FERMIONIC T-DUALITY

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NB and Juan Maldacena, arXiv:0807.3196
"Fermionic T-duality, dual superconformal symmetry, and the amplitude/Wilson loop connection"

Alday and Maldacena used invariance of $AdS_5 \times S^5$ metric under T-duality to relate planar $\mathcal{N}=4$ $d=4$ sYM amplitudes and Wilson line correlation functions at strong 't Hooft coupling.

$$\frac{1}{y^2}(dx^m dx_m + dy\,dy) \to y^2 d\tilde{x}^m d\tilde{x}_m + \frac{dy\,dy}{y^2} = \frac{1}{(y')^2}(d\tilde{x}^m d\tilde{x}_m + dy' dy') \qquad \boxed{y' = \frac{1}{y}}$$

But full $AdS_5 \times S^5$ background was not invariant since T-duality mapped

$F^{RR}_{01234} \to F^{RR}_4$ and mapped $\varphi \to \varphi + 4 \log y$.

So amplitude/Wilson line relation was only proven in limit $r_{AdS} \to \infty$ where effects of F^{RR} and φ can be ignored.

Drummond, Henn, Korchemsky, Smirnov, Sokatchev used perturbative computations to show that planar $\mathcal{N}=4$ $d=4$ sYM amplitudes and Wilson line correlation functions are also related at weak 't Hooft coupling.

Question: Can Alday-Maldacena result be extended to planar amplitudes at arbitrary 't Hooft coupling?

By including **bosonic** and **fermionic** T-dualities, invariance of AdS_5 metric can be extended to invariance of full $AdS_5 \times S^5$ background.

When superstring background has "abelian" fermionic isometry (i.e. $q^2=0$), usual **bosonic** T-duality has straightforward generalization to **fermionic** T-duality.

Fermionic T-duality leaves g_{mn} and b_{mn} invariant but transforms F^{RR} and φ.

After performing **bosonic** T-duality on four x^m variables and performing **fermionic** T-duality on eight θ^{aj} variables, the $AdS_5 \times S^5$ background is invariant.

Ignoring the issue of polarization dependence, this **explains** the relation of planar sYM amplitudes and Wilson lines at **arbitrary** 't Hooft **coupling**.

Fermionic T-duality has other applications such as mapping superstring in flat background to superstring in self-dual graviphoton background.

Review of Bosonic T-Duality (a la Buscher)

Suppose string theory background is invariant under the constant shift $x^1 \to x^1 + c \Rightarrow x^1$ only appears with derivatives in sigma model.

$$S = \int d^2z\, (g_{mn} + b_{mn})\, \partial x^m \bar{\partial} x^n \qquad \boxed{l_{mn} = g_{mn} + b_{mn}}$$

$$= \int d^2z \left[g_{11}\, \partial x^1 \bar{\partial} x^1 + l_{1m}\, \partial x^1 \bar{\partial} x^m + l_{m1}\, \partial x^m \bar{\partial} x^1 + l_{mn}\, \partial x^m \bar{\partial} x^n \right] \qquad \boxed{m,n \neq 1}$$

Integrate out $\tilde{x}^1$. Use that $\partial \bar{A} - \bar{\partial} A = 0 \Rightarrow A = \partial x^1$ and $\bar{A} = \bar{\partial} x^1$ for some x^1.

$$= \int d^2z \left[g_{11} A \bar{A} + l_{1m} A \bar{\partial} x^m + l_{m1} \partial x^m \bar{A} + l_{mn} \partial x^m \bar{\partial} x^n + \tilde{x}^1 (\partial \bar{A} - \bar{\partial} A) \right]$$

Integrate out A and $\bar{A}$.

$$= \int d^2z \left[\frac{1}{g_{11}} \partial \tilde{x}^1 \bar{\partial} \tilde{x}^1 + \frac{l_{1m}}{g_{11}} \partial \tilde{x}^1 \bar{\partial} x^m - \frac{l_{m1}}{g_{11}} \partial x^m \bar{A} + \left(l_{mn} - \frac{l_{1n} l_{m1}}{g_{11}} \right) \partial x^m \bar{\partial} x^n \right]$$

Comment: In Buscher procedure using non-compact x^1 variable, $\bar\partial A - \partial \bar A = 0$ implies $A = \partial x^1$ and $\bar A = \bar\partial x^1$ only if $\int_c (A\,dz + \bar A\,d\bar z) = 0$ around all non-trivial cycles C.

This is not an issue for genus zero amplitudes which have no non-trivial cycles.

But on higher genus surfaces, T-duality is only a symmetry if x^1 is a compact variable.

When x^1 is compact, $\bar\partial A - \partial \bar A = 0$ implies that $A = \partial x^1$ and $\bar A = \bar\partial x^1$ where $\int_c (A\,dz + \bar A\,d\bar z)$ is the winding number of x^1 around the cycle C.

After T-dualizing $x^1 \to \tilde x^1$,

$$S = \int d^2 z \left[\frac{1}{g_{11}} \partial \tilde x^1 \bar\partial \tilde x^1 + \frac{l_{1m}}{g_{11}} \partial \tilde x^1 \bar\partial x^m - \frac{l_{m1}}{g_{11}} \partial x^m \bar\partial \tilde x^1 + \left(l_{mn} - \frac{l_{1n} l_{m1}}{g_{11}} \right) \partial x^m \bar\partial x^n \right]$$

implies that background fields $l_{mn} = g_{mn} + b_{mn}$ transform as

$$g'_{11} = \frac{1}{g_{11}}, \quad l'_{1m} = \frac{l_{1m}}{g_{11}}, \quad l'_{m1} = -\frac{l_{m1}}{g_{11}}, \quad l'_{mn} = l_{mn} - \frac{l_{1n} l_{m1}}{g_{11}}$$

Integration over A and $\bar A$ produces a Jacobian factor $(\det g_{11})^{-1}$ which is absorbed by transforming the dilaton as

$$\varphi' = \varphi - \tfrac{1}{2} \log g_{11}.$$

$-$ sign in l'_{m1} implies that Dirichlet/Neumann boundary conditions for x^1 variable are switched on a D-brane.

Bosonic T-duality changes dimension of D-brane.

Fermionic T-Duality

In spacetime-supersymmetric sigma models (GS, pure spinor, hybrid, ...), supergravity background fields are combined into spacetime superfields $G_{MN}(Y)$ and $B_{MN}(Y)$ such that the action is

$$S = \int d^2z \left[(G_{MN} + B_{MN}) \partial Y^M \bar{\partial} Y^N + \dots \right].$$

For Type II supergravity, $Y^M = (x^m, \theta^\mu)$ where $m = 0$ to 9 and $\mu = 1$ to 32.

In pure spinor and hybrid formalisms, ... in action includes couplings to fermionic momenta and ghosts which are needed for quantization.

Torsion and curvature constraints imply that G_{MN} and B_{MN} can be expressed in terms of usual supergravity fields. For example, near flat space,

$$G_{mn} = g_{mn} + \underbrace{\chi^\mu_{(m}}_{\text{gravitino}} (\gamma_{n)}\theta)_\mu + \underbrace{F^{\mu\nu}}_{\text{R-R field strength}} (\gamma_m \theta)_\mu (\gamma_n \theta)_\nu + \dots$$

Suppose background superfields are invariant under constant fermionic shift $\theta' \to \theta' + \xi$ $\Rightarrow$ θ' only appears with derivatives in action.

Invariance implies background has "abelian" susy q satisfying $q^2 = 0$.

$$S = \int d^2z \, (G_{MN} + B_{MN}) \partial Y^M \bar{\partial} Y^N \qquad \boxed{L_{MN} = G_{MN} + B_{MN}}$$

$$= \int d^2z \left[B_{11} \partial\theta' \bar{\partial}\theta' + L_{1M} \partial\theta' \bar{\partial} Y^M + L_{M1} \partial Y^M \bar{\partial}\theta' + L_{MN} \partial Y^M \bar{\partial} Y^N \right]$$

Integrate out $\tilde{\theta}'$. Use that $\partial\bar{a} - \bar{\partial} a = 0 \Rightarrow a = \partial\theta'$ and $\bar{a} = \bar{\partial}\theta'$ for some θ'.

$$= \int d^2z \left[B_{11} a \bar{a} + L_{1M} a \bar{\partial} Y^M + L_{M1} \partial Y^M \bar{a} + L_{MN} \partial Y^M \bar{\partial} Y^N + \tilde{\theta}' (\partial\bar{a} - \bar{\partial} a) \right]$$

Integrate out a and $\bar{a}$.

$$= \int d^2z \left[-\frac{1}{B_{11}} \partial\tilde{\theta}' \bar{\partial}\tilde{\theta}' + \frac{L_{1M}}{B_{11}} \partial\tilde{\theta}' \bar{\partial} Y^M + \frac{L_{M1}}{B_{11}} \partial Y^M \bar{\partial}\tilde{\theta}' + \left(L_{MN} - \frac{L_{1N} L_{M1}}{B_{11}} \right) \partial Y^M \bar{\partial} Y^N \right]$$

Comment: In spacetime supersymmetric sigma models, θ' is single-valued around any non-trivial cycle.
(like a non-compact x' variable)

So $\partial\bar{a} - \bar{\partial} a = 0$ implies $a = \partial\theta'$ and $\bar{a} = \bar{\partial}\theta'$ only if $\int_c (a\,dz + \bar{a}\,d\bar{z}) = 0$ around every non-trivial cycle c.

This is not an issue for genus zero amplitudes which have no non-trivial cycles.

But on higher genus surfaces, fermionic T-duality is **not** a symmetry. To make it a symmetry, would need to introduce multivalued fermionic variable θ' satisfying $\theta' \to \theta' + \xi_c$ when θ' goes around cycle c.
ξ_c is a fermionic zero mode which would need to be included in the functional integral for $\int \mathcal{D}\theta'$.

After T-dualizing $\theta' \to \tilde{\theta}'$,

$$S = \int d^2z \left[-\frac{1}{B_{11}} \partial\tilde{\theta}' \bar{\partial}\tilde{\theta}' + \frac{L_{1M}}{B_{11}} \partial\tilde{\theta}' \bar{\partial} y^M + \frac{L_{M1}}{B_{11}} \partial y^M \bar{\partial}\tilde{\theta}' + \left(L_{MN} - \frac{L_{1N} L_{M1}}{B_{11}}\right) \partial y^M \bar{\partial} y^N \right]$$

implies that background superfields $L_{MN} = G_{MN} + B_{MN}$ transform as

$$B_{11}' = -\frac{1}{B_{11}}, \quad L_{1M}' = \frac{L_{1M}}{B_{11}}, \quad L_{M1}' = +\frac{L_{M1}}{B_{11}}, \quad L_{MN}' = L_{MN} - \frac{L_{1N} L_{M1}}{B_{11}}$$

Integration over fermionic a and $\bar{a}$ produces a Jacobian factor $(\det B_{11})^{+1}$ which is absorbed by transforming the dilaton as

$$\varphi' = \varphi + \tfrac{1}{2} \log B_{11}$$

$+$ sign in L_{M1}' implies that Dirichlet/Neumann boundary conditions for θ' variable are not switched on a D-brane.

Fermionic T-duality does not change dimension of D-brane.

To derive T-duality transformations of component supergravity fields, it is very convenient to use pure spinor formalism as was done by Benichou, Policastro, Troost for bosonic T-duality.

Defining $C = B_{11}|_{\theta=0}$, one finds that after T-dualization of θ',

$$g'_{mn}=g_{mn}, \quad b'_{mn}=b_{mn}, \quad \varphi'=\varphi+\frac{1}{2}\log C, \quad e^{\varphi'}F'^{\alpha\hat\beta}=e^{\varphi}F^{\alpha\hat\beta}-\epsilon^{\alpha}\epsilon^{\hat\beta}C^{-1}$$

$\alpha, \hat\beta = 1$ to 16 are $N=2$ $d=10$ spinor indices

$$F^{\alpha\hat\beta} = \gamma_m^{\alpha\hat\beta}F^m + \frac{1}{3!}\gamma_{mnp}^{\alpha\hat\beta}F^{mnp} + \frac{1}{2\cdot 5!}\gamma_{mnpqr}^{\alpha\hat\beta}F^{mnpqr}$$

$(\epsilon^{\alpha}, \epsilon^{\hat\alpha})$ is Killing spinor associated with abelian susy

Abelian susy $\Rightarrow$ $\epsilon^{\alpha}(\gamma_m)_{\alpha\beta}\epsilon^{\beta} + \epsilon^{\hat\alpha}(\gamma_m)_{\hat\alpha\hat\beta}\epsilon^{\hat\beta} = 0$

Torsion constraints $\Rightarrow$ $\epsilon^{\alpha}(\gamma_m)_{\alpha\beta}\epsilon^{\beta} - \epsilon^{\hat\alpha}(\gamma_m)_{\hat\alpha\hat\beta}\epsilon^{\hat\beta} = \partial_m C$

Using $\partial_m C = \epsilon^{\alpha}(\gamma_m)_{\alpha\beta}\epsilon^{\beta} - \epsilon^{\hat\alpha}(\gamma_m)_{\hat\alpha\hat\beta}\epsilon^{\hat\beta}$, can easily check that transformed background fields

$$g'_{mn}=g_{mn}, \quad b'_{mn}=b_{mn}, \quad \varphi'=\varphi+\frac{1}{2}\log C, \quad e^{\varphi'}F'^{\alpha\hat\beta}=e^{\varphi}F^{\alpha\hat\beta}-\epsilon^{\alpha}\epsilon^{\hat\beta}C^{-1}$$

satisfy supergravity equations of motion and are invariant under abelian susy described by Killing spinor $\epsilon'^{\alpha}=\frac{\epsilon^{\alpha}}{C}$ and $\epsilon'^{\hat\alpha}=\frac{\epsilon^{\hat\alpha}}{C}$.

Note that constant mode of C is unconstrained since when B_{11} is constant,

$$\int d^2z\, B_{11}\,\partial\theta'\bar\partial\theta' = \int d^2z\, \frac{1}{2}B_{11}\left[\partial(\theta'\bar\partial\theta') - \bar\partial(\theta'\partial\theta')\right] = 0$$

Assumes that surface terms can be ignored.

Example 1: $d=4$ Minkowski + Calabi-Yau 3-fold

Using $d=4$ hybrid formalism, sigma model in this Type II background is

$$S = \int d^2z \left[\partial x_{a\dot a}\bar\partial x^{a\dot a} + p_a \bar\partial\theta^a + \hat p_a \partial\hat\theta^a + p_{\dot a}\bar\partial\theta^{\dot a} + \hat p_{\dot a}\partial\hat\theta^{\dot a}\right] + S_{CY}$$

Choose "chiral" representation where $q_a = \frac{\partial}{\partial\theta^a}$ and $\hat q_a = \frac{\partial}{\partial\hat\theta^a}$. $\boxed{a,\dot a = 1,2}$

To T-dualize θ^a and $\hat\theta^a$, add the trivial surface term

$$S \to S + \int d^2z\, C_{ab}(\partial\theta^a\bar\partial\hat\theta^b - \bar\partial\theta^a\partial\hat\theta^b)$$ where $C_{ab} = C_{ba}$ is constant.

After T-dualization of θ^a and $\hat\theta^a$,

$$S = \int d^2z \left[\partial x_{a\dot a}\bar\partial x^{a\dot a} + p_a\bar\partial\tilde\theta^a + \hat p_a\partial\hat{\tilde\theta}^a + p_{\dot a}\bar\partial\theta^{\dot a} + \hat p_{\dot a}\partial\hat\theta^{\dot a} + (C^{-1})^{ab} p_a \hat p_b\right] + S_{CY}$$

This is action for self-dual graviphoton background with $F^{ab} = e^{-\varphi}(C^{-1})^{ab}$.

Used in topological strings and for non-anticommutative theories (Ooguri + Vafa, Seiberg)

On higher genus surfaces, flat and self-dual graviphoton backgrounds are not equivalent because of extra fermionic zero modes ξ_c.

Example 2: $AdS_5 \times S^5$ background

$AdS_5\times S^5$ sigma model is constructed from Metsaev-Tseytlin currents

$$J^A = (g^{-1}\partial g)^A \quad\text{and}\quad \bar J^A = (g^{-1}\bar\partial g)^A$$

$A = (c, \alpha, \hat\alpha)$
$c = 0$ to 9, $(\alpha,\hat\alpha) = 1$ to 16

where $g(x,\theta)$ takes values in $AdS_5\times S^5$ coset $\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$.

$$S = \int d^2z\,(G_{MN} + B_{MN})\partial Y^M \bar\partial Y^N$$
$$= \int d^2z\, r_{AdS}^2 \left[\eta_{cd} J^c \bar J^d + (\gamma^{01234})_{\alpha\hat\beta}(J^\alpha \bar J^{\hat\beta} - \bar J^\alpha J^{\hat\beta})\right]$$

Action is invariant under global $PSU(2,2|4)$ isometries $\delta g = \Sigma g$.

Can choose parametrization of g such that four translations P_m $(m=0$ to $3)$ and eight chiral susy's q_{aj} $(a=1$ to 2, $j=1$ to $4)$ act as

$x^m \to x^m + c^m$ and $\theta^{aj} \to \theta^{aj} + \xi^{aj}$ where (c^m, ξ^{aj}) are constants.

Since these shift isometries leave action invariant, can T-dualize x^m and θ^{aj} variables.

After T-dualizing four x^m variables, AdS_5 metric is invariant since

$$\frac{r_{AdS}^2}{y^2}\left(dx^m dx_m + (dy)^2\right) \to \frac{y^2}{r_{AdS}^2}\, d\tilde{x}^m d\tilde{x}_m + \frac{r_{AdS}^2 (dy)^2}{y^2} = \frac{r_{AdS}^2}{(y')^2}\left(d\tilde{x}^m d\tilde{x}_m + (dy')^2\right).$$

$\boxed{y' = \frac{r_{AdS}^2}{y}}$ (Kallosh + Tseytlin)

But R-R field-strength and dilaton change as

$$e^{\varphi} F^{\alpha\hat{\beta}} = (\gamma_{01234})^{\alpha\hat{\beta}} \to e^{\varphi'} F'^{\alpha\hat{\beta}} = (i\gamma_4)^{\alpha\hat{\beta}}$$

$$\varphi = \text{constant} \to \varphi' = \text{constant} + 4 \log y$$

If one now T-dualizes eight θ^{aj} variables, one finds

$$g''_{mn} = g'_{mn} = g_{mn}, \quad \varphi'' = \varphi' - 4\log y = \text{constant}$$

$$e^{\varphi''} F''^{\alpha\hat{\beta}} = e^{\varphi'} F'^{\alpha\hat{\beta}} - \epsilon^{\alpha}\epsilon^{\hat{\beta}} C^{-1} = (i\gamma_4)^{\alpha\hat{\beta}} - [(1+i\gamma_{0123})(i\gamma_4)]^{\alpha\hat{\beta}} = (\gamma_{01234})^{\alpha\hat{\beta}}.$$

So $AdS_5 \times S^5$ background is invariant (for amplitudes on sphere or disc).

$\Rightarrow$ Alday-Maldacena method relates planar sYM amplitudes and Wilson line correlation function at arbitrary 't Hooft coupling up to a polarization-dependent factor.

When acting on T-dualized variables $(\tilde{x}_m, \tilde{\theta}_{aj})$, original $PSU(2,2|4)$ transformations P_m and q_{aj} become trivial, and conformal and chiral superconformal boosts K^m and s^{aj} become non-local.

$$\begin{tikzcd}[column sep=small]
P_m \arrow[d] & q_{aj} \arrow[d] & K^m \arrow[d] & s^{aj} \arrow[d] & D \arrow[dd] & \bar{q}_{\dot a}^{j} \arrow[ddr] & \bar{s}^{\dot a}_{j} \arrow[ddl] & M_{mn} \arrow[dd] & R_j^{\,k} \arrow[dd] \\
\text{trivial} & & \text{non-local} & & & & & & \\
\tilde{P}_m & \tilde{q}^{aj} & \tilde{K}^m & \tilde{s}_{aj} & -D & \bar{q}^{\dot a}_{j} & \bar{s}^{j}_{\dot a} & M_{mn} & R_j^{\,k}
\end{tikzcd}$$

Dilatation D changes sign since $y' = \frac{r_{AdS}^2}{y}$.

In terms of T-dualized variables, "new" dual superconformal transf's are $(\tilde{P}_m, \tilde{q}^{aj}, \tilde{K}^m, \tilde{s}_{aj})$. "Old" dual superconformal transf's form supergroup $SU(2)\times SU(2|4)$. Dual supercons. group was simultaneously found by Drummond, Henn, Korchemsky, Sokatchev.

Conclusions

- Superstring background with abelian supersymmetry is related by fermionic T-duality to a superstring background with fields

$g'_{mn} = g_{mn}$, $b'_{mn} = b_{mn}$, $\varphi' = \varphi + \frac{1}{2}\log C$, $e^{\varphi'}F'^{\alpha\hat\beta} = e^{\varphi}F^{\alpha\hat\beta} - \epsilon^{\alpha}\epsilon^{\hat\beta}C^{-1}$

$(\epsilon^{\alpha}, \epsilon^{\hat\alpha})$ is Killing spinor satisfying $\epsilon^{\alpha}(\gamma_m)_{\alpha\beta}\epsilon^{\beta} + \epsilon^{\hat\alpha}(\gamma_m)_{\hat\alpha\hat\beta}\epsilon^{\hat\beta} = 0$

$$\epsilon^{\alpha}(\gamma_m)_{\alpha\beta}\epsilon^{\beta} - \epsilon^{\hat\alpha}(\gamma_m)_{\hat\alpha\hat\beta}\epsilon^{\hat\beta} = \partial_m C$$

- For superstring tree amplitudes, T-dual background is equivalent to original background.

- Under fermionic T-duality, $d=4$ Minkowski background is mapped to self-dual graviphoton background.

- $AdS_5 \times S^5$ background is mapped to itself after T-dualizing x^m $(m=0$ to $3)$ and θ^{aj} $(a=1$ to 2, $j=1$ to $4)$.

Applications of fermionic T-duality

- Explains "dual superconformal symmetry" of perturbative $\mathcal{N}=4$ $d=4$ super-YM amplitudes found by Drummond, Henn, Korchemsky, Sokatchev.

- Relates non-local conserved currents of $AdS_5 \times S^5$ sigma model with dual superconformal generators (Ricci, Tseytlin, Wolf; Beisert, Ricci, Tseytlin, Wolf)

- Except for unresolved issue of polarization dependence, extends to arbitrary 't Hooft coupling the Alday-Maldacena proof which relates planar $\mathcal{N}=4$ $d=4$ super-YM amplitudes and Wilson line correlation functions.

Possible future applications

- Use fermionic T-duality to study non-anticommutative structure of self-dual graviphoton background.
- Check invariance of other AdS superstring backgrounds under combination of bosonic and fermionic T-dualities.
- Use fermionic T-duality to enlarge U-duality group of supergravity backgrounds to a U-duality supergroup.
- If E_{10}/E_{11} models of Nicolai et al/West et al could be extended to supergroup models, fermionic supergravity fields might be related to fermionic generators just as bosonic supergravity fields have been related to bosonic generators of E_{10}/E_{11}.