A solvable toy model for QED at finite T

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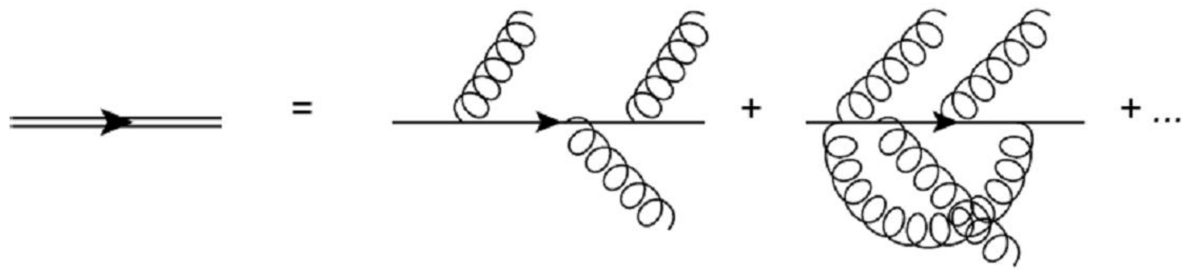
Contents

The Bloch-Nordsieck Model

Photon Mass= 0

- Long range interactions $\longrightarrow$ No free charged fermion propagator, ”photon cloud, quasi particle”



- From any (finite) EM energy infinite number of photons can be created

$$N = \lim_{\nu \to 0} \frac{I(\nu)}{h\nu} = \infty \text{ since } I(0) \neq 0$$

We need to sum over all possible photon contribution! (even $\nu \to 0$ when!)

The Bloch-Nordsieck Model

B-N solution

- An exact solution by path integral
- Hard to generalize to more complex theory

Perturbation Theory

- 1-loop correction
- Breaks down in IR

Resummations

- 2PI, Schwinger-Dyson
- Works in IR regime

A method to *treat infrared physics (?)*

The B-N Model

QED

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m - eA\!\!\!/\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$$

$$\partial\!\!\!/ = \gamma_{\mu}\partial^{\mu}$$

m: fermion mass, e: coupling constant

Bloch-Nordsieck

$$\mathcal{L} = \psi^{\dagger}\left(iu^{\mu}\partial_{\mu} - m - eu^{\mu}A_{\mu}\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$u^{\mu} \in M^4$ and $u_{\mu}u^{\mu} = 1$

- NO spinflips
- NO positrons

↓

BN = scalar field theory

The B-N Model

Free theory

$$(iu^\mu \partial_\mu - m)G_0(x-y) = \delta(x-y) \qquad \text{(EOM)}$$

The propagator in momentum space:

(retarded)

$$\hat{G}_0(p) = \frac{1}{u_\mu p^\mu - m + i\epsilon} \longleftrightarrow G_0(x-y) = 0 \text{ if } x^0 < y^0$$

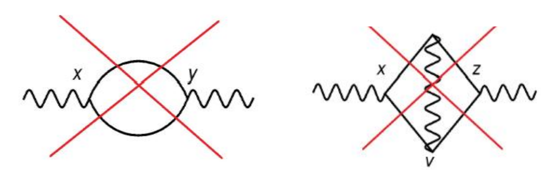
Interacting case

$$[u^\mu (i\partial_\mu + eA_\mu(x)) - m]G(x,y|A) = \delta(x-y) \qquad \text{(EOM)}$$

$$G(x,y) = \frac{\int G(x,y|A)\langle T \exp\left\{ie\int \bar{\psi}(z)A(z)\psi(z)dz\right\}\rangle_{F_0}\mathcal{D}A}{\int \langle T \exp\left\{ie\int \bar{\psi}(z)A(z)\psi(z)dz\right\}\rangle_{F_0}\mathcal{D}A}$$

$$\hat{G}(up) = \frac{1}{(up-m)^{1+\gamma}} = \frac{1}{up-m}e^{-\gamma \ln(up-m)}$$

No antifermions



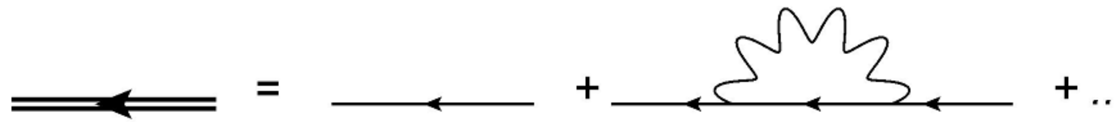
$$\gamma = \frac{e^2(3-\xi)}{8\pi^2} = \frac{\alpha(3-\xi)}{2\pi}$$

ξ is gauge fixing parameter

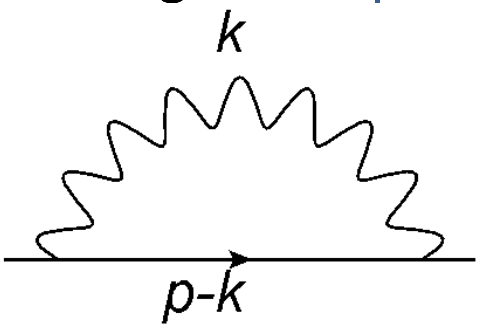
Perturbation Theory

1-loop

Expand *by the coupling* :



Divergent loop-integral (in Feynman gauge, ξ=1):



$$\equiv (-ie)^2 \int \frac{d^4k}{(2\pi)^4} u^\mu \frac{i}{(p^\alpha - k^\alpha)u_\alpha - m} u^\nu \frac{-ig_{\mu\nu}}{k^2} = -i\Sigma_{1l}$$

- The self-energy of the fermion
- Dimensional regularization
- Special frame *u=(1,0,0,0)*

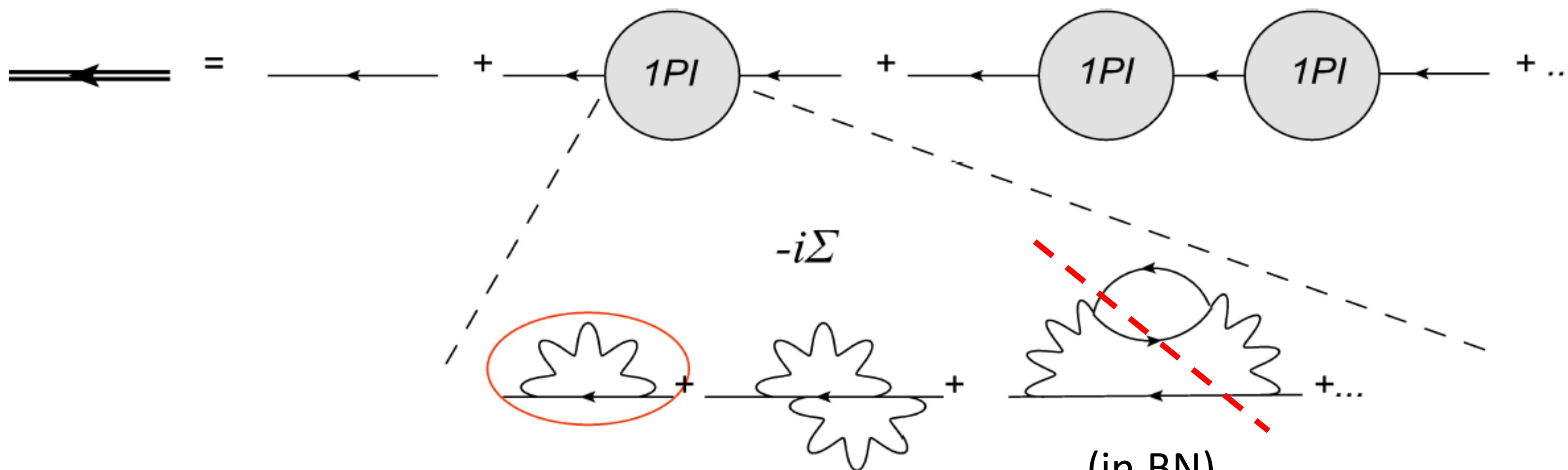
(+ UV RENORM.)

$$\Sigma_{1l}^{ren} = \Sigma_{1l} - \Sigma_{ct} = -\frac{\alpha}{\pi}(p^0 - m)\ln\frac{m - p^0}{\mu} \qquad (< \infty)$$

Perturbation Theory 1-loop

Dyson-equation $\longrightarrow$ geometric series

$$i\hat{G}(p^0) = \frac{i}{G_0^{-1} - \Sigma}$$



1-loop propagator

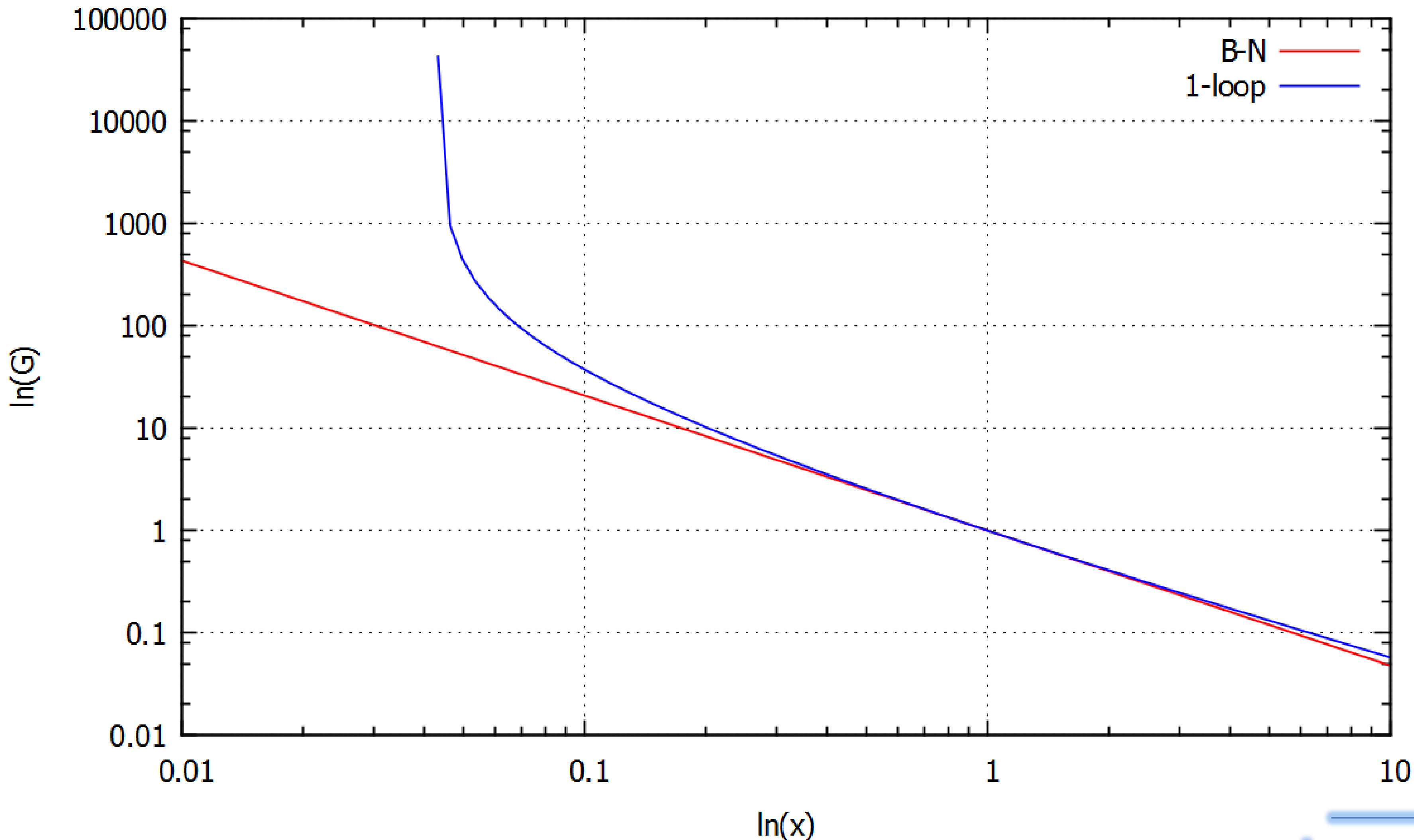
$$G^{1l} = \frac{1}{(up-m) + \gamma(up-m)\ln\frac{up-m}{\mu}} = \frac{1}{up-m}\left(1 - \gamma \ln\frac{up-m}{\mu} + \dots\right)$$

Bloch-Nordsieck propagator

$$G^{BN}(up) = \frac{1}{up-m} e^{-\gamma \ln(up-m)} = \frac{1}{up-m}(1 - \gamma \ln(up-m) + \dots)$$

$$\left(\gamma = \frac{\alpha}{\pi}\right)$$

Perturbation Theory 1-loop



$$G^{1l}(x) = \frac{1}{x(1+\gamma \ln x)}$$

$$G^{BN}(x) = \frac{1}{x^{1+\gamma}}$$

α=1

$(x := up - m)$

$(\gamma = \frac{\alpha}{\pi})$

Perturbation Theory 1-loop

Why is that?

$$G^{BN}(up) = \frac{1}{up - m} e^{-\gamma \ln(up-m)} = \frac{1}{up - m} \sum_{n=0}^{\infty} (-1)^n \frac{(\gamma \ln(up - m))^n}{n!}$$

$R = \infty$

$$G^{1l} = \frac{1}{(up - m)(1 + \gamma \ln(up - m))} = \frac{1}{up - m} \sum_{n=0}^{\infty} (-1)^n (\gamma \ln(up - m))^n$$

$|\gamma \ln(up - m)| < 1$

The Dyson-series doesn't converge everywhere

PT: BREAKS DOWN IN IR

Resummations

1PI Effective Action

$$Z[J] = \int \mathcal{D}\phi e^{i(S[\phi]+J^i\phi_i)} = e^{iW[J]}$$

$$i\frac{\delta W[J]}{\delta(iJ_i)} = \langle\phi^i\rangle \equiv \Phi^i$$

$$i\frac{\delta^2 W[J]}{\delta(iJ_i)\delta(iJ_j)} = \langle T\phi^i\phi^j\rangle_c \equiv G^{ij}$$

$$\Gamma^{1PI}[\Phi] = W[J] - J_i\frac{\delta W}{\delta J_i} = W[J] - J_i\Phi^i$$

$$Z[J] = e^{i(\Gamma^{1PI}[\Phi]+J_i\Phi^i)}$$

Gener. functional of 1PI / 2PI diagrams

$$\left(\phi^i = \{\phi(x), A^a_\mu(x), \psi^a(x), \ldots\}\right)$$

2PI Effective Action

$$Z[J,K] = \int \mathcal{D}\phi e^{i(S[\phi]+J^i\phi_i+\frac{1}{2}\phi^i K_{ij}\phi^j)} = e^{iW[J,K]}$$

$$i\frac{\delta W[J,K]}{\delta(iJ_i)} \equiv \Phi^i$$

$$i\frac{\delta W[J,K]}{\delta(iK_{ij})} \equiv \frac{1}{2}(\Phi^i\Phi^j + iG^{ij})$$

$$\Gamma^{2PI} = W[J,K] - J^i\Phi_i - \frac{1}{2}K_{ij}(\Phi^i\Phi^j + iG^{ij})$$

$$\Gamma^{2PI} = S_0[\Phi] + ic\mathrm{Tr}[\ln G^{-1} + G_0^{-1}G - 1] - i\Gamma_{int}[\Phi, G]$$

$$\frac{\delta\Gamma^{2PI}(\Phi_s, G_s)}{\delta G} = 0 \rightarrow G_s^{-1} = G_0^{-1} - 2\frac{\delta\Gamma_{int}[\Phi_s, G_s]}{\delta G}$$

$$\Sigma[G] = 2\frac{\Gamma[\Phi, G]}{\delta G}$$

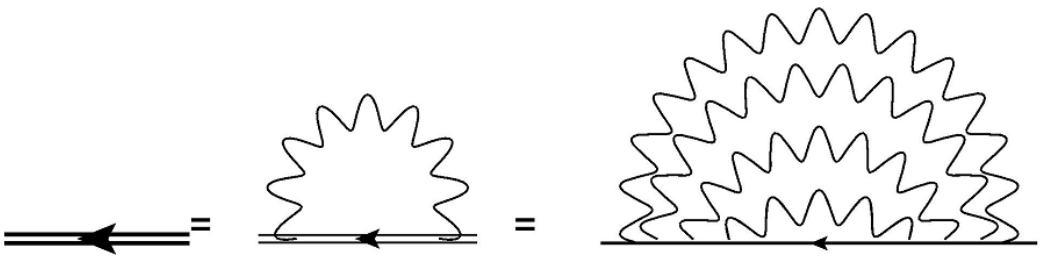
$$G^{-1} = G_0^{-1} - \Sigma[G]$$

Self-consistent Eqs.

Resummations

We need to *handle the IR regime* $\longrightarrow$ new approach: *2PI resummation*

- Summing up the photon-loops (rainbow diagram)
- **Treating *G* as *full propagator***
- "Quasi particle picture"
- Details: *A. Jakovac, Phys. Rev. D76, 125004 (2007). [hep-ph/0612268]*



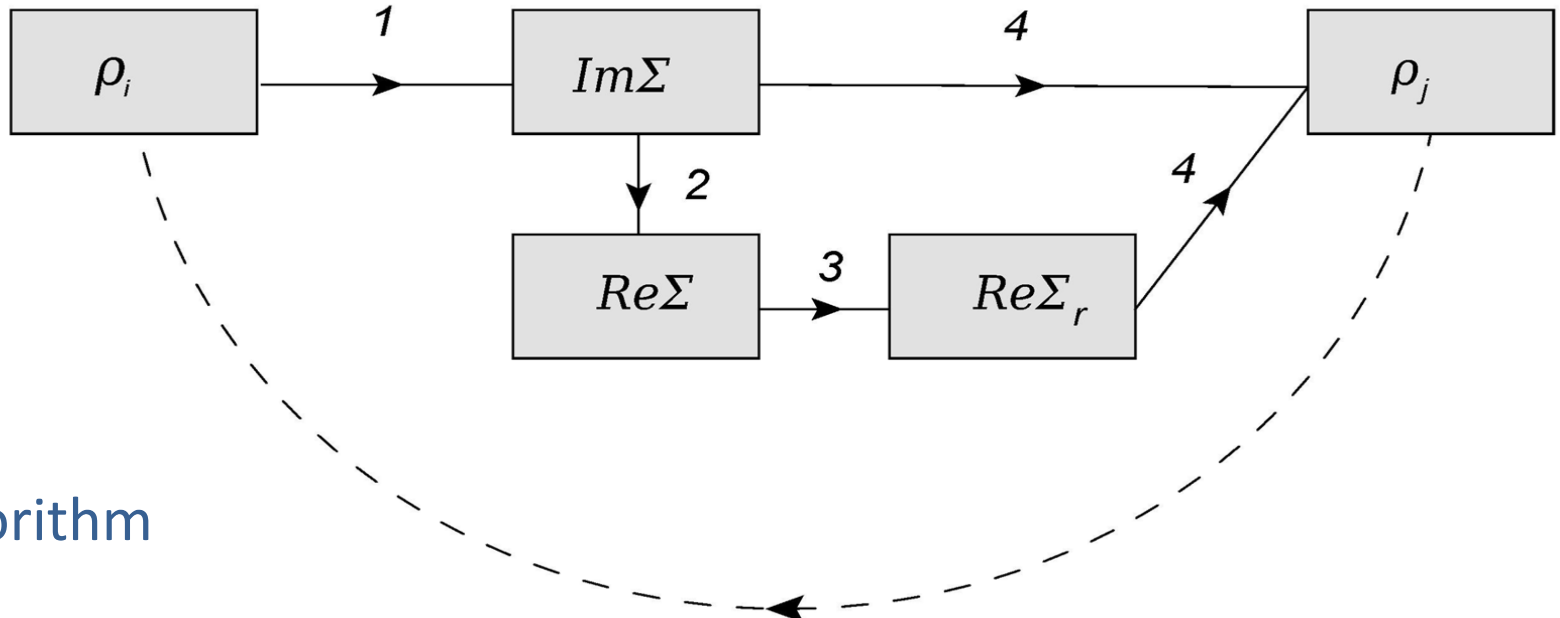
$G[\Sigma] \Leftrightarrow \Sigma[G]$

Self-consistent equations

$$\Sigma(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} G(p-k) \frac{1}{k^2} \qquad G(p) = \frac{1}{G_0^{-1}(p) - \Sigma(p)}$$

Resummations

2PI



The algorithm

1 $\mathrm{Im}\Sigma(p^0) = \frac{1}{4\pi^2}\int_0^{p^0} dk^0\; k^0 \rho^{FR}(p^0 - k^0)$

2 $\mathrm{Re}\Sigma(p^0) = \mathcal{P}\frac{1}{\pi}\int_{-\infty}^{\infty} dq^0\; \frac{\mathrm{Im}\Sigma(q^0)}{q^0 - p^0}$ (Kramers-Kronig)

3 $\mathrm{Re}\Sigma_r(p^0) = \mathrm{Re}\Sigma(p^0) - \left(\mathrm{Re}\Sigma(p_t^0) - \frac{\partial \mathrm{Re}\Sigma(p_t^0)}{\partial p^0}\Big|_{p_t^0} (p^0 - p_t^0)\right)$

4 $\rho^{FR}(p^0) = \frac{2\mathrm{Im}\Sigma_r}{\mathrm{Re}[G_0^{-1} - \Sigma_r]^2 + [\mathrm{Im}\Sigma_r]^2}$, $(\rho(x) = \langle\{\psi(x), \psi^\dagger(0)\}\rangle_0)$

The spectral function

$$\rho(x) = \langle\{\psi(x), \psi^\dagger(0)\}\rangle$$

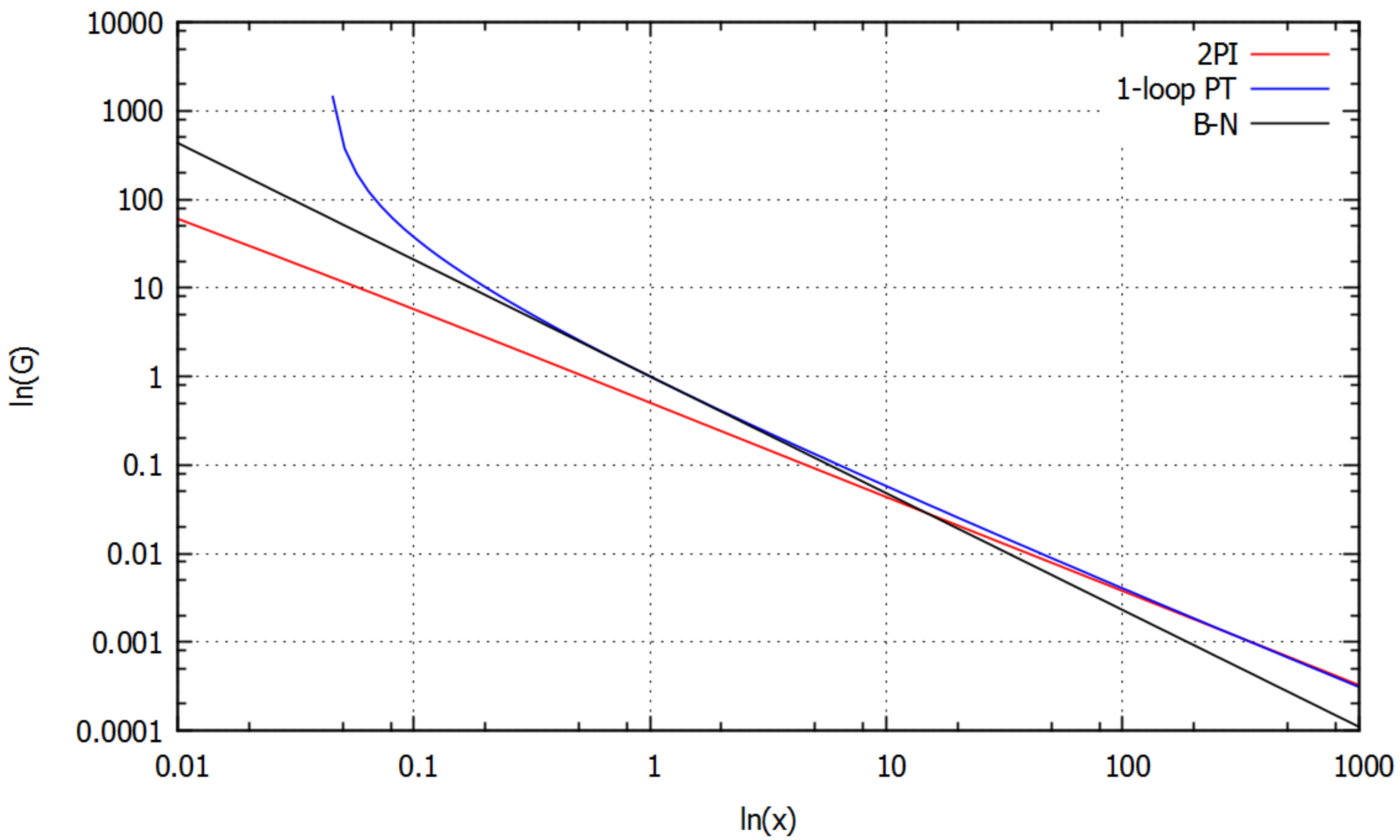
$$\rho(p) = 2i\mathrm{Im}\,G^R(p)$$

It contains the spectral information on the system
E.g. in free case:

$$\rho(p) \propto \delta(up - m)$$

Resummations

The propagator



- Doesn’t fit well
- Works in IR

Resummations

Dyson-Scwinger eq.

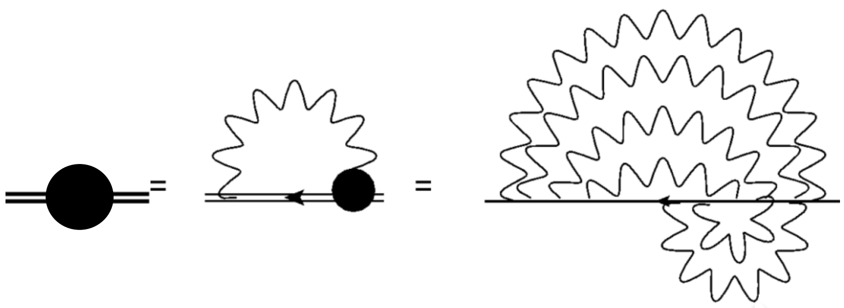
”Modified 2PI ” = 2PI + vertex corrections (D-S eq.)

We have a third equation!

1 $G\,[\Sigma]$

2 $\Sigma\,[G;\Gamma]$

3 $\Gamma\,[G]$ (vertex function)



1 $$G(p) = \frac{1}{G_0^{-1}(p) - \Sigma(p)}$$

2 $$\Sigma(p) = \frac{-ie^2}{(2\pi)^4}\int dk^4 \frac{1}{k^2 + i\epsilon} G(p-k) u_\mu \Gamma^\mu(k; p-k, p)$$

3 $$k_0 \Gamma^0(p, p-k, k) = G^{-1}(p) - G^{-1}(p-k)$$ WARD-IDENTITIES

Resummations

Dyson-Scwinger eq.

The self-energy:

$$\Sigma(p) = \frac{-ie^2}{(2\pi)^4} \int dk^4 \frac{1}{k^2 + i\epsilon} G(p-k) u_\mu \Gamma^\mu(k; p-k, p)$$

Since $\Gamma^0(p, p-k, k) = \frac{G^{-1}(p) - G^{-1}(p-k)}{k_0}$

And $u = (1, 0, 0, 0)$

$$\Sigma(p^0) = \frac{-ie^2}{(2\pi)^4} G(p^0) \int dk^4 \frac{1}{k^2 + i\epsilon} \frac{G(p^0 - k^0)}{k_0}$$

(+ D-S renorm., A. Jakovac, P. Mati PHYSICAL REVIEW D85, 085006 (2012))

From this we can get

$$(p^0 - m) G(p^0) = \frac{\alpha}{\pi} \int_{p_0}^{M} d\omega\, G(\omega)$$

Its solution:

$$G(p^0) = \frac{const.}{(p^0 - m)^{(1+\frac{\alpha}{\pi})}} \equiv \text{B-N}$$

D-S eq. = ”Exact Resummation”

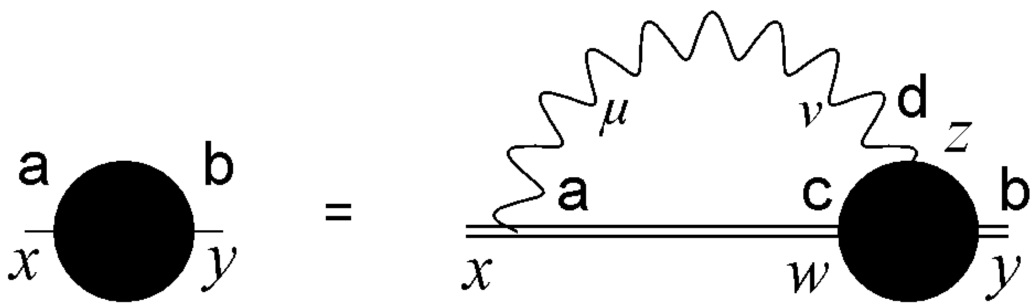
Let's go to $T \neq 0$!

Resummations

Finite Temperature

What's new?

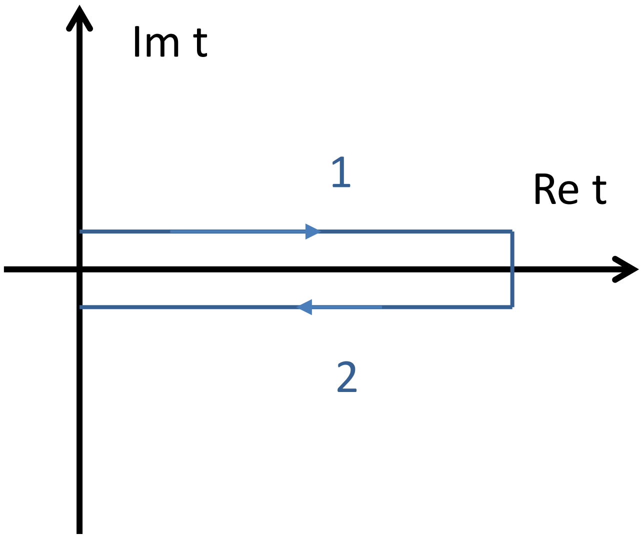
- The plasma assigns the frame of reference $\rightarrow u = (u_0, u_1, u_2, u_3)$
- F-D and B-E distributions come into picture
- New kind of loop integrals (retarded Self-energy, propagators have matrix structure; *R/A & Keldysh formalism*!)



Resummations

The path integral formalism:

$$Z[J_c] = \int \mathcal{D}O e^{i\int_c dt \int d^3x(\mathcal{L}+J_c O)} \longrightarrow Z[J_1, J_2] = Z_0 e^{-\frac{i}{2}\int d^4x \int d^4y (J^a(x) G_{ab}(x,y) J^b(y))}$$



$$iG_{ab}(x,y) = (-i)^2 \frac{1}{Z_0} \frac{\delta^2 Z[J]}{\delta J_a(x) \delta J_b(y)}\bigg|_{J=0} = \langle T_C O_a(x) O_b(y) \rangle$$

$$\mathbf{G^K} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

(linear transformation)

$$\mathbf{G^{R/A}} = \begin{pmatrix} 0 & G_{ar} \\ G_{ra} & G_{rr} \end{pmatrix}$$

(A similar matrix structure hold for the self energy)

Resummations

Finite Temperature

The propagators:

Keldysh $iG_{ab}(x) = \langle T_C O_a(x) O_b^\dagger(0) \rangle$
(a,b=1,2 contours)

(Photon)
$iG_{\mu\nu,ab}(x) = \langle T_C A_{\mu a}(x) A_{\nu b}(0) \rangle$

(Fermion)
$i\mathcal{G}_{ab}(x) = \langle T_C \psi_a(x) \psi_b^\dagger(0) \rangle$

β=1/T temperature dependence

E.g. the offdiagonals:

$$iG_{12}(k) = \pm n_\pm(k_0)\rho(k), \qquad iG_{21}(k) = (1 \pm n_\pm)(k_0)\rho(k)$$

(Where $n_\pm(k_0) = \frac{1}{e^{\beta k_0} \mp 1}$ * and $\rho(k) = iG_{21}(k) - iG_{12}(k)$)

R/A (relations)

$$G_{rr} = \frac{G_{21} + G_{12}}{2}, \quad G_{11} = G_{ra} + G_{12}, \quad \rho = iG_{ra} - iG_{ar}$$

*

The Dirac-Fermi distribution is zero in this model:

- The fermion is a hard probe of the soft photon fields, so it's NOT part of the thermal medium
- It can be checked: at one-loop order it violates causality

Resummations

The Self sonsistent equations at finite T:

$G[\Sigma]$ **1** $$G_{ra}(p) = G^{(0)}_{ra}(p) + G^{(0)}_{ra}(p)\Sigma_{ar}(p)G_{ra}(p)$$

$\Sigma[G;\Gamma]$ **2** $$\Sigma_{ab}(p) = i(-1)^{a+1}e^2u_\mu \int \frac{d^4k}{(2\pi)^4}\mathcal{G}_{ac}(p-k)G^{\mu\nu}_{ad}(k)\Gamma_{\nu;dcb}(k;p-k,p)$$

$\Gamma[G]$ **3** $$\Gamma_{abc}(k;p,q) = \frac{1}{uk}\left[\delta_{ab}\mathcal{G}^{-1}_{bc}(q) - \delta_{ac}\mathcal{G}^{-1}_{bc}(p)\right]$$

We're only interested in the *spectral function*

$$w\bar{\rho}(w) = -\frac{\alpha}{\pi}\int dq\, f(q,u)\,\bar{\rho}(w-q)$$

$$f(q) = \frac{u_0^2 - u^2}{2}\int_{u_0-u}^{u_0+u}\frac{ds}{us^2}\,(1 + n(\frac{q}{s}))$$

Notes:

- The UV renorm. is the same as at T=0
- Finite T contribution is not singular

Resummations

Finite Temperature

Solution for the |u|=0

$$w\bar{\rho}(w) = -\frac{\alpha}{\pi}\int dq\, f(q,u)\,\bar{\rho}(w-q) \longrightarrow w\bar{\rho}(w) = -\frac{\alpha}{\pi}\int dq\,(1+n(q))\bar{\rho}(w-q)$$

By inverse Fourier transformation (convolution $\longrightarrow$ product) : $i\partial_t\bar{\rho}(t) = \frac{iT\alpha}{\tanh(\pi t T)}\bar{\rho}(t)$

$$\bar{\rho}(t) = \bar{\rho}_0\,(\sinh \pi t T)^{\alpha/\pi}$$

Fourier transform: *using analytic continuation*

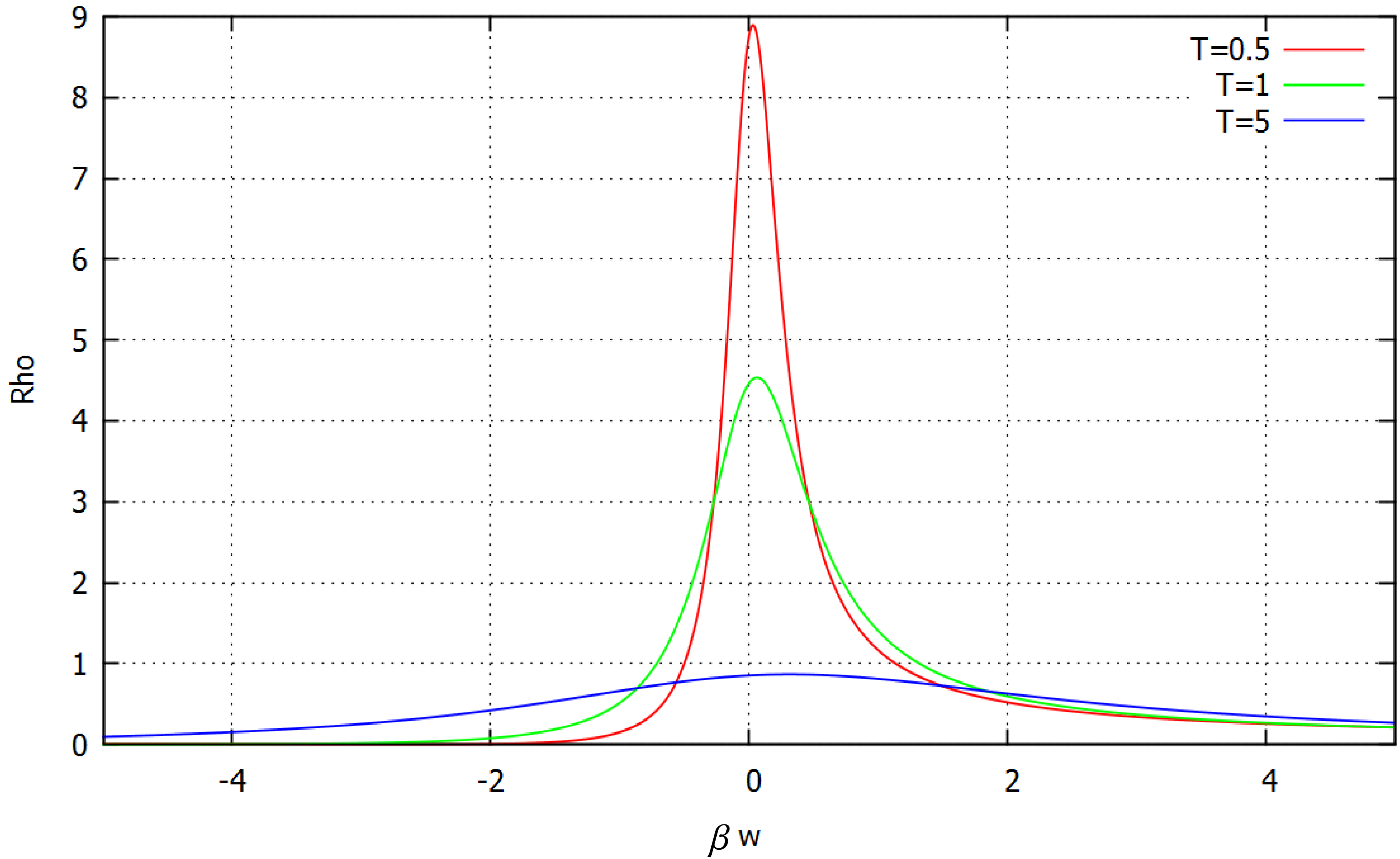
$$\bar{\rho}(w) = \frac{e^{\beta w/2}\sin\alpha}{\cosh(\beta w) - \cos\alpha}\,\frac{1}{\left|\Gamma\left(1+\frac{\alpha}{2\pi}+i\frac{\beta w}{2\pi}\right)\right|^2}$$

$$\bar{\rho}(\beta w \gg 1) \sim \frac{e^{\beta w}}{2\cosh(\beta w)}\,\frac{1}{w^{1+\frac{\alpha}{\pi}}} \xrightarrow{T\to 0} \Theta(w)\,w^{-1-\frac{\alpha}{\pi}}$$

It gives backl T=0 case!

Resummations

Finite Temperature



- Spreads out as T increases
- Peak shifted by finite thermal correction

Resummations

Finite Temperature

Solution for the |u| finite case

The asymptotic can be factorized:

$$\bar{\rho}(t) = Z(t)\bar{\rho}_{u=0}(t;\alpha_{eff})$$

$$\alpha_{eff}(u) = \alpha\frac{u_0^2 - u^2}{2u}\ln\frac{u_0+u}{u_0-u} \qquad Z(t) = \exp\left\{\frac{U^2\alpha}{2\pi u}\int_{u_0-u}^{u_0+u}\frac{ds}{s^2}\ln\frac{\sinh\pi t T s}{(\sinh\pi T t)^s}\right\}$$

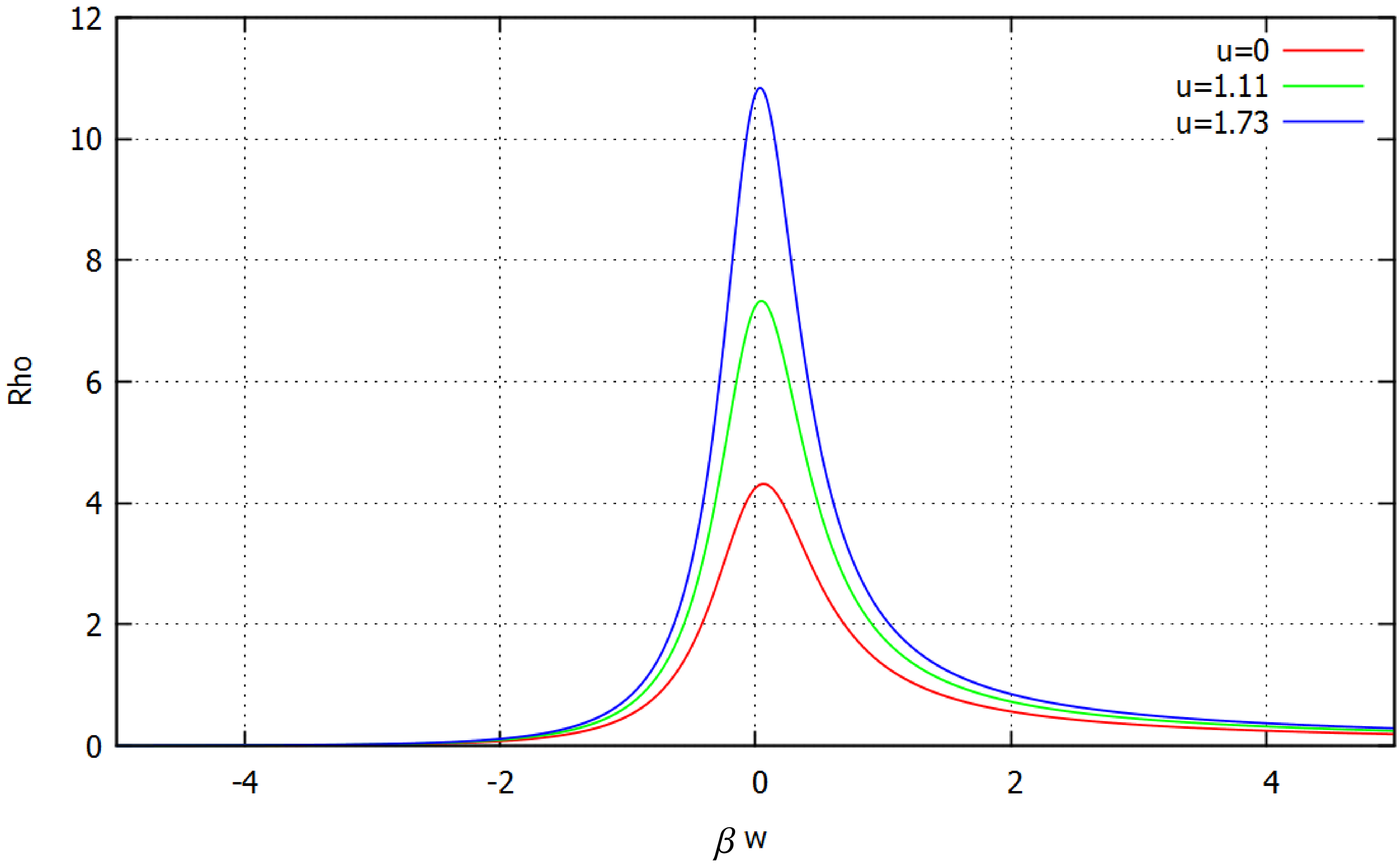
We need the spectral function in the momentum space:

$$\bar{\rho}(t) = Z(t)\bar{\rho}_{u=0}(t;\alpha_{eff}) \xrightarrow{\text{FT}} \bar{\rho}(w) = \int_{-\infty}^{\infty}\frac{dw}{2\pi}Z(w-w')\bar{\rho}_{u=0}(w';\alpha_{eff})$$

This can be solved only numerically

Resummations

Finite Temperature



- Peak increases with *u*
- The half-width decreasing with $u \longrightarrow$ particle lifetime increases

Conclusion, Outlook

- B-N: Exactly solvable model
- We made different levels of approximations
- The resummations are valid even in IR
- We found a new way to solve the model
- Possible to extend to finite temperature (closed form for u=0 case!)

Outlook:

- adapting the method to QED
- applications at ELI (strong fields vs. nonperurbative method)
- examine bounded states (IR physics questions)
- adapting the method to QCD (???)

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THANK YOU!