Fermionic T-Duality

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NB and Juan Maldacena, arXiv: 0807.3196
“Fermionic T-duality, dual superconformal symmetry, and the amplitude/Wilson loop connection”

Aladay and Maldacena used invariance of $AdS_5 \times S^5$ metric under T-duality to relate planar $N=4$ $d=4$ SYM amplitudes and Wilson line correlation functions at strong 't Hooft coupling.

\[ \frac{1}{y^2} (dx^m dx_m + dy dy) \rightarrow y^2 d\bar{x}^m d\bar{x}_m + \frac{dy dy}{y^2} = \frac{1}{(y')^2} (d\bar{x}^m d\bar{x}_m + dy' dy') \]

But full $AdS_5 \times S^5$ background was not invariant since T-duality mapped $F_{01234}^{RR} \rightarrow F_4^{RR}$ and mapped $\phi \rightarrow \phi + 4 \log y$.

So amplitude/Wilson line relation was only proven in limit $r_{AdS} \rightarrow \infty$ where effects of $F^{RR}$ and $\phi$ can be ignored.

Drummond, Henn, Korchemsky, Smirnov, Sokatchev used perturbative computations to show that planar $N=4$ $d=4$ SYM amplitudes and Wilson line correlation functions are also related at weak 't Hooft coupling.

Question: Can Alday-Maldacena result be extended to planar amplitudes at arbitrary 't Hooft coupling?
By including bosonic and fermionic T-dualities, invariance of \( AdS_5 \) metric can be extended to invariance of full \( AdS_5 \times S^5 \) background.

When superstring background has "abelian" fermionic isometry (i.e. \( q^2 = 0 \)), usual bosonic T-duality has straightforward generalization to fermionic T-duality. Fermionic T-duality leaves \( g_{mn} \) and \( b_{mn} \) invariant but transforms \( F^{RR} \) and \( \Phi \).

After performing bosonic T-duality on four \( x^m \) variables and performing fermionic T-duality on eight \( \Theta^j \) variables, the \( AdS_5 \times S^5 \) background is invariant.

Ignoring the issue of polarization dependence, this explains the relation of planar SYM amplitudes and Wilson lines at arbitrary 't Hooft coupling.

Fermionic T-duality has other applications such as mapping superstring in flat background to superstring in self-dual graviphoton background.

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**Review of Bosonic T-Duality (ala Buscher)**

Suppose string theory background is invariant under the constant shift \( x' \rightarrow x' + c \Rightarrow x' \) only appears with derivatives in sigma model.

\[
S = \int d^2 \xi \left( g_{mn} b_{mn} \right) \partial x^m \partial x^n
\]

\[
= \int d^2 \xi \left[ g_{ii} \partial x'^i \partial x^i + \frac{l_{ii}}{g_{ii}} \partial x'^i \partial x^i + l_{ii} \partial x^m \partial x^n + l_{mn} \partial x^m \partial x^n \right]
\]

Integrate out \( \partial x' \). Use that \( \partial A - \partial A = 0 \Rightarrow A = \partial x' \) and \( \partial A = \partial A' \) for some \( x' \).

\[
= \int d^2 \xi \left[ g_{ii} A \partial A + l_{ii} \partial x^m A + l_{mn} \partial x^m \partial A + l_{mn} \partial x^m \partial x^n + \partial x' \left( \partial A - \partial A \right) \right]
\]

Integrate out \( A \) and \( \partial A \).

\[
= \int d^2 \xi \left[ \frac{l_{ii}}{g_{ii}} \partial x' \partial x^i + \frac{l_{ii}}{g_{ii}} \partial x'^i \partial x^m - \frac{l_{ii}}{g_{ii}} \partial x^m A + \left( l_{mn} - \frac{l_{im} l_{jn}}{g_{ij}} \right) \partial x^m \partial x^n \right]
\]
In Buscher procedure using non-compact $x'$ variable, \( \overline{\partial} A - \partial \overline{A} = 0 \) implies \( A = \partial x' \) and \( \overline{A} = \overline{\partial} x' \) only if \( \int_C (A d \overline{z} + \overline{A} d z) = 0 \) around all non-trivial cycles \( C \).

This is not an issue for genus zero amplitudes which have no non-trivial cycles.

But on higher genus surfaces, T-duality is only a symmetry if \( x' \) is a compact variable.

When \( x' \) is compact, \( \overline{\partial} A - \partial \overline{A} = 0 \) implies that \( A = \partial x' \) and \( \overline{A} = \overline{\partial} x' \) where \( \int_C (A d \overline{z} + \overline{A} d z) \) is the winding number of \( x' \) around the cycle \( C \).

After T-dualizing \( x' \to \overline{x}' \),

\[
S = \int d \overline{z} \left[ \frac{i}{g_{11}} \partial_\tau \overline{x}' \partial x'^* + \frac{b_{11}}{g_{11}} \partial_\tau \overline{x}' \partial x'^* - \overline{\partial}_\tau \overline{x}' \partial x'^* + \left( b_{11} - \frac{b_{11} b_{11}}{g_{11}} \right) \partial_\tau \overline{x}' \partial x'^* \right]
\]

implies that background fields \( b_{11} \to g_{11} + b_{11} \) transform as

\[
g_{11}' = \frac{1}{g_{11}}, \quad b_{11}' = \frac{b_{11}}{g_{11}}, \quad l_{1m}' = \frac{l_{1m}}{g_{11}}, \quad l_{m1}' = -\frac{l_{m1}}{g_{11}}, \quad l_{mn}' = l_{mn} - \frac{b_{11} l_{mn}}{g_{11}}
\]

Integration over \( A \) and \( \overline{A} \) produces a Jacobian factor \( (\det g_{11})^{-1} \) which is absorbed by transforming the dilaton as

\[
\varphi' = \varphi - \frac{1}{2} \log g_{11}.
\]

= sign in \( l_{1m}' \) implies that Dirichlet/Neumann boundary conditions for \( x' \) variable are switched on a D-brane.

Bosonic T-duality changes dimension of D-brane.
Fermionic T-Duality

In spacetime-supersymmetric sigma models (GS, pure spinor, hybrid, ...), supergravity background fields are combined into spacetime superfields $G_{MN}(Y)$ and $B_{MN}(Y)$ such that the action is

$$S = \int d^2 \bar{z} \left[ (G_{MN} + B_{MN}) \partial Y^m \bar{\partial} Y^n + ... \right].$$

For Type II supergravity, $Y^m = (x^m, \theta^m)$ where $m = 0$ to 9 and $\mu = 1$ to 32. In pure spinor and hybrid formalisms, ... in action includes couplings to fermionic momenta and ghosts which are needed for quantization.

Torsion and curvature constraints imply that $G_{MN}$ and $B_{MN}$ can be expressed in terms of usual supergravity fields. For example, near flat space, $G_{MN} = g_{MN} + \chi^m_{\nu} (\gamma^\nu \theta^m) + F^{\mu\nu} (\gamma^\nu \theta^m) (\gamma^\mu \theta^m) + ...$

gravitino $\downarrow$ R-R field strength

Suppose background superfields are invariant under constant fermionic shift $\theta' \rightarrow \theta' + \phi \Rightarrow \theta' only appears with derivatives in action.

Invariance implies background has "abelian" susy $q$ satisfying $q^2 = 0$.

$$S = \int d^2 \bar{z} \left[ (G_{MN} + B_{MN}) \partial Y^m \bar{\partial} Y^n \right] = \int d^2 \bar{z} \left[ B_{\bar{m}} \partial \theta' \bar{\partial} \theta' + L_{m1} \partial \theta' \bar{\partial} Y^m + L_{m1} \partial Y^m \bar{\partial} \theta' + L_{mN} \partial Y^m \bar{\partial} Y^n \right]$$

Integrate out $\theta'$. Use that $\partial \bar{a} \rightarrow \partial \bar{a} = 0 \Rightarrow a = \partial \theta'$ and $\bar{a} = \bar{\partial} \theta'$ for some $\theta'$.

$$= \int d^2 \bar{z} \left[ B_{\bar{m}} a \bar{a} + L_{m1} a \bar{\partial} Y^m + L_{m1} \partial Y^m \bar{a} + L_{mN} \partial Y^m \bar{\partial} Y^n + \bar{\theta}' (\partial \bar{a} - \bar{\partial} a) \right]$$

Integrate out $a$ and $\bar{a}$.

$$= \int d^2 \bar{z} \left[ -\frac{i}{\theta'} \partial \bar{\theta}' \bar{\partial} \theta' + \frac{L_{m1}}{B_{\bar{m}}} \partial \theta' \bar{\partial} Y^m + \frac{L_{m1}}{B_{\bar{m}}} \partial Y^m \bar{\partial} \theta' + (L_{mN} - \frac{L_{m1} L_{m1}}{B_{\bar{m}}}) \partial Y^m \bar{\partial} Y^n \right]$$
Comment: In spacetime supersymmetric sigma models, $\theta'$ is single-valued around any non-trivial cycle. (like a non-compact $x'$ variable)

So $\partial a - \partial \bar{a} = 0$ implies $a = 2\theta'$ and $\bar{a} = \bar{\theta'}$ only if $\int_C (a \partial \bar{\theta} + \bar{a} \partial \theta) = 0$ around every non-trivial cycle $C$.

This is not an issue for genus zero amplitudes which have no non-trivial cycles.

But on higher genus surfaces, fermionic T-duality is not a symmetry. To make it a symmetry, would need to introduce multivalued fermionic variable $\theta'$ satisfying $\theta' \rightarrow \theta' + \xi_C$ when $\theta'$ goes around cycle $C$.

$\xi_C$ is a fermionic zero mode which would need to be included in the functional integral for $\int d\theta'$.

After T-dualizing $\theta' \rightarrow \bar{\theta'}$,

$$S = \int d^2 \bar{z} \left[ - \frac{1}{8\pi} \partial \phi \bar{\partial} \phi + \frac{L_{im}}{8\pi} \partial \phi \bar{\partial} \phi + \frac{L_{mi}}{8\pi} \bar{\partial} \phi \partial \phi + (L_{mn} - \frac{L_{im} L_{mi}}{8\pi}) \partial \phi \bar{\partial} \phi \right]$$

implies that background superfields $L_{mn} = G_{mn} + B_{mn}$ transform as

$$B'_{\bar{m} \bar{n}} = - \frac{1}{B_{\bar{m} \bar{n}}} , \quad L'_{1n} = \frac{L_{1n}}{B_{\bar{m} \bar{n}}} , \quad L'_{1m} = + \frac{L_{1m}}{B_{\bar{m} \bar{n}}} , \quad L'_{mn} = L_{mn} - \frac{L_{im} L_{mi}}{B_{\bar{m} \bar{n}}}$$

Integration over fermionic $a$ and $\bar{a}$ produces a Jacobian factor $(\det B_{\bar{m} \bar{n}})^{-1}$ which is absorbed by transforming the dilaton as

$$\phi' = \phi + \frac{1}{2} \log B_{\bar{m} \bar{n}}$$

+ sign in $L'_{1m}$ implies that Dirichlet/Neumann boundary conditions for $\theta'$ variable are not switched on a D-brane.

Fermionic T-duality does not change dimension of D-brane.
To derive T-duality transformations of component supergravity fields, it is very convenient to use pure spinor formalism as was done by Benichou, Policastro, Troost for bosonic T-duality.

Defining $C = B_{11} \big|_{\theta = 0}$, one finds that after T-dualization of $\Theta'$,

\[ g'_{mn} = g_{mn}, \quad b'_{mn} = b_{mn}, \quad \Phi' = \Phi + \frac{1}{2} \log C, \quad e^\alpha F^{\alpha} = e^\alpha F^{\alpha} - \epsilon^\alpha \epsilon^\beta C^{-1} \]

$\alpha, \beta = 1$ to 16 are $N=2$ d=10 spinor indices

\[ F^{\alpha} = \gamma_m{}^{\alpha} F^m + \frac{1}{3!} \gamma_m{}^{\alpha} F^{mn} + \frac{1}{2 \cdot 5!} \gamma_m{}^{\alpha} F^{mnp} + \frac{1}{6 \cdot 9!} \gamma_m{}^{\alpha} F^{mnop} \]

$(\epsilon^\alpha, \epsilon^\beta)$ is Killing spinor associated with abelian susy

Abelian susy $\Rightarrow$ $\epsilon^\alpha (\gamma_m)_{d\rho} \epsilon^\rho - \epsilon^\beta (\gamma_m)_{d\sigma} \epsilon^\sigma = 0$

Torsion constraints $\Rightarrow$ $\epsilon^\alpha (\gamma_m)_{d\rho} \epsilon^\rho - \epsilon^\beta (\gamma_m)_{d\sigma} \epsilon^\sigma = \partial_m C$

Using $\partial_m C = \epsilon^\alpha (\gamma_m)_{d\rho} \epsilon^\rho - \epsilon^\beta (\gamma_m)_{d\sigma} \epsilon^\sigma$, can easily check that transformed background fields

$g'_{mn} = g_{mn}, \quad b'_{mn} = b_{mn}, \quad \Phi' = \Phi + \frac{1}{2} \log C, \quad e^\alpha F^{\alpha} = e^\alpha F^{\alpha} - \epsilon^\alpha \epsilon^\beta C^{-1}$

satisfy supergravity equations of motion and are invariant under abelian susy described by Killing spinor $\epsilon' = \frac{\epsilon^\alpha}{C}$ and $\epsilon'^\beta = \frac{\epsilon^\beta}{C}$.

Note that constant mode of $C$ is unconstrained since when $B_{11}$ is constant,

\[ \int d^2 \theta \bar{\theta} \tilde{\theta} \theta' = \int d^2 \theta \bar{\theta} \tilde{\theta} \theta' = \frac{1}{2} B_{11} [\bar{\theta} (\theta' \bar{\theta}') - \bar{\theta} (\theta' \bar{\theta}')] = 0 \]

Assumes that surface terms can be ignored.
Example 1: $d=4$ Minkowski + Calabi-Yau 3-fold
Using $d=4$ hybrid formalism, sigma model in this Type II background is
\[ S = \int d^2 \xi \left[ x^a_{,\xi} \partial x^a + p^a \partial \theta^a + \hat{p}^a \partial \hat{\theta}^a + p^a \partial \phi^a + \hat{p}^a \partial \hat{\phi}^a \right] + S_{cy} \]
Choose “chiral” representation where $q_a = \frac{2}{i} \phi^a$ and $\hat{q}_a = \frac{2}{i} \bar{\phi}^a$.
To T-dualize $\theta^a$ and $\hat{\theta}^a$, add the trivial surface term
\[ S \rightarrow S + \int d^2 \xi \ C_{ab} \left( \partial \theta^a \bar{\partial} \theta^b - \bar{\partial} \theta^a \partial \theta^b \right) \]
where $C_{ab} = C_{ba}$ is constant.
After T-dualization of $\theta^a$ and $\hat{\theta}^a$,
\[ S = \int d^2 \xi \left[ x^a_{,\xi} \partial x^a + p^a \partial \phi^a + \hat{p}^a \partial \phi^a + \hat{p}^a \partial \phi^a + (C^{-1})^{ab} \hat{p}^a \hat{p}^b \right] + S_{cy} \]
This is action for self-dual graviphoton background with $F^{ab} = \epsilon^{abc} (C^{-1})^{bc}$.
Used in topological strings and for non-anticommutative theories (Ooguri + Vafa, Seiberg).
On higher genus surfaces, flat and self-dual graviphoton backgrounds are not equivalent because of extra fermionic zero modes $S_c$.

Example 2: $AdS_5 \times S^5$ background
$AdS_5 \times S^5$ sigma model is constructed from Metsaev-Tseytlin currents
\[ J^A = (g^{-1} \partial g)^A \text{ and } \bar{J}^A = (g^{-1} \bar{\partial} g)^A \]
where $g(x,\theta)$ takes values in $AdS_5 \times S^5$ coset
\[ S = \int d^2 \xi \left( G_{MN} + B_{MN} \right) \partial Y^M \bar{\partial} Y^N \]
\[ = \int d^2 \xi \ \sqrt{-\text{det} g_{\text{ads}}} \left[ \gamma_{cd} J^c J^d + (C^{01234})_{\hat{a}\hat{b}} \left( J^{\hat{a}} \bar{J}^{\hat{b}} - J^{\hat{a}} J^{\hat{b}} \right) \right] \]
Action is invariant under global $PSU(2,2|4)$ isometries $\delta g = \Sigma_i \delta_i g$.
Can choose parametrization of $g$ such that four translations
\[ P^m \ (m=0 \text{ to } 3) \]
are translations
\[ x^m \rightarrow x^m + c^m \]
and eight chiral susy's $q_{aj} \ (a=1 \text{ to } 2, j=1 \text{ to } 4)$ act as
\[ \Theta^a \rightarrow \Theta^a + \xi^a j \]
where $(c^m, \xi^a j)$ are constants.
Since these shift isometries leave action invariant, can T-dualize $x^m$ and $\Theta^a j$ variables.
After T-dualizing four $x^m$ variables, AdS$_5$ metric is invariant since
$$\frac{r_{\text{ads}}}{y^2} \left( dx^m dx^n + (dy)^2 \right) \rightarrow \frac{y^2}{r_{\text{ads}}} \left( dx^m dx^n + \frac{r_{\text{ads}}}{y^2} (dy)^2 \right) = \frac{r_{\text{ads}}}{y^2} \left( dx^m dx^n + (dy)^2 \right).$$
But R-R field-strength and dilaton change as
$$e^x F^m = (\gamma_{01234})^{-\frac{5}{4}} \rightarrow e^{y'} F^m = (i \gamma_\eta)^{-\frac{5}{4}}$$
$$\Phi = \text{constant} \rightarrow \Phi' = \text{constant} + 4 \log y$$

If one now T-dualizes eight $\theta^a$ variables, one finds
$$g''_{mn} = g'_{mn} = g_{mn}, \quad \Phi'' = \Phi' - 4 \log y = \text{constant}$$
$$e^{y''}F'' = e^{y'}F' - e^{y}e^{\Phi} C^1 = (i \gamma_\eta)^{-\frac{5}{4}} - [(1+i\gamma_{0123})(i\gamma_\eta)]^{-\frac{5}{4}} = (\gamma_{01234})^{-\frac{5}{4}}.$$  
So AdS$_5 \times$S$^5$ background is invariant (for amplitudes on sphere or disc).

$\Rightarrow$ Alday-Maldacena method relates planar SYM amplitudes and Wilson line correlation function at arbitrary 't Hooft coupling up to a polarization-dependent factor.

When acting on T-dualized variables $(\tilde{x}_m, \tilde{\theta}^a_j)$, original PSU(2,2|4) transformations $P_m$ and $q_a j$ become trivial, and conformal and chiral superconformal boosts $K^m$ and $S^a j$ become non-local.

<table>
<thead>
<tr>
<th>$P_m$, $q_a j$</th>
<th>$K^m$, $S^a j$</th>
<th>$D$</th>
<th>$\tilde{q}_j^a$, $\tilde{S}_j^a$</th>
<th>$M_{mn}$</th>
<th>$R_j^k$</th>
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<tbody>
<tr>
<td>trivial</td>
<td>non-local</td>
<td>D</td>
<td>X</td>
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<tr>
<td>$\tilde{P}_m$, $\tilde{q}_a j$, $\tilde{K}^m$, $\tilde{S}_a j$</td>
<td>$-D$</td>
<td>$\tilde{q}_j^a$, $\tilde{S}_j^a$</td>
<td>$M_{mn}$</td>
<td>$R_j^k$</td>
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Dilatation $D$ changes sign since $y' = \frac{r_{\text{ads}}}{y}$.

In terms of T-dualized variables, "new" dual superconformal transf's are $(\tilde{P}_m, \tilde{q}_a j, \tilde{K}^m, \tilde{S}_a j)$. "Old" dual superconformal transf's form supergroup SU(2) $\times$ SU(2|4). Dual superconf. group was simultaneously found by Drummond, Henn, Korchemsky, Sokatchev.
Conclusions

- Superstring background with abelian supersymmetry is related by fermionic T-duality to a superstring background with fields
  \[ g'_{mn} = g_{mn}, \quad b'_{mn} = b_{mn}, \quad \Phi' = \Phi + \frac{1}{2} \log \mathcal{C}, \quad e^\Phi e^{\hat{F}} = e^\Phi e^{\hat{F} - e^{-2} e^{\hat{F}} \mathcal{C}} \]
  \((e^\Phi, e^{\hat{F}})\) is Killing spinor satisfying
  \[ e^\Phi (\gamma_m)_{ab} e^\Phi + e^{\hat{F}} (\gamma_m)_{ab} e^{\hat{F}} = 0 \]
  \[ e^\Phi (\gamma_m)_{ap} e^p - e^{\hat{F}} (\gamma_m)_{ap} e^{\hat{F}} = \partial_m \mathcal{C} \]

- For superstring tree amplitudes, T-dual background is equivalent to original background.

- Under fermionic T-duality, d=4 Minkowski background is mapped to self-dual graviphoton background.

- \(AdS_5 \times S^5\) background is mapped to itself after T-dualizing \(x^m (m=0 \text{ to } 3)\) and \(\Theta_{ab} (a=1 \text{ to } 2, \, b=1 \text{ to } 4)\).

Applications of fermionic T-duality

- Explains “dual superconformal symmetry” of perturbative \(N=4\) d=4 super-YM amplitudes found by Drummond, Henn, Korchemsky, Sokatchev.

- Relates non-local conserved currents of \(AdS_5 \times S^5\) sigma model with dual superconformal generators (Ricci, Tseytlin, Wolf, Beisert, Ricci, Tseytlin, Wolf)

- Except for unresolved issue of polarization dependence, extends to arbitrary ’t Hooft coupling the Alday-Maldacena proof which relates planar \(N=4\) d=4 super-YM amplitudes and Wilson line correlation functions.
Possible future applications

- Use fermionic T-duality to study non-anticommutative structure of self-dual graviphoton background.

- Check invariance of other AdS superstring backgrounds under combination of bosonic and fermionic T-dualities.

- Use fermionic T-duality to enlarge U-duality group of supergravity backgrounds to a U-duality supergroup.

- If $E_{10}/E_{11}$ models of Nicolai et al./West et al. could be extended to supergroup models, fermionic supergravity fields might be related to fermionic generators just as bosonic supergravity fields have been related to bosonic generators of $E_{10}/E_{11}$. 