Dual superconformal symmetry of scattering amplitudes in $\mathcal{N}=4$ super Yang-Mills theory

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Based on work in collaboration with

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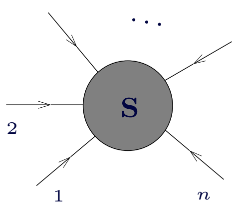
arXiv[hep-th]: 0707.0243, 0709.2368, 0712.1223, 0712.4138, 0803.1466, 0807.1095, 0808.0491

On-shell scattering amplitudes in $\mathcal{N}=4$ SYM

✔ Scattering amplitudes in $\mathcal{N}=4$ SYM

$\mathcal{A}_n =$



✗ Quantum numbers of on-shell states $|i\rangle = |p_i, h_i, a_i\rangle$: momentum ($p_i^2 = 0$), helicity (h_i), color (a_i)

✗ IR divergences $\mapsto$ dimensional regularization
$\mathcal{A}_n = \mathrm{Div}(p_i, 1/\epsilon, \mu) \times \mathrm{Fin}(p_i)$ $\rightarrow$ subject of this talk

✗ Perturbative expansion in 't Hooft coupling $a = g^2 N/8\pi^2$:

$$\mathcal{A}_n(p_i, h_i) = \mathcal{A}_{n;0} + a \sum_H \mathcal{A}^H_{n;1}\, M^H_{n;1}(p_i) + O(a^2)$$

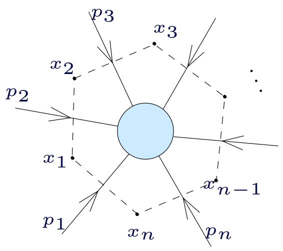
✔ Simplest example: Maximally Helicity Violating (MHV) amplitudes, e.g. for gluons: $(--+\ldots+)$, $(-+-+\ldots+)$, etc.
Unique helicity structure (tree):

$$\mathcal{A}_n^{\mathrm{MHV}}(p_1^-, p_2^-, p_3^+, \ldots, p_n^+) = \mathcal{A}_{n;0}^{\mathrm{MHV}}\; M_n^{\mathrm{MHV}}(p_i)\,, \qquad M_n^{\mathrm{MHV}} = 1 + aM_n^{(1)} + O(a^2)$$

✔ $\mathcal{N}=4$ SYM is a (super)conformal theory $\Rightarrow$ conformal symmetry of $\mathcal{A}_n(p_i)$? Two problems:
(i) Conformal boosts realized on momenta are 2nd-order differential operators [Witten'03]
(ii) IR divergences break conformal symmetry

Dual conformal symmetry I

- ✔ Hidden symmetry of $\mathcal{A}_n$ of dynamical origin
- ✔ Linear action on the particle momenta in dual space [Broadhurst'95],[Drummond,Henn,Smirnov,ES'06]



- ✗ $p_i = x_i - x_{i+1} \equiv x_{i\ i+1} \quad \Leftrightarrow \quad \sum_i p_i = 0 \ \text{ if } \ x_{n+1} \equiv x_1$
- ✗ $p_i^2 = 0 \quad \Leftrightarrow \quad x_{i\ i+1}^2 = 0$
- ✗ Simple change of variables, not a Fourier transform!
- ✗ Conformal group $SO(4,2)$ acting on the dual coordinates x_i $\Longrightarrow$ dual conformal symmetry.

- ✔ Example: MHV amplitudes

$$\ln M_n^{\rm MHV} = \ln Z_n(x_i, \epsilon, \mu) + \ln F_n(x_i) + O(\epsilon)\,, \quad \ln Z_n = \sum_{l\geq 1} a^l \sum_{i=1}^{n} (-x_{i,i+2}^2 \mu^2)^{l\epsilon} \left(\frac{\Gamma^{(l)}_{\rm cusp}}{(l\epsilon)^2} + \frac{\Gamma^{(l)}}{l\epsilon} \right)$$

- ✔ Duality MHV amplitude/Wilson loop [Alday,Maldacena'07], [Drummond,Korchemsky,ES'07], [Brandhuber,Heslop,Travaglini'07]

$$\ln F_n = \ln \langle 0 | {\rm Tr\,P} \exp \left(ig \oint_{C_n} dx^\mu A_\mu(x) \right) |0\rangle + {\rm const} + O(\epsilon)$$

WL has conformal invariance in dual space $\Rightarrow$ Anomalous CWI : [Drummond,Henn,Korchemsky, ES'07]

$$K^\mu \ln F_n = \frac{1}{2} \Gamma_{\rm cusp}(a) \sum_{i=1}^{n} \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} x_{i,i+1}^\nu \quad \Rightarrow \quad \text{Fixes } \ln F_n \text{ for } n = 4,5 \text{ but not for } n \geq 6$$

Dual conformal symmetry II

- ✔ Can we generalize dual conformal symmetry to non-MHV amplitudes?
  Need to study the helicity structures and the loop corrections
- ✔ Spinor helicity formalism: commuting spinors λ^α (helicity -1/2), $\tilde\lambda^{\dot\alpha}$ (helicity 1/2) [Xu,Zhang,Chang'87]

$$p_i^2 = 0 \ \Leftrightarrow \ p_i^{\alpha\dot\alpha} \equiv p_i^\mu (\sigma_\mu)^{\alpha\dot\alpha} = \lambda_i^\alpha \, \tilde\lambda_i^{\dot\alpha}$$

- ✔ Simplest case: MHV tree level [Parke, Taylor'86]

$$\mathcal{A}_{n;0}^{\rm MHV}(\ldots i^- \ldots j^- \ldots) = \delta^{(4)}(\sum_{k=1}^n p_k) \, \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \langle 2\, 3\rangle \ldots \langle n\, 1\rangle}\,, \qquad \langle i\, j \rangle = -\langle j\, i\rangle = \epsilon^{\alpha\beta} \lambda_{i\,\alpha}\, \lambda_{j\,\beta}$$

Is it dual conformal?

- ✔ Dual conformal transformations of spinors [Drummond,Henn,Korchemsky, ES'08]
  - ✗ Conformal group = Poincaré + inversion:

$$I[x^\mu] = \frac{x^\mu}{x^2} \equiv x^{-1}\,, \qquad I[x_i - x_j] = x_i^{-1} (x_i - x_j) x_j^{-1}$$

  - ✗ Transforming spinors: $p_i^{\alpha\dot\alpha} = (x_i - x_{i+1})^{\alpha\dot\alpha} = \lambda_i^\alpha \, \tilde\lambda_i^{\dot\alpha} \quad \Rightarrow$

$$I[\lambda_i^\alpha] = \frac{\lambda_i^\alpha (x_i)_{\alpha\dot\alpha}}{x_i^2} \equiv \lambda_i\, x_i^{-1} = \lambda_i^\alpha \frac{(x_{i+1})_{\alpha\dot\alpha}}{x_i^2} \ \Rightarrow \ I[\langle i\ i+1\rangle] = \langle i | \frac{x_{i+1}}{x_i^2} x_{i+1}^{-1} | i+1 \rangle = \frac{\langle i\ i+1\rangle}{x_i^2}$$

Dual conformal symmetry III

- ✔ What about the momentum conservation delta function $\delta^{(4)}(\sum_{i=1}^{n} p_i)$?
  - ✗ Cyclic symmetry : $\sum_{i=1}^{n} p_i = 0 \;\Leftrightarrow\; \sum_{i=1}^{n} (x_i - x_{i+1}) = 0$ iff $x_{n+1} \equiv x_1$
  - ✗ Relax cyclicity, $x_1 \neq x_{n+1}$, and then impose it by

$$\delta^{(4)}(x_1 - x_{n+1}) \;\rightarrow\; \text{manifestly dual conformal}$$

- ✔ Split-helicity tree amplitudes: all negative-helicity gluons appear contiguously
  - ✗ Known explicitly from recursion relations [Britto, Cachazo, Feng, Roiban, Spradlin, Volovich, Witten]
  - ✗ Example: split-helicity MHV tree amplitude – manifestly dual conformal!

$$\mathcal{A}_n^{\rm MHV}(--+\ldots+) = \delta^{(4)}(x_1 - x_{n+1})\,\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}$$

  - ✗ All split-helicity tree amplitudes are dual conformal
  - ✗ Non-split-helicity amplitudes are dual conformal not on their own, but as parts of superamplitudes
  - ✗ Superamplitudes in dual superspace exhibit dual superconformal symmetry.

Superamplitudes in on-shell superspace I

- ✔ $\mathcal{N}=4$ gluon supermultiplet $\rightarrow$ PCT self-conjugate $\rightarrow$ holomorphic (chiral) description

$$\Phi(p,\eta) = G^+(p) + \eta^A\Gamma_A(p) + \eta^A\eta^B S_{AB}(p) + \eta^A\eta^B\eta^C\epsilon_{ABCD}\bar\Gamma^D(p) + \eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}G^-(p)$$

  η^A ($SU(4)$ index $A = 1\ldots4$, helicity $1/2$) are Grassmann variables of on-shell superspace

- ✔ Superamplitudes $\mathcal{A}_n(\Phi(1)\ldots\Phi(n))$ = expansion in powers of η_i^A
- ✔ Example: Nair's description of tree MHV amplitudes [Nair'88]

$$\mathcal{A}_n^{\rm MHV} = \frac{\delta^{(4)}(\sum_{i=1}^{n} p_i)\,\delta^{(8)}(\sum_{j=1}^{n} \lambda_{j\,\alpha}\,\eta_j^A)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle} = \frac{\delta^{(4)}(\sum_{i=1}^{n} p_i)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}\Big(\langle 12\rangle^4\,\eta_1^4\eta_2^4\eta_3^0\ldots\eta_n^0 + \ldots\Big)$$

- ✔ On-shell $\mathcal{N}=4$ supersymmetry
  - ✗ generators

$$q_\alpha^A = \lambda_\alpha\eta^A\,, \qquad \bar q_{A\,\dot\alpha} = \tilde\lambda_{\dot\alpha}\frac{\partial}{\partial\eta^A}$$

  - ✗ algebra (with $p^2=0$)

$$\{q_\alpha^A, \bar q_{B\,\dot\alpha}\} = \delta_B^A\,\lambda_\alpha\tilde\lambda_{\dot\alpha} = \delta_B^A\,p_{\alpha\dot\alpha}$$

Superamplitudes in on-shell superspace II

- ✔ Invariance of the superamplitude: $p_{\alpha\dot{\alpha}}\mathcal{A}_n = q^A_\alpha \mathcal{A}_n = 0 \ \Rightarrow$

$$\mathcal{A}_n(\lambda,\tilde{\lambda},\eta) = \delta^{(4)}(\sum_{i=1}^n p_i)\ \delta^{(8)}(\sum_{j=1}^n \lambda_j\,\eta_j)\Big[\mathcal{A}_n^{(0)} + \mathcal{A}_n^{(4)} + \ldots + \mathcal{A}_n^{(4n-16)}\Big]$$

- ✔ $\mathcal{A}_n^{(4k)}(\eta)$ – homogeneous polynomials in η of degree $4k$:
  $k=0 \ \to$ MHV, $k=1 \ \to$ Next-to-MHV, $\ldots$, $k=n-4 \ \to \ \overline{\text{MHV}}$
- ✔ Simplest case – All-order MHV superamplitude:

$$\mathcal{A}_n^{\rm MHV}(\lambda,\tilde{\lambda},\eta) = \delta^{(4)}(\sum_{i=1}^n p_i)\ \delta^{(8)}(\sum_{j=1}^n \lambda_{j\,\alpha}\,\eta_j^A)\Big[\frac{M_n(p)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}\Big]$$

- ✔ Define 'ratio' R = general/MHV superamplitude:

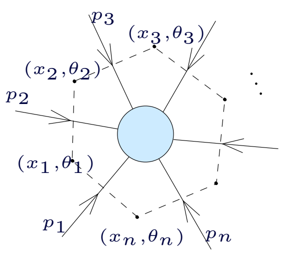
$$\mathcal{A}_n = \mathcal{A}_n^{\rm MHV}\times\Big[R_n(\lambda,\tilde{\lambda},\eta)+O(\epsilon)\Big] = \mathcal{A}_n^{\rm MHV}\ \Big[1+R_n^{(4)}+\ldots+R_n^{(4n-16)}+O(\epsilon)\Big]$$

  $R_n^{(4k)}$: finite homogeneous polynomials in $\eta \to$ helicity structures and loop corrections for all n–particle amplitudes.

- ✔ Conjecture: all $R_n^{(4k)}$ are exactly dual conformal. The conformal anomaly is confined to the IR divergent MHV prefactor.

Dual $\mathcal{N}=4$ superconformal symmetry

- ✔ Chiral dual superspace $(x_{\alpha\dot{\alpha}}\,,\,\theta^A_\alpha\,,\,\lambda_\alpha)$:



  - ✗ $p=\sum_{i=1}^n p_i = 0 \ \to \quad p_i = x_i - x_{i+1}\,,\quad x_{n+1}=x_1$
  - ✗ $q=\sum_{i=1}^n \lambda_i\eta_i = 0 \ \to \quad \lambda_{i\,\alpha}\,\eta_i^A = (\theta_i-\theta_{i+1})^A_\alpha\,,\quad \theta_{n+1}=\theta_1$

- ✔ Dual $\mathcal{N}=4$ superconformal symmetry in dual superspace
  - ✗ $\mathcal{N}=4$ super-Poincaré algebra

$$Q_{A\,\alpha} = \sum_{i=1}^n \frac{\partial}{\partial\theta_i^{A\,\alpha}}\,,\quad \bar{Q}^A_{\dot{\alpha}} = \sum_{i=1}^n \theta_i^{A\,\alpha}\frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}}\,,\ P_{\alpha\dot{\alpha}} = \sum_{i=1}^n \frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}}\,;\qquad \{Q_{A\,\alpha},\bar{Q}^B_{\dot{\alpha}}\} = \delta^B_A\,P_{\alpha\dot{\alpha}}$$

  - ✗ Conformal inversion: $I[x_i]=x_i^{-1}\,,\quad I[\theta_i]=\theta_i x_i^{-1}\,,\quad I[\lambda_i]=\lambda_i x_i^{-1}$
  - ✗ From Poincaré to conformal supersymmetry:
    - Conformal boosts: $K = IPI$
    - Special conformal supersymmetry : $S = I\bar{Q}I$, $\bar{S} = IQI \equiv \bar{q}$

Dual superconformal symmetry: MHV superamplitudes

- ✔ Impose cyclicity, $x_{n+1} = x_1$, $\theta_{n+1} = \theta_1$, through delta functions. Then, only in $\mathcal{N} = 4$,

$$I[\delta^{(4)}(x_1 - x_{n+1})] = x_1^8\, \delta^{(4)}(x_1 - x_{n+1}) \qquad I[\delta^{(8)}(\theta_1 - \theta_{n+1})] = x_1^{-8}\, \delta^{(8)}(\theta_1 - \theta_{n+1})$$

- ✔ MHV superamplitude in dual superspace

$$\mathcal{A}_n^{\rm MHV}(x, \theta, \lambda) = \frac{\delta^{(4)}(x_1 - x_{n+1})\, \delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 1\,2\rangle\langle 2\,3\rangle \ldots \langle n\,1\rangle}\, M_n(x_{ij})$$

  - ✗ Tree – manifestly dual superconformal covariant.
  - ✗ Loops – IR divergent factor $M_n(x_{ij})$ satisfies anomalous dual conformal Ward identity

- ✔ Part of the superconformal algebra $(Q, \bar{S}, P)$ is a symmetry of the whole amplitude, and $(\bar{Q}, S, K, D)$ only of the tree and the helicity structures (due to anomalies):

$$\bar{Q}\mathcal{A} = \bar{S}\mathcal{A} = 0 \ \Rightarrow\ \{\bar{Q}, \bar{S}\}\mathcal{A} = (D - C)\mathcal{A} = 0 \ \Rightarrow\ D = C$$

  $D =$ conformal weight $\mapsto$ anomalous; $\qquad C =$ helicity $\mapsto$ protected ?
  Needs better understanding !

Dual superconformal symmetry: NMHV superamplitudes I

- ✔ General superamplitude: $\mathcal{A}_n = \mathcal{A}_n^{\rm MHV}(a, 1/\epsilon)\left[1 + R_n^{(4)} + \ldots + R_n^{(4n-16)} + O(\epsilon)\right]$
  - ✗ $\mathcal{A}_n^{\rm MHV}$ is IR divergent and satisfies an anomalous dual CWI $\Leftrightarrow$ Wilson loop
  - ✗ Conjecture: $R_n^{(4)}$ are finite dual (super)conformal invariants [Drummond,Henn,Korchemsky,ES'08]

- ✔ Evidence: One-loop NMHV superamplitudes
  - ✗ n–gluon NMHV known [Bern, Dixon, Kosower'04]
  - ✗ New result: One-loop NMHV superamplitude $\Leftrightarrow$ dual (super)conformal invariant

$$R_n^{(4)} = \sum_{p,q,r=1}^{n} c_{pqr}\, \delta^{(4)}(\Xi_{pqr})\, M_{pqr}(x_{ij})$$

  - dual superconformal covariant

$$\Xi_{pqr} = \langle p|x_{pq}x_{qr}|\theta_{rp}\rangle + \langle p|x_{pr}x_{rq}|\theta_{qp}\rangle = \langle p|x_{pq}x_{qr}\sum_{i=p}^{r-1}|i\rangle\eta_i + \langle p|x_{pr}x_{rq}\sum_{i=p}^{q-1}|i\rangle\eta_i$$

  - dual conformal covariant with matching conformal and helicity weights

$$c_{pqr} = \frac{\langle q-1\,q\rangle\langle r-1\,r\rangle}{x_{qr}^2\langle p|x_{pr}x_{r\,q-1}|q-1\rangle\langle p|x_{pr}x_{r\,q}|q\rangle\langle p|x_{pq}x_{q\,r-1}|r-1\rangle\langle p|x_{pq}x_{q\,r}|r\rangle}$$

  - dual conformal invariant $M_{pqr}(x_{ij}) = 1 + aM_{pqr}^{(\rm one-loop)} + ?\ O(a^2)$, made of finite combinations of one-loop scalar box integrals

Dual superconformal symmetry: NMHV superamplitudes II

- ✔ The superstructure

$$\delta^{(8)}(\sum_{i=1}^{n} \lambda_{i\,\alpha}\,\eta_i^A)\; c_{pqr}\;\delta^{(4)}(\Xi_{pqr}) = \mathcal{H}_{m_1m_2m_3}\;\eta_{m_1}^4\;\eta_{m_2}^4\;\eta_{m_2}^4 + \ldots$$

encodes all helicity structures for gluons, gluinos, scalars.

$\mathcal{H}_{m_1m_2m_3}$ $\Leftrightarrow$ 3-mass-box coefficients [Bern,Dixon, Kosower'04]

- ✔ Expanding in η_i breaks manifest dual conformal symmetry, except for split-helicity terms. The non-split-helicity ones transform into each other
- ✔ NMHV tree-level superamplitudes: new, manifestly Lorentz covariant form of the NMHV tree superamplitude obtained by setting all $M_{pqr}(x_{ij}) = 1$

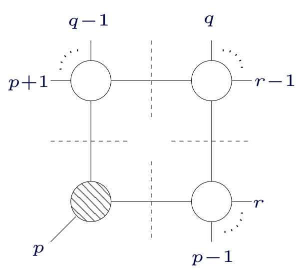
$$\mathcal{A}_{n;0}^{\mathrm{NMHV}} = \delta^{(4)}(\sum_{i=1}^{n} \lambda_i\tilde{\lambda}_i)\;\delta^{(8)}(\sum_{j=1}^{n} \lambda_j\,\eta_j)\;\sum_{p,q,r=1}^{n} c_{pqr}\;\delta^{(4)}(\Xi_{pqr})$$

No need for reference spinor ! [Cachazo, Svrcek, Witten'04],[Georgio, Glover, Khoze'04]

- ✔ "Twistor coplanarity" [Witten'03], [Cachazo, Svrcek, Witten'04]
is a direct corollary of dual $\bar{Q}$ supersymmetry.

Generalized unitarity in superspace

- ✔ Generalized unitarity – efficient method for computing one-loop corrections [Britto, Cachazo, Feng'04]
- ✔ Supersymmetrization – replace the sum over exchange particles by a Grassmann integral $\int d^4\eta$
- ✔ Allows to compute the complete one-loop NMHV superamplitude [Drummond,Henn,Korchemsky,ES'08]
- ✔ Example: 3-mass-box coefficients



- ✗ 3-particle MHV tree superamplitude $\leftrightarrow$ complexify the momenta, $\tilde{\lambda} \neq \bar{\lambda}$
- ✗ Grassmann Fourier transform of the 3-particle $\overline{\mathrm{MHV}}$ tree superamplitude (needed to get the right degree in η)

$$\mathcal{A}_{3;0}^{\overline{\mathrm{MHV}}} = i(2\pi)^4\delta^{(4)}(\sum_{i=1}^{3} \lambda_i^{\alpha}\,\tilde{\lambda}_i^{\dot{\alpha}})\,\frac{\delta^{(4)}(\eta_1[23]+\eta_2[31]+\eta_3[12])}{[12][23][31]}$$

The result is exactly the 3-point dual superconformal invariant $c_{pqr}\;\delta^{(4)}(\Xi_{pqr})$

- ✔ 4-mass box coefficients contribute to NNMHV amplitudes. New type of superconformal invariant.

Conclusions and outlook

- ✔ Dual superconformal symmetry is a universal feature of $\mathcal{N} = 4$ scattering amplitudes
- ✔ Its field-theory origin is unknown (dynamical). Recent explanation from string theory.
  [Berkovits, Maldacena'08], [Beisert,Ricci,Tseytlin'08]
- ✔ Probably the "tip of an iceberg" of an (infinite?) set of (non-local?) symmetries $\longrightarrow$ integrability?
- ✔ non-MHV amplitudes $\longrightarrow$ finite exactly dual conformal functions. Can we find differential equations for them? $\longrightarrow$ integrability?
- ✔ The MHV/Wislon loop duality does not see the helicity structure. Need to supersymmetrize the WL and test if it is dual to non-MHV superamplitudes.
- ✔ Search for the complete $n-$particle tree superamplitude.