Nonlinear Fluid Dynamics from Gravity

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Outline

References

Talk based on

- arXiv: 0712.2456, S. Bhattacharyya, V. Hubeny, S.M. , M. Rangamani
- 0803.2526, above + R.Loganayagam, G. Mandal, T. Morita and H. Reall
- 0806.0006, S. Bhattacharyya, R. Loganayagam, S.M. , S. Nampuri, S. Trivedi and S. Wadia
- To appear, S. Bhattacharyya, R. Loganayagam, I. Mandal, S.M., A. Sharma
- To appear, N. Banerjee , J.Bhattacharya, S.Bhattacharyya, S. Dutta, V. Hubeny, R. Loganayagam, S.M., M. Rangamani, P.Surowka.

Closely Related work

- 0712.2451, R. Baier, P. Romatschke, D. Son, A.O.Starinets, M. Stephanov
- 0801.3701, R. Loganayagam
- 0802.3224, M. Van Raamsdonk
- 0804.2453, S. Dutta
- 0805.3774, M. Heller, P. Surowka, R. Loganayagam, M. Spalinski, S.E.Vazquez
- 0806.4602, M. Haack , A. Yarom.

Immediate precursors: important work by Son, Starinets, Kovtun, Policastro, Janik and collaborators. Also 0708.1770 (S. Bhattacharyya, S. Lahiri, R. Loganayagam, S.M.)

Trace dynamics at Large N

- Consider any large N gauge theory. Let $\rho_m(x) = Tr\ O_m(x)$ denote set of all single trace gauge invariant operators of the theory.
- According to general lore, in the large N limit the gauge theory path integral may be rewritten as

$$\int \prod_m \mathcal{D}\rho_m(x) \exp\left[-N^2 S(\rho_m)\right]$$

Consequently large N gauge theories are effectively classical when rewritten in terms of trace variables.

Outline and References
Introduction
Spacetimes dual to boundary fluid flows
Global Structure and Entropy Current
Rotating Black holes
Two Generalizations
Discussion

Trace Dynamics from Supergravity

- Maldacena 1997: The classical large N evolution equations for $\mathcal{N} = 4$ Yang Mills are IIB SUGRA on $AdS_5 \times S^5$. $\rho_m(x,t)$ to be read off from the boundary values of bulk fields.
- Evolution equations of 10d bulk fields elegant and local. Map to unfamiliar, nonlocal and complicated looking evolution equations for $\rho_m(x,t)$.
- Would be nice to better understand the implied four dimensional dynamics for ρ_n. This talk: study ρ_n dynamics in a universal sector in a long distance limit. Will show that the bulk equations imply local and familiar boundary dynamics of $\rho_m(x)$ in this limit.

Universal Sector

- Consider any 2 derivative theory of gravity interacting with other fields, that admits AdS_{d+1} space as a solution.
- Every such theory admits a consistent truncation to Einstein gravity with a negative cosmological constant. All fields other than the Einstein frame graviton are simply set to their background AdS_{d+1} values under this truncation.
- Dual implication: Simple universal dual dynamics for the stress tensor of all the (infinitely many) large N field theories with a 2 derivative bulk dual. Most of the rest of this talk: study this simple universal sector subsector at long wavelengths.

Einstein's Equations imply Navier Stokes

- In this talk we conjecture and largely demonstrate that the set of all regular long wavelength solutions to Einstein's equations with a negative cosmological constant in $d+1$ dimensions is identical to the set of solutions of the boundary Navier Stokes equations (with holographically determined values of transport coefficients) in d dimensions.
- Thus Einstein Equations (1915) $\rightarrow$ Navier Stokes equations (1822), adding to the list of connections uncovered by string theory between classic but apparently unrelated equations of physics.

Boosted Black Branes

$$R_{MN} - \frac{R}{2} g_{MN} = \frac{d(d-1)}{2} g_{MN} : \quad : M, N = 1 \dots d+1$$

Simplest soln : AdS_{d+1} space

$$ds^2 = \frac{dr^2}{r^2} + r^2 g_{\mu\nu} dx^\mu dx^\nu ; \quad : \mu, \nu = 1 \dots d$$

($g_{\mu\nu}$ = constant boundary metric). Another solution: black brane at temperature T and velocity u_μ

$$ds^2 = \frac{dr^2}{r^2 f(r)} + r^2 \mathcal{P}_{\mu\nu} dx^\mu dx^\nu - r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu$$

$$f(r) = 1 - \left(\frac{4\pi T}{d\ r}\right)^d ; \quad \mathcal{P}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

Outline and References
Introduction
Spacetimes dual to boundary fluid flows
Global Structure and Entropy Current
Rotating Black holes
Two Generalizations
Discussion

$u_\mu(x)$ and $T(x)$

- The boundary stress tensor for the boosted black brane is

$$T_{\mu\nu} = K T^d \left(g_{\mu\nu} + d u_\mu u_\nu\right) ; \quad K = \frac{1}{16\pi G_{d+1}} \left(\frac{4\pi}{d}\right)^d$$

- Note that

$$T_{\mu\nu}(x) u^\nu(x) = K' T(x)^d u^\mu(x), \quad K' = (1-d)K$$

  (u^μ is the unique timelike eigenvector).
- We will use this equation to define the velocity and temperature field of any locally asymptotically *AdS* solution of Einsteins equations. Simple physical interpretation.

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Our Question

- Consider an arbitrary evolution $T_{\mu\nu}(x)$ on a boundary with metric $g_{\mu\nu}(x)$. Let $\Delta(x)$ denote the minimum length scale of variation of $T_{\mu\nu}(x)$ and $g_{\mu\nu}(x)$. Let $\epsilon(x) = \frac{1}{T(x)\Delta(x)}$.
- If $\epsilon(x) \ll 1$ then $T_{\mu\nu}(x)$, $g_{\mu\nu}(x)$ 'slowly varying' (vary on length scales large comp to the equilibriation length, $\frac{1}{T}$).
- Question: Given arbitrary slowly varying boundary stress tensor $T_{\mu\nu}(x)$. What are its boundary 'equations of motion', i.e. under what conditions can $T_{\mu\nu}(x)$ be obtained from a regular solutions to Einstein's equations? What is the bulk metric dual to any $T_{\mu\nu}(x)$ that satisfies these conditions?
- Address this question: perturbatively construct families of (we conjecture all) 'slowly varying' bulk spacetimes.

The tubewise approximation

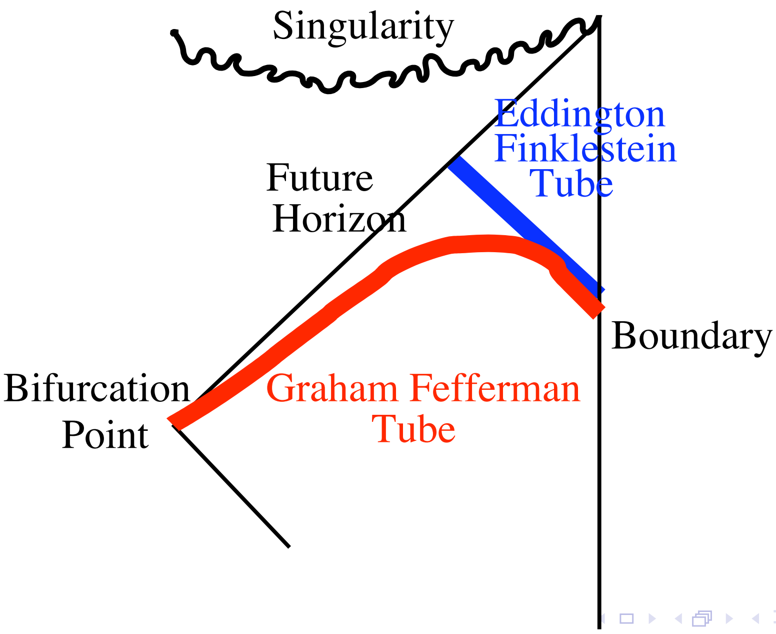
- We expect slowly varying boundary configurations to be locally thermalized. Suggests bulk solution tubewise approximated by black branes. But along which tubes?
- Naive guess: lines of constant x^μ in Schwarschild (Graham Fefferman) coordinates, i.e. metric approximately

$$ds^2 = \frac{dr^2}{r^2 f(r)} + r^2 \mathcal{P}_{\mu\nu}(x) dx^\mu(x) dx^\nu(x) - r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu$$

$$f(r) = 1 - \left(\frac{4\pi T(x)}{d\ r}\right)^d ; \quad \mathcal{P}_{\mu\nu} = g_{\mu\nu}(x) + u_\mu(x) u_\nu(x)$$

- Wrong. Metrics not regular. Bad starting point for perturbation theory. Also intuitively problem with causality.

Penrose diagram



Zero order metric

- Causality suggests the use of tubes centered around ingoing null geodesics. In particular we try

$$ds^2 = g^{(0)}_{MN} dx^M dx^N = -2u_\mu(x) dx^\mu dr + r^2 \mathcal{P}_{\mu\nu}(x) dx^\mu dx^\nu - r^2 f(r, T(x)) u_\mu(x) u_\nu(x) dx^\mu dx^\nu$$

- Metric generally regular but not solution to Einstein's equations. However solves equations for constant $u^\mu, T, g_{\mu\nu}$. Consequntly appropriate starting point for a perturbative soln of equations in the parameter $\epsilon(x)$.

Perturbation Theory: Redn to ODEs

- That is we set

$$g_{MN} = g^{(0)}_{MN}(\epsilon x) + \epsilon g^{(1)}_{MN}(\epsilon x) + \epsilon^2 g^{(2)}_{MN}(\epsilon x) \ldots$$

  and attempt to solve for $g^{(n)}_{MN}$ order by order in ϵ.
- Perturbation expansion surprisingly simple to implement. Nonlinear partial differential equation $\rightarrow$ 15 ordinary differential equations, in the variable r at each order and each boundary point.

Outline and References
Introduction
Spacetimes dual to boundary fluid flows
Global Structure and Entropy Current
Rotating Black holes
Two Generalizations
Discussion

Perturbation Theory: Constraint Equations

- Gauge choice: $g_{r\mu}(x) = -u_\mu(x), \quad g_{rr} = 0$. 10 undetermined metric components $g^{(n)}_{\mu\nu}$ at each order. Naively 15 but actually 14 independent Einstein equations. Split up into 4 constraint equations and 10 dynamical equations.
- The constraint equations at n^{th} order are independent of $g^{(n)}_{\mu\nu}$: they are $\nabla^\mu T^{(n-1)}_{\mu\nu} = 0$, where $T^{(n-1)}_{\mu\nu} = 0$ is the boundary stress tensor dual to the solution upto $(n-1)^{th}$ order.

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Perturbation Theory: Dynamical Equations

- The dynamical equations take the form $M\, g^{(n)} = s^{(n)}$. Here M is a ‘homogeneous’ differential operator in r that is the same at every order. $s^{(n)}$ is a source function that is independent of $g^{(n)}$ and is determined by the solution to $(n-1)^{th}$ order.
- It turns out to be possible to exactly solve the equation $M\, g^{(n)} = s^{(n)}$ for an arbitrary source function $s^{(n)}$. For any given source function s^n there is a family of solutions to the equation (which differ by solutions of the homogeneous equation $M\, g = 0$)

Outline and References
Introduction
Spacetimes dual to boundary fluid flows
Global Structure and Entropy Current
Rotating Black holes
Two Generalizations
Discussion

Perturbation Theory: Uniqueness of Solns

However provided that the source function is regular at the ‘horizon’ and dies off sufficiently fast at infinity (conditions that are true for $s^{(n)}$ generated in perturbation theory), the solution to this equation is unique subject to the following requirements:

1. That the solution is dual to the specified boundary metric $g_{\mu\nu}(x)$, velocity field $u_\mu(x)$ and the temperature $T(x)$. (condition on the large r behaviour of the solution).
2. That the solution is regular at the zeroth order horizon (condition at $r = \frac{4\pi T}{d}$)

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Perturbation Theory: Navier Stokes Equations

- We may now construct the stress tensor $T^{(n)}_{\mu\nu}$ dual to our perturbative solution. $T^{(n)}_{\mu\nu}$ is uniquely determined as a function of n^{th} order in derivatives of $g_{\mu\nu}$, u_μ and T.
- Recall that the constraint equations are $\nabla^\mu T_{\mu\nu} = 0$. But this equation, together with the specification of $T_{\mu\nu}$ as a function of derivatives of $g_{\mu\nu}$, u_μ and T has a name: the (generalized) Navier Stokes equation.

Outline and References
Introduction
Spacetimes dual to boundary fluid flows
Global Structure and Entropy Current
Rotating Black holes
Two Generalizations
Discussion

Perturbation Theory: Summary

- Summary: Explicit map from the space of solutions of the generalized Navier Stokes equations in d dimensions to long wavelenth solutions of Einstein's equations.
- Requirement of regularity of the horizon ensures this map is locally one to one in solution space.
- Naive Graham Fefferman counting: $\frac{d(d+1)}{2} - 1$ parameter solution. Rougly parameterized by fluctuation fields $g^n_{\mu\nu}$. However $\frac{d(d-1)}{2} - 1$ of these modes - the tensor sector - fixed by the requirement of regularity.
- Remaining solutions parameterized by d velocities and temperatures. Closed dynamical system.

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Explicit Results at second order

We have explicitly implemented our perturbation theory to second order.

$$
\begin{aligned}
ds^2 &= -2u_\mu dx^\mu \left(dr + r\, A_\nu dx^\nu\right) + r^2 g_{\mu\nu} dx^\mu dx^\nu \\
&\quad - \left[\omega_\mu{}^\lambda \omega_{\lambda\nu} + \frac{1}{d-2}\mathcal{D}_\lambda \omega^\lambda{}_{(\mu} u_{\nu)} - \frac{1}{d-2}\mathcal{D}_\lambda \sigma^\lambda{}_{(\mu} u_{\nu)} \right. \\
&\quad \left. + \frac{\mathcal{R}}{(d-1)(d-2)} u_\mu u_\nu \right] dx^\mu dx^\nu \\
&\quad + \frac{1}{(br)^d}\left(r^2 - \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta}\right) u_\mu u_\nu dx^\mu dx^\nu \\
&\quad + 2(br)^2 F(br)\left[\frac{1}{b}\sigma_{\mu\nu} + F(br)\sigma_\mu{}^\lambda \sigma_{\lambda\nu}\right] dx^\mu dx^\nu \ldots
\end{aligned}
$$

Explicit Results at second order

$$
\begin{aligned}
&- 2(br)^2 \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1} P_{\mu\nu} K_1(br) - \frac{u_\mu u_\nu}{(br)^{d-2}} \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{(d-1)} K_2(br) \\
&+ \frac{2\, L(br)}{(br)^{d-2}} \left[P^\lambda_\mu \mathcal{D}_\alpha \sigma^\alpha{}_\lambda u_\nu + P^\lambda_\nu \mathcal{D}_\alpha \sigma_\lambda{}^\alpha u_\mu\right] dx^\mu dx^\nu \\
&- 2(br)^2 H_1(br) \left[u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} + \sigma_\mu{}^\lambda \sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1} P_{\mu\nu} \right. \\
&\qquad \left. + C_{\mu\alpha\nu\beta} u^\alpha u^\beta\right] dx^\mu dx^\nu \\
&+ 2(br)^2 H_2(br) \left[u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} + \omega_\mu{}^\lambda \sigma_{\lambda\nu} - \sigma_\mu{}^\lambda \omega_{\lambda\nu}\right] dx^\mu dx^\nu
\end{aligned}
$$

Explicit results at second order

Where

$$F(br) \equiv \int_{br}^{\infty} \frac{y^{d-1}-1}{y(y^d-1)} dy \; ; \; L(br) \equiv \int_{br}^{\infty} \xi^{d-1} d\xi \int_{\xi}^{\infty} dy \, \frac{y-1}{y^3(y^d-1)}$$

$$H_2(br) \equiv \int_{br}^{\infty} \frac{d\xi}{\xi(\xi^d-1)} \int_{1}^{\xi} y^{d-3} dy \left[1+(d-1)yF(y)+2y^2F'(y)\right]$$

$$K_1(br) \equiv \int_{br}^{\infty} \frac{d\xi}{\xi^2} \int_{\xi}^{\infty} dy \, y^2 F'(y)^2 \; ; \; H_1(br) \equiv \int_{br}^{\infty} \frac{y^{d-2}-1}{y(y^d-1)} dy$$

$$K_2(br) \equiv \int_{br}^{\infty} \frac{d\xi}{\xi^2} \Big[1-\xi(\xi-1)F'(\xi)-2(d-1)\xi^{d-1} + \left(2(d-1)\xi^d-(d-2)\right) \int_{\xi}^{\infty} dy \, y^2 F'(y)^2\Big]$$

Second order boundary stress tensor

The dual stress tensor corresponding to this metric is given by $(4\pi T = b^{-1} d)$

$$\begin{aligned} T_{\mu\nu} &= p\left(g_{\mu\nu} + d u_\mu u_\nu\right) \\ &\quad - 2\eta \left[\sigma_{\mu\nu} - \tau_\pi u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} - \tau_\omega \left(\sigma_\mu{}^\lambda \omega_{\lambda\nu} - \omega_\mu{}^\lambda \sigma_{\lambda\nu}\right)\right] \\ &\quad + \xi_\sigma \left[\sigma_\mu{}^\lambda \sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1} P_{\mu\nu}\right] + \xi_C C_{\mu\alpha\nu\beta} u^\alpha u^\beta \end{aligned}$$

$$p = \frac{1}{16\pi G_{d+1} b^d} \quad ; \quad \eta = \frac{s}{4\pi} = \frac{1}{16\pi G_{d+1} b^{d-1}}$$

$$\tau_\pi = (1 - H_1(1))b \quad ; \quad \tau_\omega = H_1(1)b \quad ; \quad \xi_\sigma = \xi_C = 2\eta b$$

Properties of soln: Stress tensor

- Schematic form of 2nd order stress tensor:

$$T_{\mu\nu} = aT^d(g_{\mu\nu} + du_\mu u_\nu) + bT^{d-1}\sigma_{\mu\nu} + T^{d-2}\sum_{i=1}^{5} c_i S^i_{\mu\nu}$$

- a is a thermodynamic parameter. b is related to the viscosity: we find $\eta/s = 1/(4\pi)$. c_i coefficients of the five traceless symmetric Weyl covariant two derivative tensors are second order transport coefficients. Values disagree with the predictions of the Israel Stewart formalism.
- Recall that results universal. Should yield correct order of magnitude estimate of transport coefficients in any strongly coupled CFT.

Outline and References
Introduction
Spacetimes dual to boundary fluid flows
Global Structure and Entropy Current
Rotating Black holes
Two Generalizations
Discussion

Properties of soln: Weyl covariance

- Weyl covariance: result is written in terms of covariant derivative built out of the 'gauge' field

$$\mathcal{A}_\nu \equiv u^\lambda \nabla_\lambda u_\nu - \frac{\nabla_\lambda u^\lambda}{d-1} u_\nu$$

R. Loganayagam . arXiv:0801.3701 [hep-th]

Can use the fact that $\mathcal{A}_\nu$ transforms like a gauge field under Weyl transformation to define a Weyl covariant derivative $\mathcal{D}$ that acts on a weight w tensor $Q^{\mu\cdots}_{\nu\ldots}$ as

$$\begin{aligned}\mathcal{D}_\lambda\, Q^{\mu\cdots}_{\nu\ldots} \equiv\ & \nabla_\lambda\, Q^{\mu\cdots}_{\nu\ldots} + w\, \mathcal{A}_\lambda Q^{\mu\cdots}_{\nu\ldots} \\ & + \left[g_{\lambda\alpha}\mathcal{A}^\mu - \delta^\mu_\lambda \mathcal{A}_\alpha - \delta^\mu_\alpha \mathcal{A}_\lambda\right] Q^{\alpha\cdots}_{\nu\ldots} + \ldots \\ & - \left[g_{\lambda\nu}\mathcal{A}^\alpha - \delta^\alpha_\lambda \mathcal{A}_\nu - \delta^\alpha_\nu \mathcal{A}_\lambda\right] Q^{\mu\cdots}_{\alpha\ldots} - \ldots\end{aligned}$$

Event Horizons

- Our solutions are singular at $r = 0$. Quite remarkably it is possible to demonstrate that under certain conditions these solutions have event horizons. The event horizon manifold $r = r(x)$, may explicitly be determined order by order in the derivative expansion. This horizon shields the $r = 0$ singularity from the boundary.
- Need some knowledge of the long time behaviour of the solution. Sufficient, though far from necessary, to assume fluid flows that reduce to constant temperature and velocity at late times. Not very strong assumption. Probably true of all finite fluctuations about uniform motion in $d \geq 2$.

Event Horizon in the derivative expansion

- The event horizon of the dual bulk geometry is the unique null manifold that reduces to the event horizon $r = \frac{4\pi T}{d} = \frac{1}{b}$ of the dual uniform black brane at late times.
- It turns out to be simple to construct this event horizon manifold in the derivative expansion: explicitly

$$r_H = \frac{1}{b} + b\left(\lambda_1 \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \lambda_2 \omega_{\alpha\beta}\omega^{\alpha\beta} + \lambda_3 \mathcal{R}\right) + \ldots$$

$$\lambda_1 = \frac{2(d^2 + d - 4)}{d^2(d-1)(d-2)} - \frac{K_2(1)}{d(d-1)}$$

$$\lambda_2 = -\frac{d+2}{2d(d-2)} \quad \text{and} \quad \lambda_3 = -\frac{1}{d(d-1)(d-2)}$$

- We can put our control over the event horizon to practical use. Recall that a d dimensional event horizon is generated by a $d-1$ dimensional family of null geodesics. Let α^i $i = 1\ldots(d-1)$ label these geodesics. Let λ be any future directed coordiante along the geodesics.
- The line element on the event horizon takes the form

$$ds^2 = g^{eh}_{ij} d\alpha^i d\alpha^j$$

  Define the area $d-1$ form as $a = \sqrt{g^{eh}} d\alpha^1 \wedge d\alpha^2 \ldots d\alpha^{d-1}$.
- Now $da = \phi d\lambda \wedge d\alpha^1 \ldots d\alpha^{d-1}$ The classic area increase theorem of black hole physics implies the assertion that $\phi \geq 0$.

Outline and References
Introduction
Spacetimes dual to boundary fluid flows
Global Structure and Entropy Current
Rotating Black holes
Two Generalizations
Discussion

- Consider the pullback of a to the boundary using the map generated by the radial ingoing null geodesics described above. The boundary hodge dual of pullback of this $d-1$ form is a current whose divergence may be shown to be non negative.
- Consequently fluid dynamics dual to gravity is equipped with a local current whose divergence is always non negative, and which agrees with the thermodynamic entropy current in equilibrium. This ‘entropy current’ is a local ‘Boltzman H’ function whose non negative divergence rigorously estabilishes the locally irreversable nature of the dual fluid flows.

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Entropy Current at second order

Explicitly this entropy current is given to second order by

$$4\, G_{d+1}\, b^{d-1}\, J^{\mu}_{S} = [\, 1 \,+\, b^2 \,(\, A_1\, \sigma^{\alpha\beta}\, \sigma_{\alpha\beta} \,+\, A_2\, \omega^{\alpha\beta}\, \omega_{\alpha\beta} \,+\, A_3\, \mathcal{R}\,)\,]\, u^{\mu} \\ +\, b^2\, [\, B_1\, \mathcal{D}_{\lambda}\, \sigma^{\mu\lambda} \,+\, B_2\, \mathcal{D}_{\lambda}\, \omega^{\mu\lambda}\,]$$

where

$$A_1 = \frac{2}{d^2}(d+2) - \frac{K_1(1)d + K_2(1)}{d}\,, \qquad A_2 = -\frac{1}{2d}, \qquad B_2 = \frac{1}{d-2}$$

$$B_1 = -2A_3 = \frac{2}{d(d-2)}$$

Rotating Kerr AdS Black holes

- It is possible to test all the general constructions described above against a class of exact solutions of general relativity. Solutions: stationary rotating black holes in global *AdS* space.
- Labelled by mass plus $[d/2]$ commuting angular momenta. They are dual to the flow of a conformal fluid on S^{d-1}. The dual velocity field of these solutions turns out to be that of rigid rotations. The temperature field is also simple, but I do not describe it.

Rotating Black Holes from Fluid Dynamics

Upon transforming these exact solutions into our fluid dynamical gauge $g_{rr} = 0$, $g_{r\mu} = -u_\mu$ they take a very simple form

$$ds^2 = -2u_\mu dx^\mu \left(dr + r\, A_\nu dx^\nu\right) + r^2 g_{\mu\nu} dx^\mu dx^\nu$$
$$- \left[\omega_\mu{}^\lambda \omega_{\lambda\nu} + \frac{1}{d-2}\mathcal{D}_\lambda \omega^\lambda{}_{(\mu} u_{\nu)} + \frac{1}{(d-1)(d-2)}\mathcal{R} u_\mu u_\nu\right] dx^\mu dx^\nu$$
$$+ \frac{r^2 u_\mu u_\nu}{b^d \det\left[r\, \delta^\mu_\nu - \omega^\mu{}_\nu\right]} dx^\mu dx^\nu$$

Expanding in boundary derivatives one finds

1. The perturbative second order fluid dynamical stress tensor is exact to all orders for all these solutions.
2. The perturbative second order construction of the bulk metric is also exact for the special case of the Schwarzschild (non rotating) metric black hole.
3. In the general case the exact metric may be expanded in the derivative expansion. This expansion contains terms of all orders. The expansion is convergent; outside the horizon the radius of convergence is $\sim T$. Truncating to second order we find perfect agreement with our perturbative results. Similarly for entropy. construction to second order. Similar statement for entropy.

Forcing and Charges

- One may attempt to generalize our construction to a bulk Lagrangian with additional fields. Gain: more solutions, wider dynamical behaviour. Price: reduced universality
- Additional fields of two sorts. Bulk gauge fields that correspond to conserved boundary charge. Plus all others.
- Gauge fields enlarge the set of fluid dynamical variables to include charge densities. Other fields yield new solns only when non normalizable part is turned on. Operator coupling leads to forcing function for fluid dynamics

Outline and References
Introduction
Spacetimes dual to boundary fluid flows
Global Structure and Entropy Current
Rotating Black holes
Two Generalizations
Discussion

Dilaton Forcing

- Example of 2nd kind: $d = 4$ Einstein Dilaton System.
- Long wavelength solution of the Einstein dilaton system with a given specified slowly varying boundary dilaton field may be obtained by perturbation theory analogeous to above. Have been explicitly constructed to second order.
- Solutions are in one to one correspondence with the forced Navier Stokes equations

$$\nabla_\mu T^{\mu\nu} = -\frac{(\pi T)^3}{16\pi G_5} \nabla^\nu \phi (u.\partial)\phi + \dots$$

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Simple Solutions

- A simple class of solutions to these equations are given by the dilaton chosen as a slowly varying function of time. If the fluid is initially at rest, it stays at rest but slowly heats up according to
$$\frac{dT}{dt} = \frac{(\dot{\phi})^2}{12\pi}.$$
The dual bulk solution has a dilaton pulses falling into the black hole, and at leading order is the Vaidya solution. Note that varying - whether increasing or decreasing - the dilaton heats up the gauge theory. Consistent with entropy. Speculations about the continuation to weak coupling.

The Einstein Maxwell System

- Generalization with gauge fields: IIB SUGRA on $AdS_5 \times S^5$. Sector with only a diagonal combination of $SO(6)$ gauge fields admits a consistent truncation to the $U(1)$ Einstein Maxwell system with a gauge Chern Simons term.
- Can set up perturbation theory to determine the most general long wavelength solution for this system. Based on boosted charged branes with varying charge densities, temperatures and velocities.
- End result: reduction to the charged Navier Stokes equations, $\nabla^\mu T_{\mu\nu} = 0$, $\nabla^\mu J_\mu = 0$ with a holographically determined expression for charge current and stress tensor in series in derivatives of temperature, charge

Exact Charged Rotating Black hole Solution

$$
\begin{aligned}
ds^2 &= -2u_\mu dx^\mu \left(dr + r\, A_\nu dx^\nu\right) + r^2 g_{\mu\nu} dx^\mu dx^\nu \\
&- \left[\omega_\mu{}^\lambda \omega_{\lambda\nu} + \frac{1}{d-2}\mathcal{D}_\lambda \omega^\lambda{}_{(\mu} u_{\nu)} + \frac{\mathcal{R}}{6} u_\mu u_\nu\right] dx^\mu dx^\nu \\
&+ \left[\left(\frac{2m}{\rho^2} - \frac{q^2}{\rho^4}\right) u_\mu u_\nu - \frac{q}{2\rho^2} u_{(\mu} l_{\nu)}\right] dx^\mu dx^\nu
\end{aligned}
$$

$$
\rho^2 \equiv r^2 + \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta} \quad ; \quad l_\mu \equiv \epsilon_{\mu\nu\lambda\sigma} u^\nu \omega^{\lambda\sigma} \quad ; \quad A_\mu = -\frac{\sqrt{3}q}{\rho^2} u_\mu
$$

which has a stress tensor and charged current

$$
T_{\mu\nu} = \frac{m}{8\pi G_5}(4u_\mu u_\nu + g_{\mu\nu}) - \frac{\sqrt{3}q}{8\pi G_5} u_{(\mu} l_{\nu)} \quad ; \quad J_\mu = \frac{\sqrt{3}q}{8\pi G_5} u_\mu
$$

- Have worked out 1st order metric and gauge field. Second order in progress.
- First order results match expansion of exact charged rotating black hole solutions. As above we find perfect agreement. The Chern Simons term is important here, clearing up an old puzzle.
- Fluid expansion for black branes governed by $\frac{1}{\Delta(x) r_H(x)}$: $r_H(x)$ = coordinate location of the local event horizon. $\Delta(x) r_H(x)$ can remain large even when the branes become extremal. Interesting possibility of extremal fluid dynamics.
- While the extremal fluid stress tensors and currents may simply be given by the naive extremal limit of the nonextremal result, the same is not true of the full bulk metric. Full story not yet clear.

Ruminations

Our results probably admit several generalizations (e.g. nonconformal, plasmaballs). They also start making contact with several important questions

- Turbulence in gravity: e.g. rotating black holes.
- The breaking of time reversal invariance.
- Cosmic censorship and singularities in equilibriation.
- Thermal nature of solutions of $S(\rho_m)$ with order N^2 energy.
- The thermodynamical nature of spacetime?

Hope to be able to get a concrete hold on these important questions.