Dual superconformal symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory

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Based on work in collaboration with
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On-shell scattering amplitudes in $\mathcal{N} = 4$ SYM

- Scattering amplitudes in $\mathcal{N} = 4$ SYM
- Quantum numbers of on-shell states $|i\rangle = |p_i, h_i, a_i\rangle$: momentum ($p_i^2 = 0$), helicity ($h_i$), color ($a_i$)
- IR divergences $\mapsto$ dimensional regularization
  $A_n = \text{Div}(p_i, 1/\epsilon, \mu) \times \text{Fin}(p_i) \rightarrow$ subject of this talk
- Perturbative expansion in 't Hooft coupling $a = g^2 N/8\pi^2$:
  $A_n(p_i, h_i) = A_{n;0} + a \sum_{H} A_{n;1}^H M_{n;1}^H(p_i) + O(a^2)$

- Simplest example: Maximally Helicity Violating (MHV) amplitudes, e.g. for gluons: $(- - + \ldots +), (- + - \ldots +)$, etc.
  Unique helicity structure (tree):
  $A_n^{\text{MHV}}(p_1^-, p_2^-, p_3^+, \ldots, p_n^+) = A_{n;0}^{\text{MHV}} M_n^{\text{MHV}}(p_i), \quad M_n^{\text{MHV}} = 1 + a M_n^{(1)} + O(a^2)$

- $\mathcal{N} = 4$ SYM is a (super)conformal theory $\Rightarrow$ conformal symmetry of $A_n(p_i)$? Two problems:
  (i) Conformal boosts realized on momenta are 2nd-order differential operators [Witten'03]
  (ii) IR divergences break conformal symmetry
Dual conformal symmetry I

- Hidden symmetry of $\mathcal{A}_n$ of dynamical origin
- Linear action on the particle momenta in dual space

- $p_i = x_i - x_{i+1} \equiv x_{i+1} - x_i$ \ \ \ $\leftrightarrows \sum_i p_i = 0$ if $x_{n+1} \equiv x_1$
- $p_i^2 = 0$ \ \ \ $\Leftrightarrow x_{i+1}^2 = 0$
- Simple change of variables, not a Fourier transform!
- Conformal group $SO(4,2)$ acting on the dual coordinates $x_i$ \ \ \ $\Rightarrow$ dual conformal symmetry.

- Example: MHV amplitudes

$$\ln M_n^{\text{MHV}} = \ln Z_n(x_i, \epsilon, \mu) + \ln F_n(x_i) + O(\epsilon), \quad \ln Z_n = \sum_{i=1}^n \alpha_i \sum_{i=1}^n (1 + 2 \mu_i) \ln \left( \frac{\Gamma_{\text{cusp}}(l_i)}{l_i} + \frac{\Gamma(l_i)}{l_i} \right)$$

- Duality MHV amplitude/Wilson loop

$$\ln F_n = \ln \langle 0 | \text{Tr} P \exp \left( ig \oint_{C_n} dx^{\mu} A_{\mu}(x) \right) | 0 \rangle + \text{const} + O(\epsilon)$$

WL has conformal invariance in dual space $\Rightarrow$ Anomalous CWI :

$$K^\mu \ln F_n = \frac{1}{2} \Gamma_{\text{cusp}}(a) \sum_{i=1}^n \ln \frac{x_{i+1}^2}{x_{i-1}^2 x_{i+1}} \Rightarrow \text{Fixes } \ln F_n \text{ for } n = 4, 5 \text{ but not for } n \geq 6$$

Dual conformal symmetry II

- Can we generalize dual conformal symmetry to non-MHV amplitudes?
- Need to study the helicity structures and the loop corrections

- Spinor helicity formalism: commuting spinors $\lambda^\alpha$ (helicity -1/2), $\bar{\lambda}^{\dot{\alpha}}$ (helicity 1/2)

$$p_i^2 = 0 \ \ \ Leftrightarrow \ p_i^{\alpha \dot{\alpha}} \equiv p_i^{\mu} (\sigma_\mu)^{\alpha \dot{\alpha}} = \lambda_i^\alpha \bar{\lambda}_i^{\dot{\alpha}}$$

- Simplest case: MHV tree level

$$\mathcal{A}_{n;0}^{\text{MHV}} (\ldots i^- \ldots j^- \ldots) = \delta^{(4)}(\sum_{k=1}^n p_k) \frac{\langle i j \rangle^4}{(1 2)(2 3)\ldots(n 1)}, \quad \langle i j \rangle = -\langle j i \rangle = \epsilon^{\alpha \beta \lambda_i \lambda_j}$$

Is it dual conformal?

- Dual conformal transformations of spinors

$$I[x^{\mu}] = x^{\mu} x^{-1}, \quad I[x_i - x_j] = x_i^{-1} (x_i - x_j) x_j^{-1}$$

$$I[\lambda_i^{\alpha \dot{\alpha}}] = (x_i - x_{i+1})^{\alpha \dot{\alpha}} = \lambda_i^{\alpha \dot{\alpha}} \bar{\lambda}_i^{\dot{\alpha} \alpha} \ \ \ \Rightarrow$$

$$I[\lambda_i^\alpha] = \frac{\lambda_i^\alpha (x_i x_{i+1})^{\alpha \dot{\alpha}}}{x_i^2} \equiv \lambda_i x_i^{-1} = \lambda_i^\alpha \frac{(x_{i+1} x_i)^{\alpha \dot{\alpha}}}{x_i^2} \Rightarrow I[(i i + 1)] = \langle i x_{i+1}^{-1} x_i | i + 1 \rangle = \frac{(i i + 1)}{x_i^2}$$
Dual conformal symmetry III

- What about the momentum conservation delta function $\delta^{(4)}(\sum_{i=1}^{n} p_i)$?
- Cyclic symmetry: $\sum_{i=1}^{n} p_i = 0 \iff \sum_{i=1}^{n} (x_i - x_{i+1}) = 0$ iff $x_{n+1} \equiv x_1$
- Relax cyclicity, $x_1 \neq x_{n+1}$, and then impose it by
  $$\delta^{(4)}(x_1 - x_{n+1}) \rightarrow \text{manifestly dual conformal}$$

- Split-helicity tree amplitudes: all negative-helicity gluons appear contiguously
  - Known explicitly from recursion relations [Britto, Cachazo, Feng, Roiban, Spradlin, Volovich, Witten]
  - Example: split-helicity MHV tree amplitude – manifestly dual conformal!
    $$A_{n}^{\text{MHV}}(- - + \ldots +) = \delta^{(4)}(x_1 - x_{n+1}) \frac{\langle 1 2 \rangle^4 \langle 1 2 \rangle \langle 2 3 \rangle \ldots \langle n 1 \rangle}{\langle 1 2 \rangle \langle 2 3 \rangle \ldots \langle n 1 \rangle}$$
  - All split-helicity tree amplitudes are dual conformal
  - Non-split-helicity amplitudes are dual conformal not on their own, but as parts of superamplitudes
  - Superamplitudes in dual superspace exhibit dual superconformal symmetry.

Superamplitudes in on-shell superspace I

- $\mathcal{N} = 4$ gluon supermultiplet $\rightarrow$ PCT self-conjugate $\rightarrow$ holomorphic (chiral) description
  $$\Phi(p, \eta) = G^+(p) + \eta^A \Gamma_A(p) + \eta^A \eta^B S_{AB}(p) + \eta^A \eta^B \eta^C \epsilon_{ABCD} \bar{\Gamma}^D(p) + \eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} G^-(p)$$
  $\eta^A$ ($SU(4)$ index $A = 1 \ldots 4$, helicity 1/2) are Grassmann variables of on-shell superspace

- Superamplitudes $A_{n}(\Phi(1) \ldots \Phi(n)) = \text{expansion in powers of } \eta^A_i$

- Example: Nair’s description of tree MHV amplitudes [Nair’88]
  $$A_{n}^{\text{MHV}} = \frac{\delta^{(4)}(\sum_{i=1}^{n} p_i) \delta^{(8)}(\sum_{j=1}^{n} \lambda_j \alpha \eta^A_j)}{(1 2) \langle 1 2 \rangle \langle 2 3 \rangle \ldots \langle n 1 \rangle} = \frac{\delta^{(4)}(\sum_{i=1}^{n} p_i)}{(1 2) \langle 1 2 \rangle \langle 2 3 \rangle \ldots \langle n 1 \rangle} \left( \langle 1 2 \rangle^4 \eta^A_1 \eta^A_2 \eta^A_3 \ldots \eta^A_n \ldots \right)$$

- On-shell $\mathcal{N} = 4$ supersymmetry
  - generators
    $$q^A_\alpha = \lambda_\alpha \eta^A, \quad \bar{q}_{A \dot{\alpha}} = \bar{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \eta^A}$$
  - algebra (with $p^2 = 0$)
    $$\{ q^A_\alpha, \bar{q}_{B \dot{\alpha}} \} = \delta^A_B \lambda_\alpha \bar{\lambda}_{\dot{\alpha}} = \delta^A_B p_{\alpha \dot{\alpha}}$$
Superamplitudes in on-shell superspace II

- Invariance of the superamplitude: \( p_{\alpha \dot{\alpha}} A_n = q_{\alpha \dot{\alpha}}^A A_n = 0 \) ⇒
  \[
  A_n(\lambda, \tilde{\lambda}, \eta) = \delta^{(4)} \left( \sum_{i=1}^{n} p_i \right) \delta^{(8)} \left( \sum_{j=1}^{n} \lambda_j \eta_j \right) \left[ A_n^{(0)} + A_n^{(4)} + \ldots + A_n^{(4n-16)} \right]
  \]

- \( A_n^{(4k)}(\eta) \) – homogeneous polynomials in \( \eta \) of degree \( 4k \):
  \[ k = 0 \rightarrow \text{MHV}, \quad k = 1 \rightarrow \text{Next-to-MHV}, \ldots, \quad k = n - 4 \rightarrow \text{MHV} \]

- Simplest case – All-order MHV superamplitude:
  \[
  A_n^{\text{MHV}}(\lambda, \tilde{\lambda}, \eta) = \delta^{(4)} \left( \sum_{i=1}^{n} p_i \right) \delta^{(8)} \left( \sum_{j=1}^{n} \lambda_j \eta_j^A \right) \left[ M_n(p) / \langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle \right]
  \]

- Define ‘ratio’ \( R = \text{general/MHV superamplitude} \):
  \[
  A_n = A_n^{\text{MHV}} \times \left[ R_n(\lambda, \tilde{\lambda}, \eta) + O(\epsilon) \right] = A_n^{\text{MHV}} \left[ 1 + R_n^{(4)} + \ldots + R_n^{(4n-16)} + O(\epsilon) \right]
  \]
  \( R_n^{(4k)} \): finite homogeneous polynomials in \( \eta \) → helicity structures and loop corrections for all \( n \)-particle amplitudes.

- Conjecture: all \( R_n^{(4k)} \) are exactly dual conformal. The conformal anomaly is confined to the IR divergent MHV prefactor.

Dual \( \mathcal{N} = 4 \) superconformal symmetry

- Chiral dual superspace \( (x_{\alpha \dot{\alpha}}, \theta^A_{\alpha \dot{\alpha}}, \lambda_{\alpha}) \):

  \[ p = \sum_{i=1}^{n} p_i = 0 \rightarrow p_i = x_i - x_{i+1}, \quad x_{n+1} = x_1 \]

  \[ q = \sum_{i=1}^{n} \lambda_i \eta_i = 0 \rightarrow \lambda_i \eta_i^A = (\theta_i - \theta_{i+1})_A, \quad \theta_{n+1} = \theta_1 \]

- Dual \( \mathcal{N} = 4 \) superconformal symmetry in dual superspace

  \( \mathcal{N} = 4 \) super-Poincaré algebra

  \[
  Q_{A \alpha} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta_{i A \alpha}}, \quad \tilde{Q}_{\dot{A} \dot{\alpha}} = \sum_{i=1}^{n} \theta_{A \alpha}^{i \dot{A} \dot{\alpha}} \partial_{x_{i A \alpha}}, \quad P_{\alpha \dot{\alpha}} = \sum_{i=1}^{n} \partial_{x_{i A \alpha}}, \quad \{ Q_{A \alpha}, \tilde{Q}_{\dot{B} \dot{\alpha}} \} = \delta_{AB} P_{\alpha \dot{\alpha}}
  \]

  \[ \text{Conformal inversion: } I[x_i] = x_i^{-1}, \quad I[\theta_{i}] = \theta_{i} x_{i}^{-1}, \quad I[\lambda_{i}] = \lambda_{i} x_{i}^{-1} \]

  \[ \text{From Poincaré to conformal supersymmetry: } \]
  - Conformal boosts: \( K = IP \)
  - Special conformal supersymmetry: \( S = I\tilde{Q} I, \tilde{S} = I Q I \equiv \tilde{q} \)
Dual superconformal symmetry: MHV superamplitudes

- Impose cyclicity, \(x_{n+1} = x_1, \theta_{n+1} = \theta_1\), through delta functions. Then, only in \(\mathcal{N} = 4\),

\[
I[\delta^{(4)}(x_1 - x_{n+1})] = x_1^8 \delta^{(4)}(x_1 - x_{n+1}) \quad I[\delta^{(8)}(\theta_1 - \theta_{n+1})] = x_1^{-8} \delta^{(8)}(\theta_1 - \theta_{n+1})
\]

- MHV superamplitude in dual superspace

\[
\mathcal{A}^{\text{MHV}}_n(x, \theta, \lambda) = \frac{\delta^{(4)}(x_1 - x_{n+1}) \delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle} M_n(x_{ij})
\]

- Tree – manifestly dual superconformal covariant.
- Loops – IR divergent factor \(M_n(x_{ij})\) satisfies anomalous dual conformal Ward identity

- Part of the superconformal algebra \((\bar{Q}, S, P)\) is a symmetry of the whole amplitude, and \((\bar{Q}, S, K, D)\) only of the tree and the helicity structures (due to anomalies):

\[
\bar{Q}A = \bar{S}A = 0 \Rightarrow \{\bar{Q}, \bar{S}\}A = (D - C)A = 0 \Rightarrow D = C
\]

\(D = \text{conformal weight} \leftrightarrow \text{anomalous; } C = \text{helicity} \leftrightarrow \text{protected?}

Needs better understanding!

Dual superconformal symmetry: NMHV superamplitudes

- General superamplitude: \(A_n = \mathcal{A}^{\text{MHV}}_n(a, 1/\epsilon) \left[1 + R_4^{(4)} + \ldots + R_n^{(4n-16)} + O(\epsilon)\right]\)

- \(\mathcal{A}^{\text{MHV}}_n\) is IR divergent and satisfies an anomalous dual CWI \(\Leftrightarrow\) Wilson loop

- Conjecture: \(R_4^{(4)} = \text{finite dual (super)conformal invariants}\) [Drummond, Herrn, Korchemsky, ES’08]

- Evidence: One-loop NMHV superamplitudes

- \(n\)-gluon NMHV known [Bern, Dixon, Kosower’04]

- New result: One-loop NMHV superamplitude \(\Leftrightarrow\) dual (super)conformal invariant

\[
R_4^{(4)} = \sum_{p,q,r=1}^n c_{pqr} \delta^{(4)}(\Xi_{pqr}) M_{pqr}(x_{ij})
\]

- dual superconformal covariant

\[
\Xi_{pqr} = \langle p|_{x_{pq} x_{qr}}|\theta_{rp}\rangle + \langle p|_{x_{pr} x_{rq}}|\theta_{qp}\rangle = \langle p|_{x_{pq} x_{qr}} \sum_{i=p}^{r-1} |i\rangle \eta_i + \langle p|_{x_{pr} x_{rq}} \sum_{i=p}^{q-1} |i\rangle \eta_i
\]

- dual conformal covariant with matching conformal and helicity weights

\[
c_{pqr} = \frac{(q - 1) (r - 1)}{x_{qr}^2 \langle p|_{x_{pr} x_{rq}}|q - 1\rangle \langle p|_{x_{pqr}}|q\rangle \langle p|_{x_{pq} x_{qr}}|r - 1\rangle \langle p|_{x_{pq} x_{qr}}|r\rangle}
\]

- dual conformal invariant \(M_{pqr}(x_{ij}) = 1 + a M_{pqr}^{(\text{one-loop})} + O(a^2)\), made of finite combinations of one-loop scalar box integrals.
Dual superconformal symmetry: NMHV superamplitudes II

- The superstructure

\[ \delta^{(8)} \left( \sum_{i=1}^{n} \lambda_i \alpha_i \hat{\eta}_i^A \right) c_{pqr} \delta^{(4)}(\Xi_{pqr}) = \mathcal{H}_{m_1 m_2 m_3} \eta_{m_1}^A \eta_{m_2}^B \eta_{m_3}^C + \ldots \]

encodes all helicity structures for gluons, gluinos, scalars.

\[ H_m^1 m_2 m_3 = \text{3-mass-box coefficients} \quad \text{[Bern, Dixon, Kosower'04]} \]

- Expanding in \( \eta_i \) breaks manifest dual conformal symmetry, except for split-helicity terms. The non-split-helicity ones transform into each other.

- NMHV tree-level superamplitudes: new, manifestly Lorentz covariant form of the NMHV tree superamplitude obtained by setting all \( M_{pqr}(x_{ij}) = 1 \)

\[ A_{n;0}^{\text{NMHV}} = \delta^{(4)} \left( \sum_{i=1}^{n} \lambda_i \tilde{\lambda}_i \right) \delta^{(8)} \left( \sum_{j=1}^{n} \eta_j \right) \sum_{p,q,r=1}^{n} c_{pqr} \delta^{(4)}(\Xi_{pqr}) \]

No need for reference spinor!

[Cachazo, Svrcek, Witten'04], [Georgio, Glover, Khoze'04]

- “Twistor coplanarity” is a direct corollary of dual \( \bar{Q} \) supersymmetry.

Generalized unitarity in superspace

- Generalized unitarity – efficient method for computing one-loop corrections [Britto, Cachazo, Feng'04]

- Supersymmetrization – replace the sum over exchange particles by a Grassmann integral \( \int d^4 \eta \)

- Allows to compute the complete one-loop NMHV superamplitude [Drummond, Henn, Korchemsky, ES'08]

- Example: 3-mass-box coefficients

- 3-particle MHV tree superamplitude \( \leftrightarrow \) complexify the momenta, \( \tilde{\lambda} \neq \lambda \)

- Grassmann Fourier transform of the 3-particle \( \bar{\text{MHV}} \) tree superamplitude (needed to get the right degree in \( \eta \))

\[ A_{3;0}^{\bar{\text{MHV}}} = i(2\pi)^4 \delta^{(4)} \left( \sum_{i=1}^{3} \lambda_i \tilde{\lambda}_i \right) \frac{\delta^{(4)}(\eta_1[23] + \eta_2[31] + \eta_3[12])}{[12][23][31]} \]

The result is exactly the 3-point dual superconformal invariant \( c_{pqr} \delta^{(4)}(\Xi_{pqr}) \)

- 4-mass box coefficients contribute to NNMHV amplitudes. New type of superconformal invariant.
Conclusions and outlook

- Dual superconformal symmetry is a universal feature of $\mathcal{N} = 4$ scattering amplitudes.
- Its field-theory origin is unknown (dynamical). Recent explanation from string theory. [Berkovits, Maldacena'08], [Beisert, Ricci, Tseytlin'08]
- Probably the “tip of an iceberg” of an (infinite?) set of (non-local?) symmetries $\rightarrow$ integrability?
- non-MHV amplitudes $\rightarrow$ finite exactly dual conformal functions. Can we find differential equations for them? $\rightarrow$ integrability?
- The MHV/Wilson loop duality does not see the helicity structure. Need to supersymmetrize the WL and test if it is dual to non-MHV superamplitudes.
- Search for the complete $n$—particle tree superamplitude.