We will be interested in gluon scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills.

Motivation: It can give non trivial information about more realistic theories but is more tractable.

- Perturbative computations are easier. Higher loop computations are possible $\rightarrow$ proposal for all loops (MHV) $n$-point amplitudes.
- The strong coupling regime can be studied, by means of the gauge/string duality, through a weakly coupled string sigma model.
Prescription for computing scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills at strong coupling by using the $AdS/CFT$ correspondence.

**Aim of these project**

- Leading $N_c$ color ordered $n$–points amplitude at $L$ loops: $A_n^{(L)}$
- The amplitudes are IR divergent.
- Dimensional regularization $D = 4 - 2\epsilon \rightarrow A_n^{(L)}(\epsilon) = 1/\epsilon^{2L} + ...$
- Focus on MHV amplitudes and scale out the tree amplitude $M_n^{(L)}(\epsilon) = A_n^{(L)}/A_n^{(0)}$.

**Gauge theory amplitudes, Dixon’s talk**

$$A_n^{L,Full} \sim \sum_{\rho} Tr(T^{a_{\rho(1)}} ... T^{a_{\rho(n)}})A_n^{(L)}(\rho(1), ..., \rho(2))$$

- BDS proposal for all loops MHV amplitudes (Bern, Dixon, Smirnov)

$$\log \mathcal{M}_n = \frac{\log n}{2} \left( -\frac{1}{8\epsilon^2} f(-2) \left( \frac{\lambda\mu^{2\epsilon}}{s_{i,i+1}^{(t)}} \right) - \frac{1}{4\epsilon} g^{(-1)} \left( \frac{\lambda\mu^{2\epsilon}}{s_{i,i+1}^{(t)}} \right) \right) + f(\lambda) F_{n}^{(1)}(k)$$

$f(\lambda), g(\lambda) \rightarrow$ cusp/collinear anomalous dimension.
**AdS/CFT duality**

**AdS/CFT duality (Maldacena)**

Four dimensional Type IIB string theory
maximally SUSY Yang-Mills ⇔ on $AdS_5 \times S^5$.

We will study scattering amplitudes at strong coupling by using the AdS/CFT duality.

- Set up the computation: Use a $D$-brane as IR cut-off.
- Actual computations: Dimensional regularization.

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**String theory set up**

$$ds^2 = R^2 \frac{dx_{3+1}^2 + dz^2}{z^2}$$

- Place a D-brane extended along $x_{3+1}$ and located at $z_{IR} \gg R$.
- The asymptotic states are open strings ending on the D-brane.
- Consider the scattering of these open strings.
- $k_{pr} = k_{z_{IR}} R$ is very large: fixed angle and very high momentum.
- Intuition from flat space (Gross and Mendei): Amplitude is dominated by a saddle point.
World-sheet with the topology of a disk with vertex operator insertions (corresponding to external states)

- Near each vertex operator, the momentum of the external state fixes the form of the solution.
- The boundary of the world-sheet sits at $z = z_{IR}$.

- T-duality in $x^{\mu}$ directions followed by a change of coordinates $r = R^2/z \to$ we end up again with $AdS_5$!

$$ds^2 = R^2 \frac{dx_{3+1}^2 + dz^2}{z^2} \to \quad ds^2 = R^2 \frac{dy_{3+1}^2 + dr^2}{r^2}$$

The world-sheet boundary is located at $r = R^2/z_{IR}$ and is a particular line constructed as follows...

- For each particle with momentum $k^{\mu}$ draw a segment joining two points separated by $\Delta y^{\mu} = 2\pi k^{\mu}$
- Concatenate the segments according to the ordering of the insertions on the disk (particular color ordering).

- As $z_{IR} \to \infty$ the boundary of the world-sheet moves to $r = 0$.
- Exactly same computation as when computing the expectation value of a Wilson-Loop given by a sequence of light-like segments!
Prescription

- \( A_n \): Leading exponential behavior of the \( n \)-point scattering amplitude.
- \( A_{min}(k_1^\mu, k_2^\mu, ..., k_n^\mu) \): Area of a minimal surface that ends on a sequence of light-like segments on the boundary.

\[
A_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}
\]

- Prefactors are subleading in \( 1/\sqrt{\lambda} \), and we don’t compute them.
- In particular our computation is blind to helicity (and hence works also for non MHV)

Four point amplitude at strong coupling

Consider \( k_1 + k_3 \rightarrow k_2 + k_4 \)

- The simplest case \( s = t \).

Need to find the minimal surface ending on such sequence of light-like segments

\[
\begin{align*}
    r(y_1, y_2) &= \sqrt{(1 - y_1^2)(1 - y_2^2)} \\
    y_0 &= y_1 y_2
\end{align*}
\]

- The "dual" AdS space possesses isometries \( SO(2, 4) \).
- This dual conformal symmetry takes this solution to the most general one!
Let’s compute the area...

- Small problem: The area diverges!
- Dimensional reduction scheme: Start with $\mathcal{N} = 1$ in $D=10$ and go down to $D = 4 - 2\epsilon$.
- For integer $D$ this is exactly the low energy theory living on $Dp$–branes ($p = D - 1$)

### Gravity dual

$$
\begin{align*}
  ds^2 &= h^{-1/2}d\lambda_D^2 + h^{1/2} \left( dr^2 + r^2 d\Omega_{9-D}^2 \right), \\
  h &= \frac{c_D \lambda_D}{r^{8-D}} \\
  \lambda_D &= \frac{\lambda \mu^2}{(4\pi e^{-\gamma})^\epsilon} \\
  c_D &= 2^{4\epsilon} \pi^{3\epsilon} \Gamma(2 + \epsilon)
\end{align*}
$$

### T-dual coordinates

$$
\begin{align*}
  ds^2 &= \sqrt{\lambda_D c_D} \left( dy_D^2 + dr^2 \right) \\
  S &= \sqrt{\frac{\lambda_D c_D}{2\pi}} \int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon}
\end{align*}
$$

- Presence of $\epsilon$ will make the integrals convergent.
- The eoms will depend on $\epsilon$ but if we plug the original solution into the new action, the answer is accurate enough.
- plugging everything into the action...

$$
S \approx -\sqrt{\frac{\lambda_D c_D}{2\pi a^\epsilon}} \ _2F_1 \left( \frac{1}{2}, \frac{1 - \epsilon}{2}; \frac{1 - \epsilon}{2}; b^2 \right) + \mathcal{O}(\epsilon)
$$

- Just expand in powers of $\epsilon$...
Final answer

\[ \mathcal{A} = e^{iS} = \exp \left[ iS_{\text{div}} + \frac{\sqrt{\lambda}}{8\pi} \left( \log \frac{s}{t} \right)^2 + \tilde{C} \right] \]

\[ S_{\text{div}} = 2S_{\text{div},s} + 2S_{\text{div},t} \]

\[ S_{\text{div},s} = -\frac{1}{\epsilon^2} \frac{1}{2\pi} \sqrt{\lambda \mu^{2\epsilon}} (-s)^\epsilon - \frac{1}{\epsilon} \frac{1}{4\pi} (1 - \log 2) \sqrt{\lambda \mu^{2\epsilon}} (\frac{1}{(-s)^\epsilon}) \]

- Should be compared to the field theory answer

\[ \mathcal{A} \sim (\mathcal{A}_{\text{div},s})^2 (\mathcal{A}_{\text{div},t})^2 \exp \left\{ \frac{f(\lambda)}{8} (\ln s/t)^2 \right\} \]

\[ \mathcal{A}_{\text{div},s} = \exp \left\{ -\frac{1}{8\epsilon^2} f(-2) \left( \frac{\lambda \mu^{2\epsilon}}{s^\epsilon} \right) - \frac{1}{4\epsilon} g(-1) \left( \frac{\lambda \mu^{2\epsilon}}{s^\epsilon} \right) \right\} \]

**SO(2, 4)** transformations fixed somehow the kinematical dependence of the finite piece.

- This dual conformal symmetry constrains the form of the amplitude

**Dual Ward identity**

\[ \mathcal{O}_K \mathcal{A} = 0 \quad \rightarrow \quad \mathcal{O}_k \text{Fin} = -\mathcal{O}_k \text{Div} \]

For \( n = 4, 5 \) the solution is unique and agrees with BDS! for \( n = 6 \) there is some freedom.

- Also perturbatively and even dual super-conformal symmetry on NMHV (Sokatchev’s talk).
- Dual super-conformal symmetry present for all values of the coupling! (Berkovits’s talk)
What about the BDS ansatz?

- Symmetries “protect” BDS from corrections, we need to consider $n > 5$. What about $n = \infty$?

We choose a zig-zag configuration that approximates the rectangular Wilson loop.

\[
\log < W_{\text{weak}}^{\text{weak rect}} > = \frac{\lambda}{8\pi} \frac{T}{L}
\]

\[
\log < W_{\text{strong}}^{\text{strong rect}} > = \sqrt{\lambda} \frac{4\pi^2}{\Gamma(\frac{1}{4})^4} \frac{T}{L}
\]

- While BDS $\rightarrow \log < W_{\text{strong}}^{\text{strong rect}} > = \frac{\sqrt{\lambda}}{4} \frac{T}{L}$. Impressive explicit computations showed that indeed BDS fails for 6 gluons at two loops! (Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich)

This computation shows a relation between Wilson loops and scattering amplitudes.

\[
\mathcal{M}(k_1, ..., k_n) \approx < W(x_1, ..., x_n) >, \quad k_i = x_{i+1} - x_i
\]

- This relation holds also at weak coupling! (for MHV amplitudes)
  - Four legs at one loop (Drummond, Korchemsky, Sokatchev)
  - $n$ legs at one loop (Brandhuber, Heslop, Travaglini)
  - Up to six legs at two loops (Drummond, Henn, Korchemsky, Sokatchev)
Other processes (something like \( meson/\text{photon} \rightarrow q + \bar{q} \))

- Brane I extending along \((x^0, x^1)\) at \(Z = Z_{IR}\).
- Brane II extending along \((x^0, x^1, z)\).
- Meson \( \rightarrow (II, II) \), quarks \( \rightarrow (I, II) \).

- \((0, \kappa) \rightarrow (\kappa/2, \kappa/2) + (-\kappa/2, \kappa/2)\).
- After T-duality, a triangle in the \((y^0, y^1)\) plane with boundary conditions for \(r\).
- \(r = 0\) in the red lines (quarks), \(r = \infty\) in the blue line (meson).

Also possible to consider \( meson \rightarrow q + \bar{q} + \text{gluons} \).

Other processes like singlets into gluons.

We could not find the solutions but the singular behavior can be understood

\[
g_{quark}(\lambda) = \frac{\sqrt{\lambda}}{4\pi} (1 - 3 \log 2)
\]
Consider a massless gauge theory

**High spin, twist two operators**

\[ \mathcal{O}_S = Tr(\phi D^S \phi), \quad S \gg 1 \implies \Delta\mathcal{O}_s = f(\lambda) \log S - B(\lambda) + \ldots \]

**Scattering amplitudes**

\[ \log \mathcal{A} = \frac{f(\lambda)}{\epsilon^2} + \frac{g(\lambda)}{\epsilon} + \ldots \]

- Is there any relation between \( B(\lambda) \) and \( g(\lambda) \)?

They are not equal but \( g_R - B_R = C_RX \), where \( X \) is a universal function (Dixon, Magnea, Sterman).

All this quantities can be computed at strong coupling in planar \( \mathcal{N} = 4 \) SYM for gluons (adjoint) and quarks (fundamental).

\[ B_{gg} = \frac{\sqrt{\lambda}}{\pi} \left( \log \left( \frac{\sqrt{\lambda}}{2\pi} \right) + 1 - 2 \log 2 \right) \]

\[ B_{qq} = \frac{\sqrt{\lambda}}{2\pi} \left( \log \left( \frac{\sqrt{\lambda}}{2\pi} \right) + 1 - 3 \log 2 \right) \]

- Universality seems to hold at strong coupling!
Try to make explicit computations for $n > 4$, e.g. $n = 6$ is a good one.

Subleading corrections in $1/\sqrt{\lambda}$? Information about helicity of the particles, etc.

Gross and Mende computed higher genus amplitudes (in flat space) using similar ideas, can we do the same?

Can we repeat the computation in other backgrounds?

Deeper relation between Wilson loops and scattering amplitudes?

Some powerful alternative to BDS?

A lot of structure (some discovered and hopefully more waiting to be discovered) behind scattering amplitudes of planar MSYM.

For $n = 4, 5$, we think we know them to all values of the coupling!

We haven’t assume/use at all the machinery of integrability.

Talk by someone at strings 2009:

Expression for all planar MSYM amplitudes at all values of the coupling.