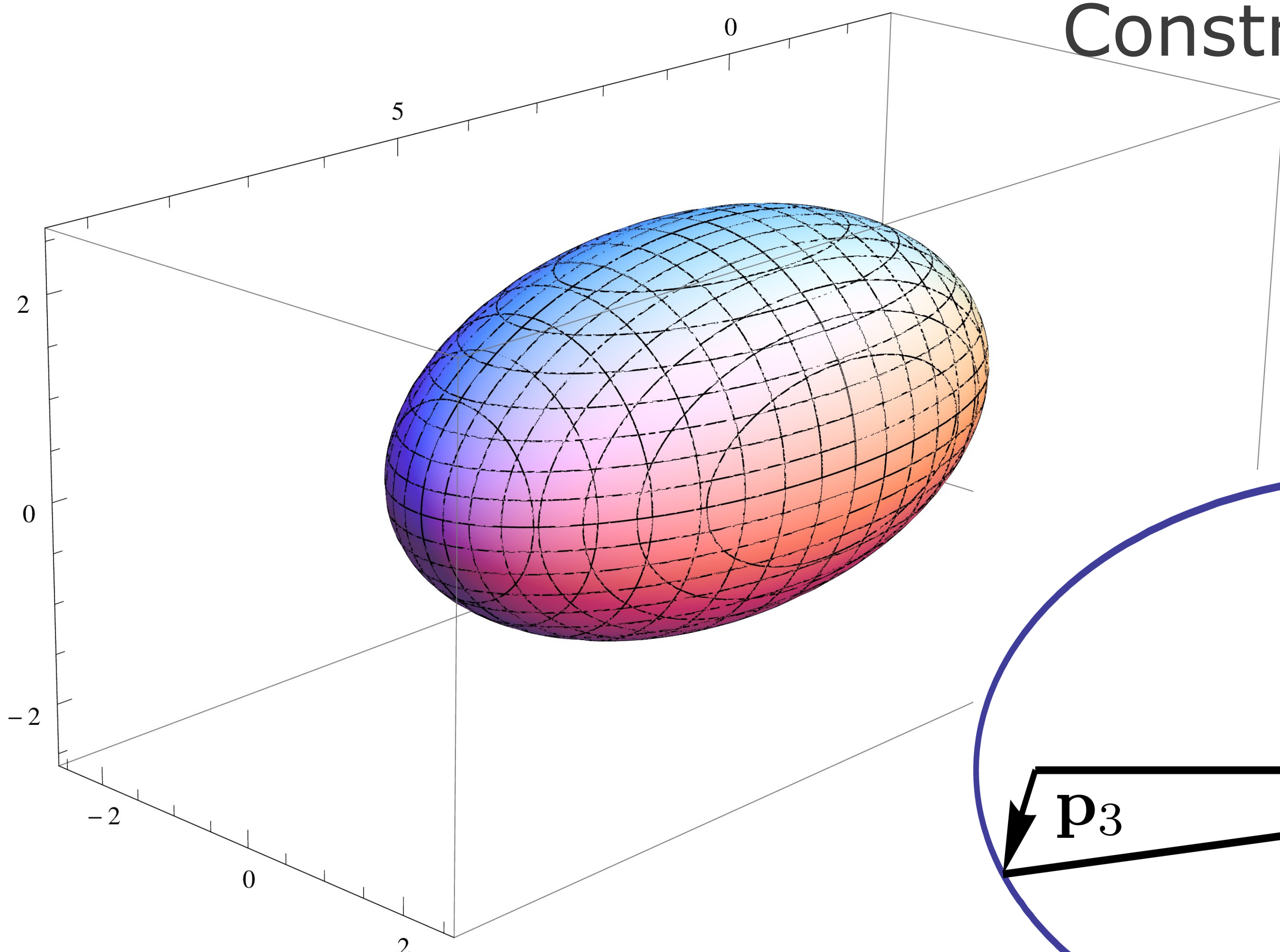
A toy-model – simulation

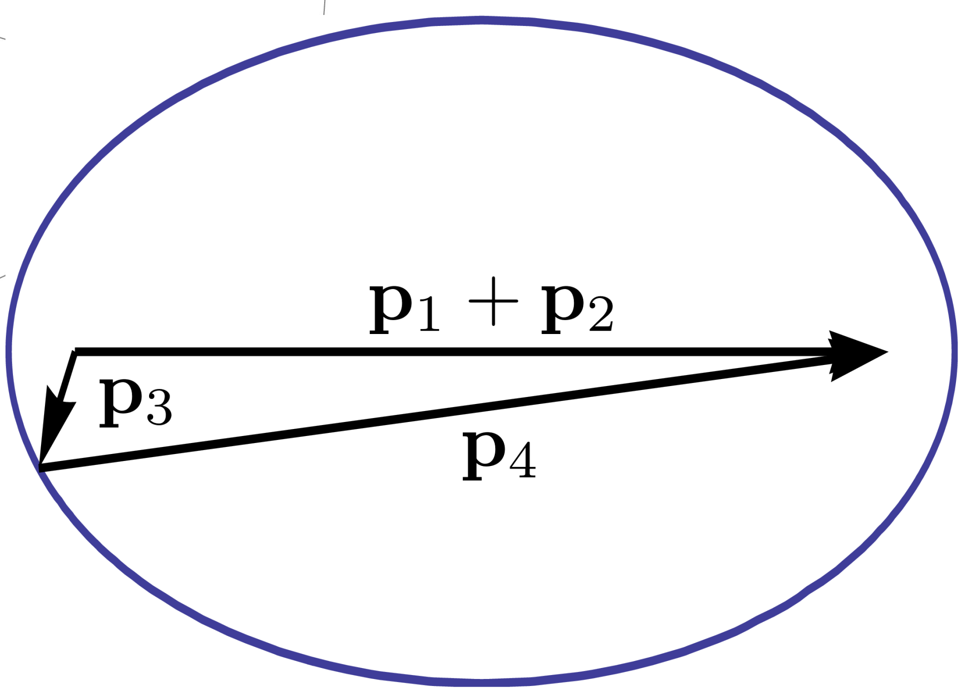
Numerical implementation:

- ensemble described by $\{(\mathbf{p}_i, E_i = c|\mathbf{p}|^n), i = 1 \ldots N\}$
- collision-by-collision evolution, NOT in real time
- random sampling on the constrained phase space
- random variables:
    - *1 outgoing energy*
    - *angle between incoming & outgoing planes*

A toy-model – simulation

Constrained surfaces





A toy-model – simulation

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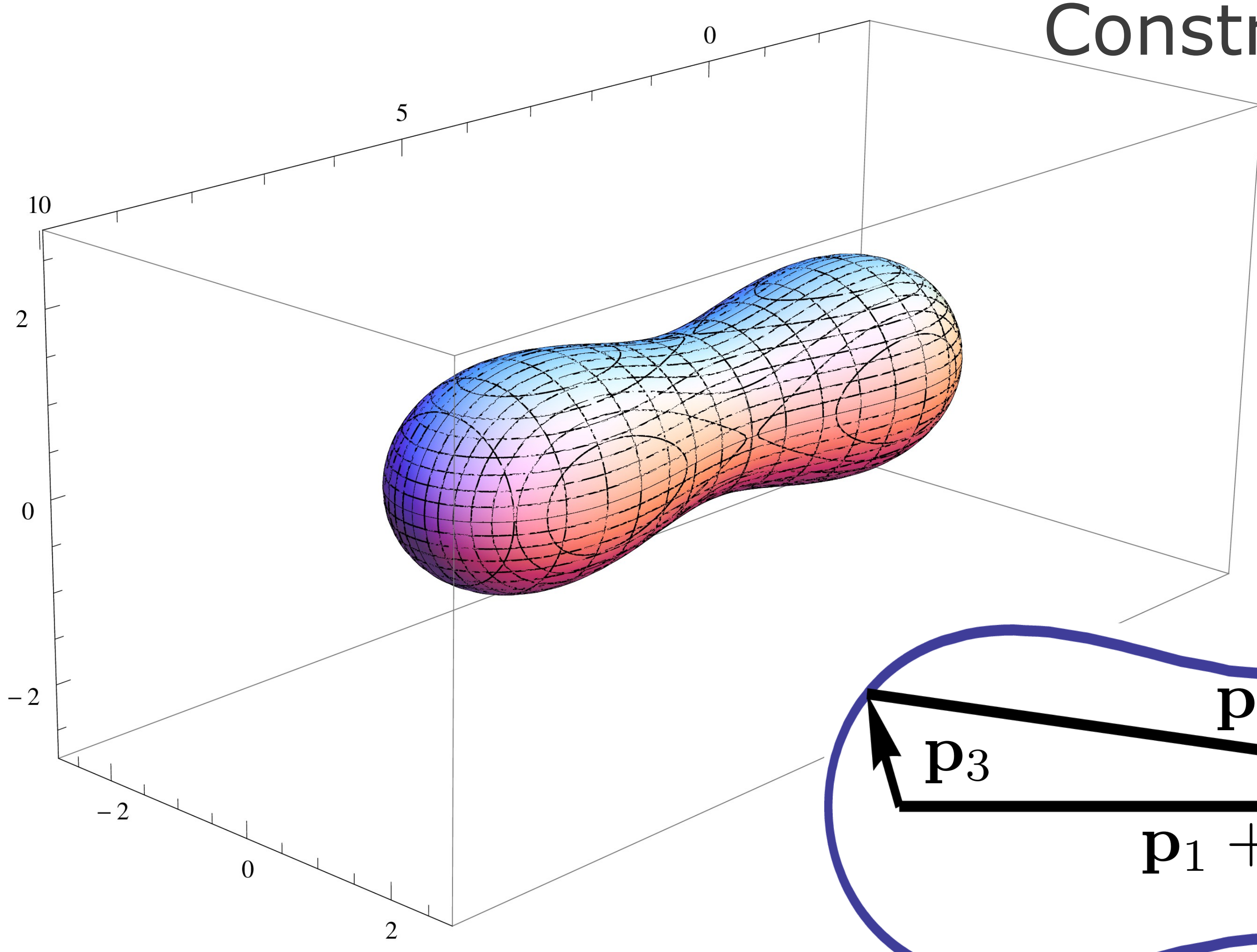

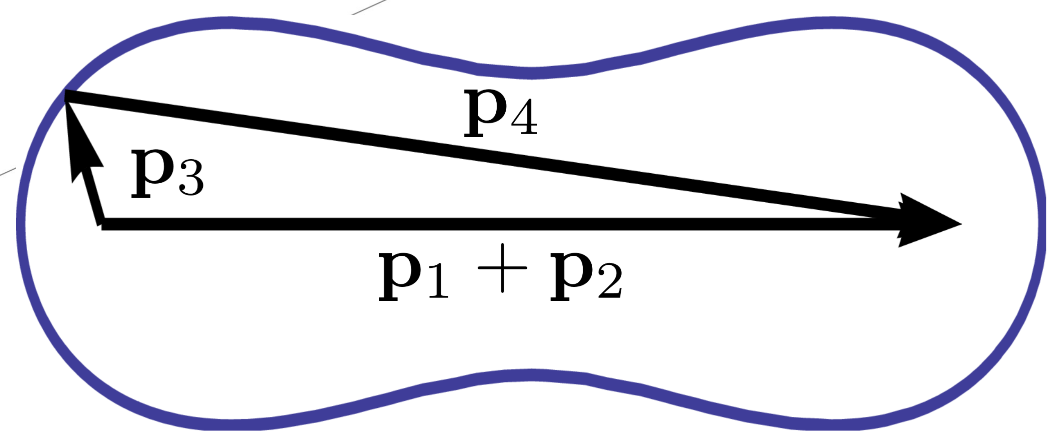

A toy-model – detailed balance

$$\dot{f}_1^{\alpha} = \sum_{\beta} \iiint_{234} \underbrace{\mathcal{W}^{\alpha\beta}_{1234}(f_3^{\alpha} f_4^{\beta} - f_1^{\alpha} f_2^{\beta})}_{=0}$$

$$f^{\alpha}(E_1) f^{\beta}(E_2) = f^{\alpha}(E_3) f^{\beta}(E_4)$$

$$h^{\alpha\beta}(E_1, E_2) = h^{\alpha\beta}(E_3, E_4)$$

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$$L^{\alpha\beta}(E_1) + L^{\alpha\beta}(E_2) = L^{\alpha\beta}(E_3) + L^{\alpha\beta}(E_4)$$

n+n(n-1)/2 independent equations

A toy-model – detailed balance

for $\alpha = \beta$

$$f^{\alpha}(E_1)f^{\alpha}(E_2) = f^{\alpha}(E_3)f^{\alpha}(E_4)$$

$$L^{\alpha\alpha}(E_1) + L^{\alpha\alpha}(E_2) = L^{\alpha\alpha}(E_3) + L^{\alpha\alpha}(E_4)$$

$$\Longrightarrow \quad f^{\alpha} \sim e^{-\gamma_{\alpha\alpha} L^{\alpha\alpha}(E)}$$

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$\Rightarrow$

$$f^{\alpha} \sim e^{-\gamma_{\alpha\alpha} L^{\alpha\alpha}(E)}$$

$$f^{\alpha}(E_1)f^{\beta}(E_2) = f^{\alpha}(E_3)f^{\beta}(E_4)$$

$$\gamma_{\alpha\alpha}L^{\alpha\alpha}(E_1) + \gamma_{\beta\beta}L^{\beta\beta}(E_2) = \gamma_{\alpha\alpha}L^{\alpha\alpha}(E_3) + \gamma_{\beta\beta}L^{\beta\beta}(E_4)$$

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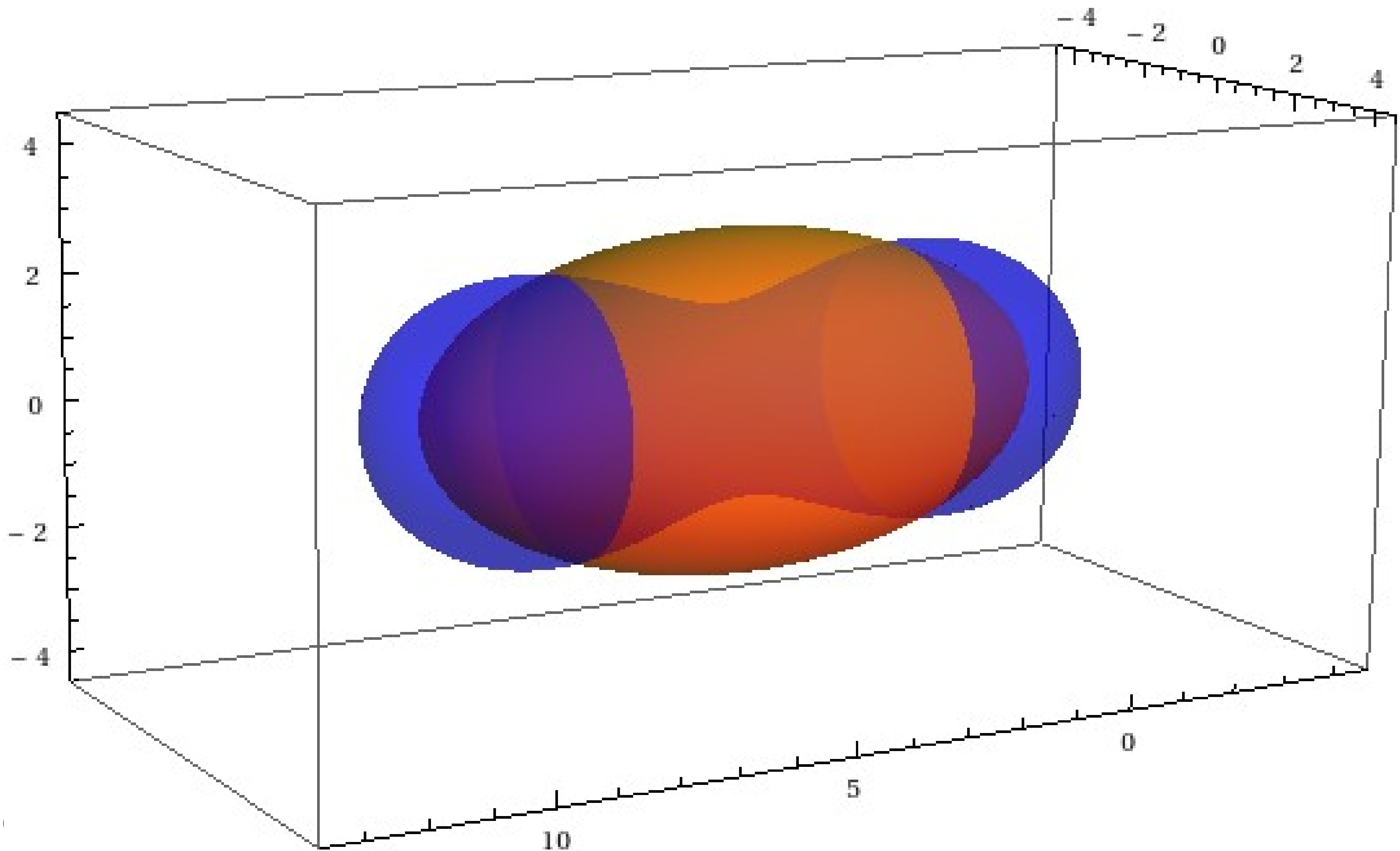
$$\gamma_{\alpha\alpha} L^{\alpha\alpha}(E_1) + \gamma_{\beta\beta} L^{\beta\beta}(E_2) = \gamma_{\alpha\alpha} L^{\alpha\alpha}(E_3) + \gamma_{\beta\beta} L^{\beta\beta}(E_4)$$

AND for $\alpha \neq \beta$

$$L^{\alpha\beta}(E_1) + L^{\alpha\beta}(E_2) = L^{\alpha\beta}(E_3) + L^{\alpha\beta}(E_4)$$

A toy-model

d=3, n=2: closed curves



A toy-model

Lot of constraints...

$$L^{\alpha\beta}(E_1) + L^{\alpha\beta}(E_2) = L^{\alpha\beta}(E_3) + L^{\alpha\beta}(E_4) \quad \alpha, \beta = 1 \ldots n$$

cannot be fulfilled for every pairs

⇒ detailed balance does not exist

⇒ no global conservations

Long time behaviour?

A toy-model – numerical results

- collisional invariants:

$$E_1 + E_2 + \frac{a^{\alpha\beta}}{\langle E \rangle^n} E_1 E_2 = E_3 + E_4 + \frac{a^{\alpha\beta}}{\langle E \rangle^n} E_3 E_4$$

- scaling distributions (NESS): $f^\alpha(E) = \frac{1}{\langle E \rangle} \phi(E/\langle E \rangle)$

$$\phi(x) \sim (1 + Ax)^{-B}$$

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determined by initial values

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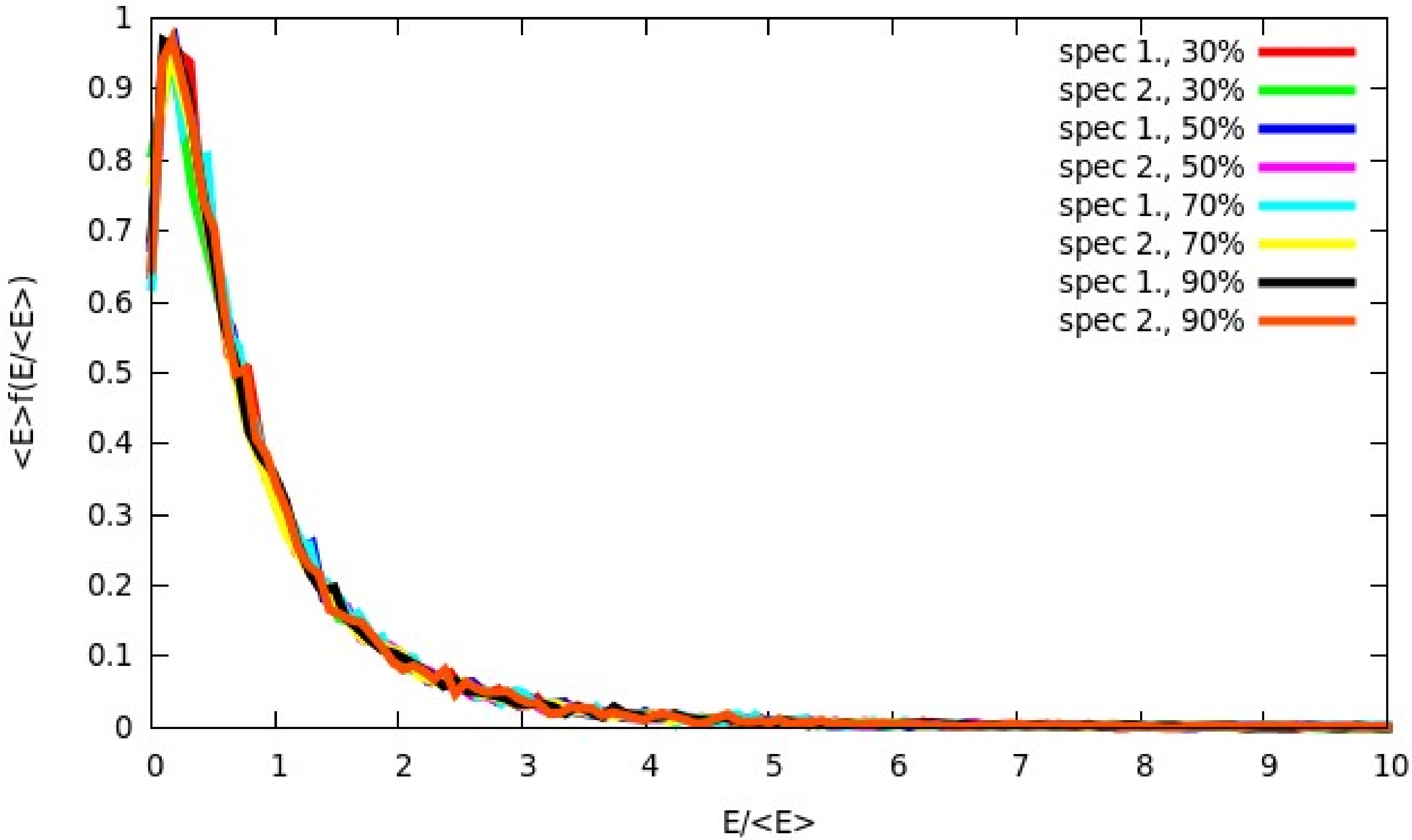

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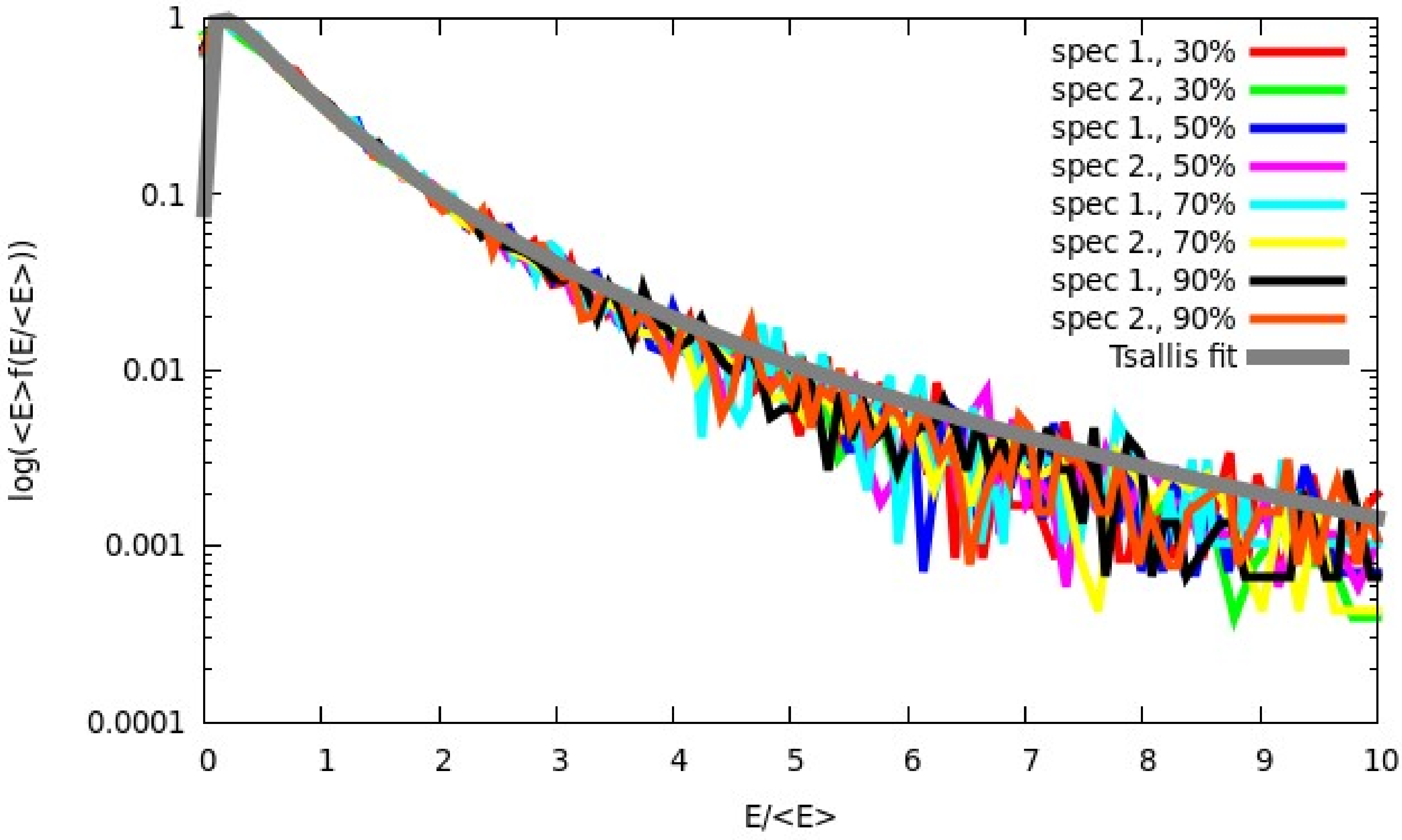

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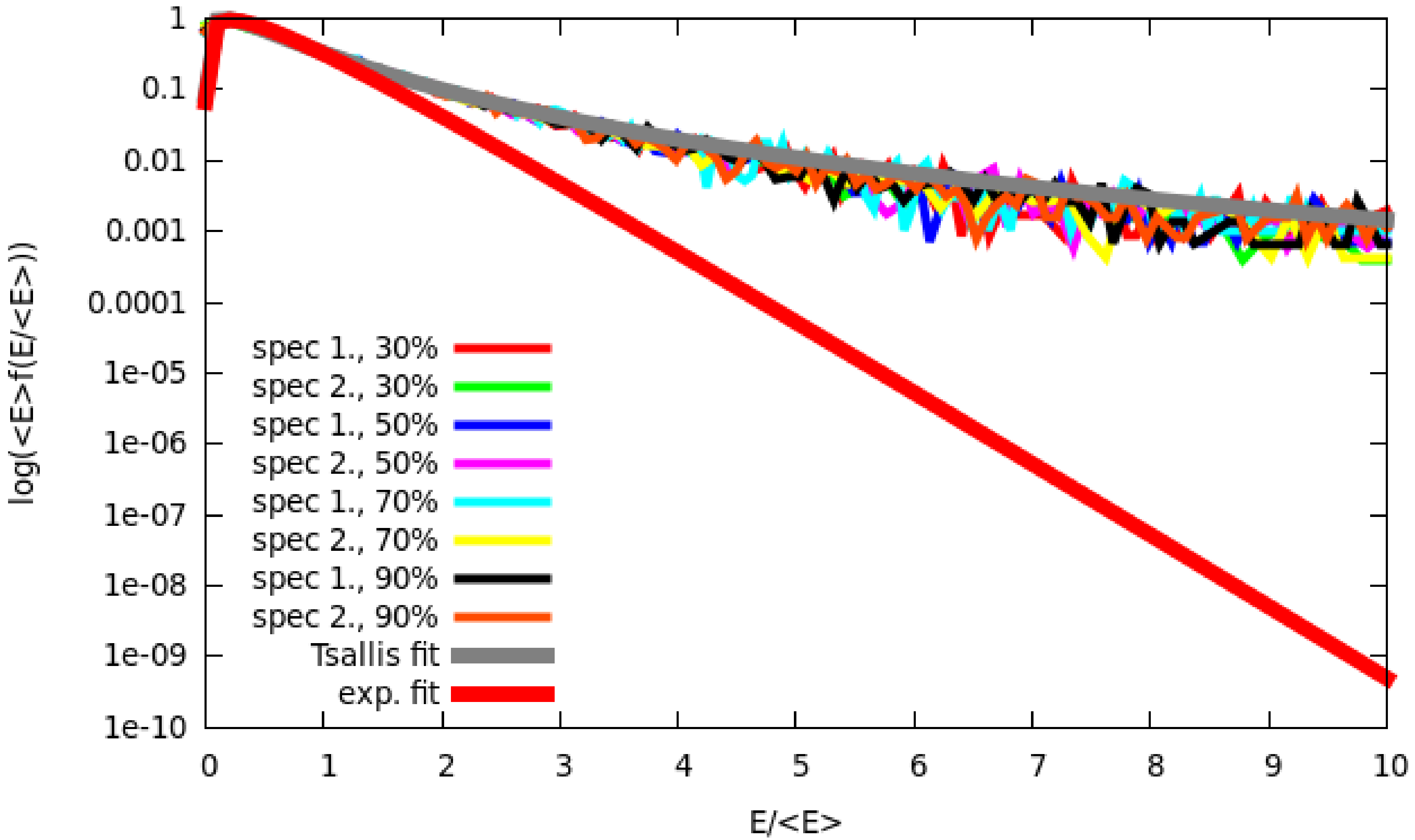

A toy-model – a ~ 1/<E>

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\langle E\rangle = \sum_{\alpha}\int_0^{\Lambda}\mathrm{d}E\,\dot{f}^{\alpha}(E)E = \langle E\rangle\sum_{\alpha\beta}I^{\alpha\beta}$$

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<E> scaled out: no time dependence

$\Rightarrow$ $\langle E\rangle \sim e^{\gamma\tau}$

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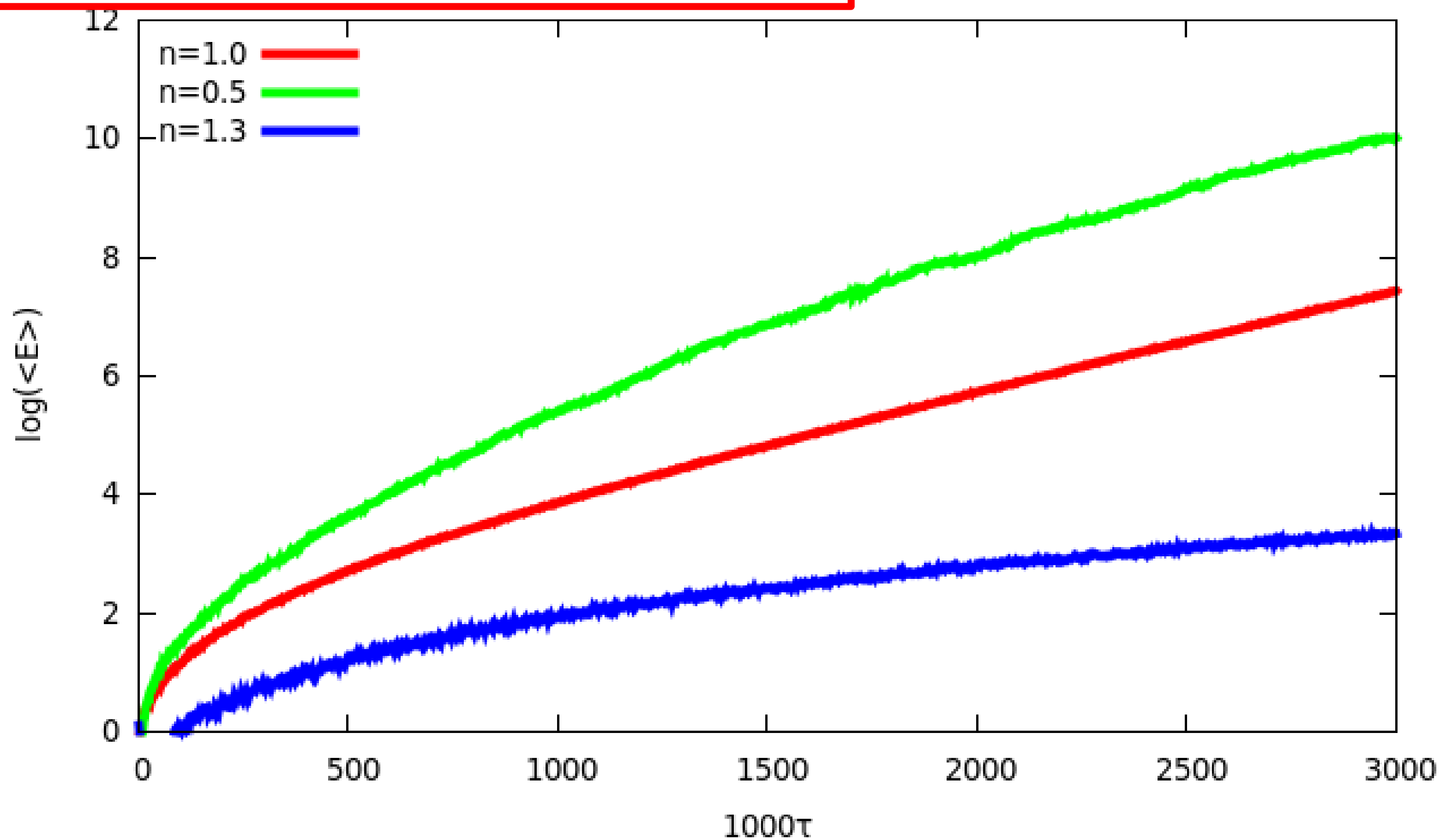
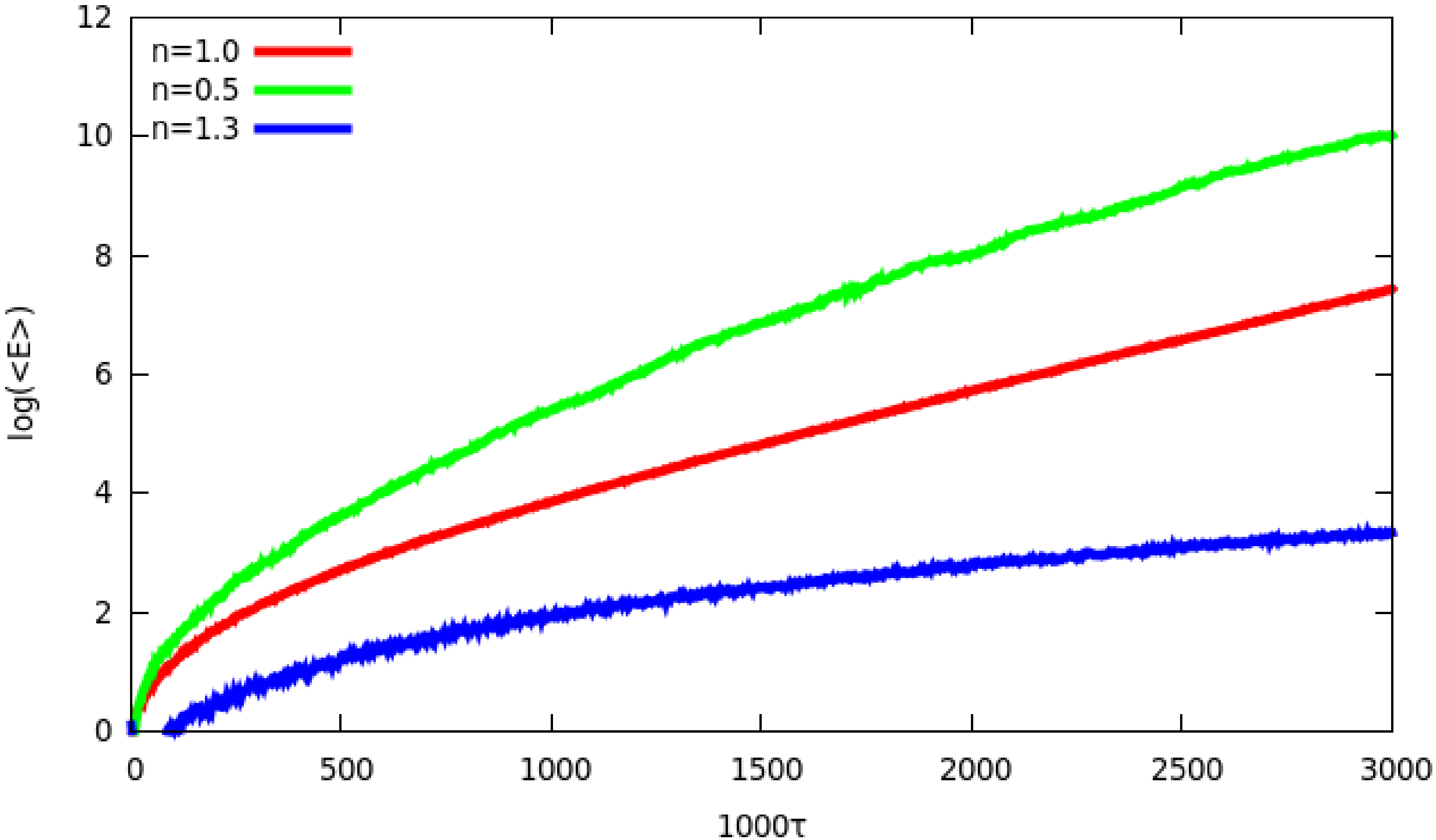


A toy-model – a ~ 1/<E>

$$S_{en} = -\sum_{\alpha} \int_0^{\Lambda} \mathrm{d}E f^{\alpha}(E) \ln f^{\alpha}(E) = \mathrm{const.} + \ln(\langle E \rangle)$$



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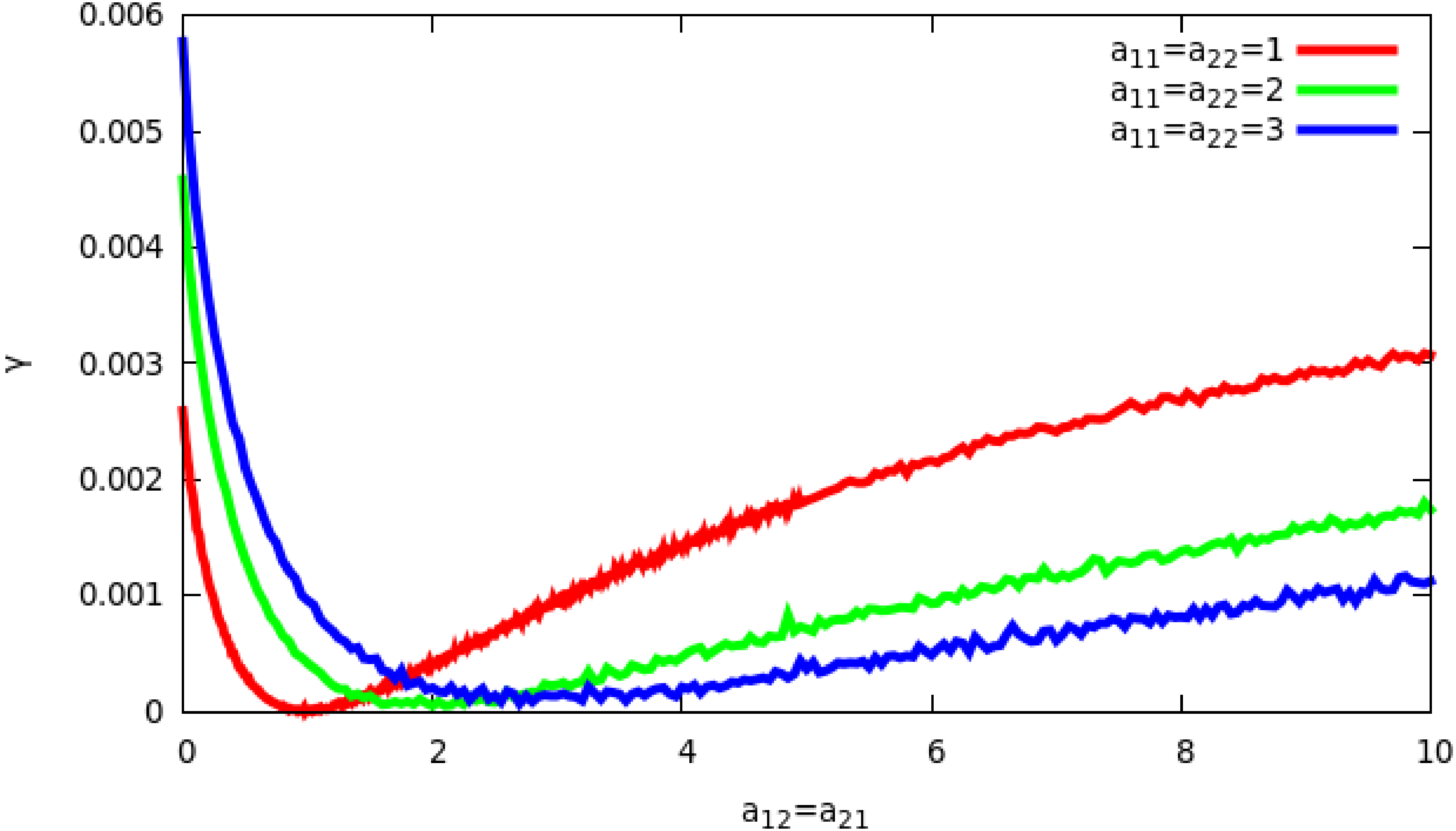

Conclusions & summary

- Simple kinetic toy-model with non-additive coll. invariants:
  - two (or more) component: frustration (disappears continuously) (balance states will be different for each component)
  - no detailed balance, but NESS in long time
  - thermal description: heating system (driving)
  - stable equation of states

- Motivation for energetically non-extensive models:
  - dense plasmas,
  - dealing with >2-part. collisions in a mean-field way

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Thank you for your attention!

Questions?