Integrability of the AdS/CFT System

Matthias Staudacher

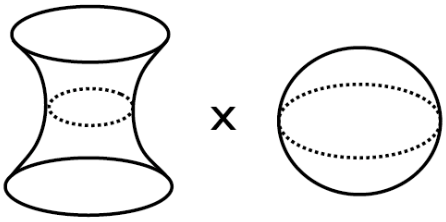
Max-Planck-Institut für Gravitationsphysik
AEI Potsdam

Strings 2008

CERN, Geneva, 21.8.2008

The AdS/CFT Correspondence

IIB Superstrings on $AdS_5 \times S^5$



string tension: $\frac{1}{\alpha'}$ string coupling: g_s

$\mathcal{N} = 4$ $SU(N)$ supersymmetric gauge theory

'T Hooft coupling: $\lambda = N g_{\text{YM}}^2$ Inverse color number: $\frac{1}{N}$

Conjecture: exact duality between these two theories

[Maldacena '97; Gubser, Klebanov, Polyakov '98; Witten '98]

Fascinating Links

- Between quantum field theories without gravity, and string theories with (both classical and quantized) gravity
- Between exactly solvable two-dimensional quantum field theory and exactly solvable four-dimensional quantum field theory
- Between gauge/string theories and solved as well as unsolved problems of theoretical solid state physics
- Between gauge/string theories and mathematics (representation theory, quantum groups and Hopf algebras, complex analysis, integral equations, quantum geometry, ...)

$\mathcal{N} = 4$ Supersymmetric Gauge Theory, I

[Brink, Schwarz, Scherk '77; Gliozzi, Scherk, Olive '77]

Fields: All fields are in the adjoint representation, they are $N \times N$ matrices.

- gauge field $\mathcal{A}_\mu$ with $\mu = 0, 1, 2, 3$ of dimension $\Delta = 1$
- field strength $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i\,[\mathcal{A}_\mu, \mathcal{A}_\nu]$, $\Delta = 2$
- $\mathbf{6}$ real scalars Φ_m, with $m = 1, \ldots, 6$, $\Delta = 1$
- $4 \times \mathbf{4}$ real fermions $\Psi_{\alpha a}, \dot{\Psi}^a_{\dot{\alpha}}$ mit $\alpha, \dot{\alpha} = 1, 2$, $a = 1, 2, 3, 4$, $\Delta = \frac{3}{2}$
- covariant derivatives: $\mathcal{D}_\mu = \partial_\mu - i\,\mathcal{A}_\mu$, $\Delta = 1$

$\mathcal{N}=4$ Supersymmetric Gauge Theory, II

Action:

$$\boxed{\begin{aligned} S=\frac{N}{\lambda}\int\frac{d^4x}{4\pi^2}\,\mathrm{Tr}\biggl(&\tfrac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}+\tfrac{1}{2}\mathcal{D}^\mu\Phi^m\,\mathcal{D}_\mu\Phi_m-\tfrac{1}{4}[\Phi^m,\Phi^n][\Phi_m,\Phi_n]\\ &+\dot\Psi^a_{\dot\alpha}\sigma^{\dot\alpha\beta}_\mu\mathcal{D}^\mu\Psi_{\beta a}-\tfrac{i}{2}\,\Psi_{\alpha a}\sigma^{ab}_m\epsilon^{\alpha\beta}[\Phi^m,\Psi_{\beta b}]-\tfrac{i}{2}\,\dot\Psi^a_{\dot\alpha}\sigma^m_{ab}\epsilon^{\dot\alpha\dot\beta}[\Phi_m,\dot\Psi^b_{\dot\beta}]\biggr)\end{aligned}}$$

Free parameters: $\lambda=Ng^2_{\mathrm{YM}}$ und N.

The "most beautiful" four-dimensional gauge theory. λ ist dimensionless.

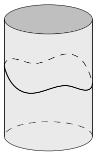
Superconformal quantum field theory. [Avdeev, Tarasov, Vladimirov '80; Grisaru, Rocek, Siegel '80]

[Sohnius, West '81; Caswell, Zanon '81; Brink, Lindgren, Nilsson '83; Mandelstam '83; Howe, Stelle, Townsend '84]

Global Symmetry: PSU$(2,2|4)$.

IIB Superstring on $AdS_5\times S^5$

Two-dimensional worldsheet with coordinates σ,τ :



Embedded into the coset space (Fermions act like "staples")

$$\frac{\widetilde{\mathrm{PSU}}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}=\overbrace{AdS_5\times S^5}\,.$$

The IIB Superstring σ-Model on $AdS_5 \times S^5$

Action:

[Metsaev, Tseytlin '98]

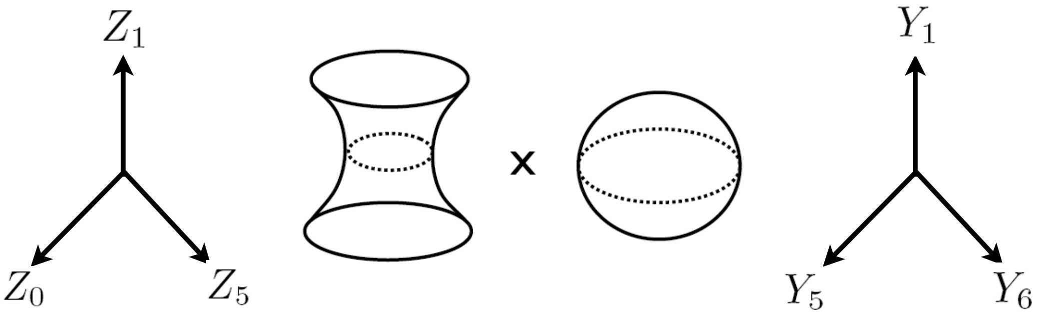
$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau \, d\sigma \, \left(\partial_a Z^M \partial^a Z_M + \partial_a Y_N \partial^a Y_N \right) + \text{Fermions}\,.$$

with

$$\begin{aligned} \mathrm{AdS}_5: \quad & -Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 - Z_5^2 = -R^2 \\ \mathrm{S}^5: \quad & Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2 = R^2 \end{aligned}$$

The quantization of this model has not yet been understood.
However, see below ...

The Spectral Problem of IIB Superstrings on $AdS_5 \times S^5$

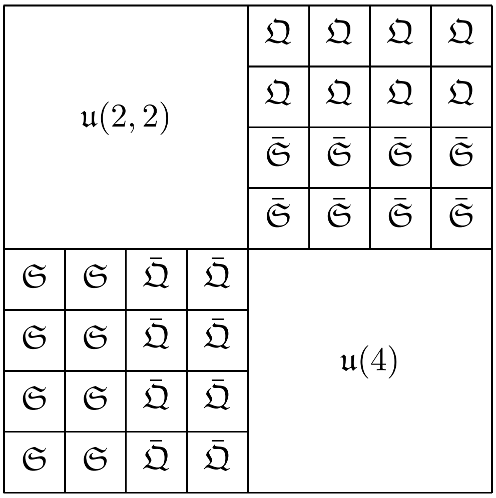


$Z_0 + i\,Z_5 = \rho_3\, e^{i\,t}$, $Z_1 + i\,Z_2 = \rho_1\, e^{i\,\alpha_1}$, $Z_3 + i\,Z_4 = \rho_2\, e^{i\,\alpha_2}$:
3 angles t, α_1, α_2 $\longrightarrow$ 3 conserved quantities E, S_1, S_2. E is the energy.

$Y_1 + i\,Y_2 = r_1\, e^{i\,\phi_1}$, $Y_3 + i\,Y_4 = r_2\, e^{i\,\phi_2}$, $Y_5 + i\,Y_6 = r_3\, e^{i\,\phi}$:
3 angles ϕ_1, ϕ_2, ϕ $\longrightarrow$ 3 conserved angular momenta J_1, J_2, J_3.

The PSU$(2,2|4)$ Symmetry of the AdS/CFT System

$\mathbf{32}$ bosonic generators and $\mathbf{32}$ fermionic generators $\mathfrak{Q}, \bar{\mathfrak{Q}}, \mathfrak{S}, \bar{\mathfrak{S}}$. $\mathfrak{su}(2,2)$: conformal algebra, $\mathfrak{su}(4)$: R-symmetry. $\mathfrak{u}(2,2|4)$ is reducible.



$$\begin{array}{|c|c|}\hline \mathfrak{u}(2,2) & \begin{array}{cccc} \mathfrak{Q} & \mathfrak{Q} & \mathfrak{Q} & \mathfrak{Q} \\ \mathfrak{Q} & \mathfrak{Q} & \mathfrak{Q} & \mathfrak{Q} \\ \bar{\mathfrak{S}} & \bar{\mathfrak{S}} & \bar{\mathfrak{S}} & \bar{\mathfrak{S}} \\ \bar{\mathfrak{S}} & \bar{\mathfrak{S}} & \bar{\mathfrak{S}} & \bar{\mathfrak{S}} \end{array} \\ \hline \begin{array}{cccc} \mathfrak{S} & \mathfrak{S} & \bar{\mathfrak{Q}} & \bar{\mathfrak{Q}} \\ \mathfrak{S} & \mathfrak{S} & \bar{\mathfrak{Q}} & \bar{\mathfrak{Q}} \\ \mathfrak{S} & \mathfrak{S} & \bar{\mathfrak{Q}} & \bar{\mathfrak{Q}} \\ \mathfrak{S} & \mathfrak{S} & \bar{\mathfrak{Q}} & \bar{\mathfrak{Q}} \end{array} & \mathfrak{u}(4) \\ \hline \end{array}$$

Instead of $\mathbf{8}$, only $\mathbf{3}+\mathbf{3}$ $\mathfrak{u}(1)$ Cartan charges: $(E, S_1, S_2|J_1, J_2, J_3)$

Conformal Energy/Dilatation weight are a part of the symmetry!

The Spectral Problem of $\mathcal{N}=4$ SYM

Conformal invariance restricts the structure of two-point functions:

$$\boxed{\langle \mathcal{O}_n(x)\, \mathcal{O}_m(0) \rangle = \frac{\delta_{nm}}{x^{2\Delta_n}}\,.}$$

Δ_n is the anomalous scaling dimension of the composite operator $\mathcal{O}_n$. This leads to the mixing problem of $\mathcal{N}=4$:

$$\boxed{\mathcal{O} = \mathrm{Tr}\left(\mathcal{X}\mathcal{Y}\mathcal{Z}\mathcal{F}_{\mu\nu}\Psi^A_\alpha(\mathcal{D}_\mu\mathcal{Z})\ldots\right)\mathrm{Tr}\left(\ldots\ldots\right)\ \ldots + \ldots}$$

The partons carry additive, protected Lorentz and R-symmetry charges S_1, S_2, J_1, J_2, J_3. Here Δ_n is related to the dilatation generator $\mathfrak{D}$:

$$\boxed{\left[\,\mathfrak{D}, \mathcal{O}_n(0)\,\right] = i\,\Delta_n\,\mathcal{O}_n(0)\,.}$$

$\Delta_n(\lambda)$ is not protected, it generically depends on the 't Hooft coupling λ.

The Spectral Problem of AdS/CFT and Integrability

A key prediction of AdS/CFT:

$$\boxed{\begin{array}{ccc} \text{string energy} & \leftrightarrow & \text{scaling dimension} \\ E(\lambda) & = & \Delta(\lambda) \end{array}}$$

- Solid Fact I: The $AdS_5 \times S^5$ string σ-model is classically integrable.

  [Bena,Polchinski,Roiban '03]

  It has been completely solved in terms of an algebraic curve.

  [Kazakov, Marshakov, Minahan, Zarembo '04, Beisert, Kazakov, Sakai, Zarembo '05]

- Solid Fact II: The full one-loop dilatation operator of $\mathcal{N} = 4$ SYM can be mapped to a quantum integrable spin chain. It has been completely diagonalized by means of the Bethe ansatz. [Minahan, Zarembo '02, Beisert, MS '03]

Mixing Problem in $\mathcal{N} = 4$ SYM and Spin Chains

Consider twist operators:

$$\boxed{\mathcal{O} = \mathrm{Tr}\left(\mathcal{D}^{S_1}\mathcal{Z}^{J_3}\right) + \ldots .}$$

$\mathcal{D} = \mathcal{D}_1 + i\,\mathcal{D}_2$ mit $\mathcal{D}_\mu = \partial_\mu + i\,A_\mu$ is a covariant lightcone derivative. The dilatation operator is regarded as the Hamiltonian of a spin chain.

The spin chain is

$$\boxed{\mathrm{Tr}\Big((\mathcal{D}^{s_1}\mathcal{Z})(\mathcal{D}^{s_2}\mathcal{Z})\ldots(\mathcal{D}^{s_{J_3-1}}\mathcal{Z})(\mathcal{D}^{s_{J_3}}\mathcal{Z})\Big)\,,}$$

where $S_1 = s_1 + s_2 + \ldots + s_{J_3-1} + s_{J_3} := M =$ Magnon number.

The Asymptotic Bethe Ansatz

[Sutherland '78; MS '04]

The excitations of the integrable gauge theory spin chain scatter according to matrix Bethe equations, where the p_k are lattice momenta:

$$\boxed{e^{ip_k L}\,|\Psi\rangle = \left(\prod_{\substack{j=1\\ j\neq k}}^{M} S(p_k,p_j)\right)\cdot|\Psi\rangle\,, \qquad E=\sum_{k=1}^{M} q_2(p_k)\,.}$$

The (asymptotic) S-matrix is assumed to be factorized.

So far, factorization was only proved in special cases (at one loop for all, and up to four loops for some operators).

However, for finite size chains we are not allowed to assume exactness of the S-matrix, as it rests on long-range interactions: → wrapping problem!

The AdS/CFT (internal) S-Matrix

[Arutyunov, Frolov, MS '04; MS '04; Beisert, MS '05; Beisert '05 + '06; Janik '06; Beisert, Hernandez, Lopez '06; Beisert, Eden, MS '06]

Die S-matrix should be unitary, and satisfy the Yang-Baxter-equation:

$$\boxed{S_{12}\,S_{21}=1\,,\qquad S_{12}\,S_{13}\,S_{23}=S_{23}\,S_{13}\,S_{12}\,.}$$

It was (ad-hoc) conjectured to also possess crossing symmetry: [Janik '06]

$$\boxed{S_{12}\,S_{\bar{1}2}=f_{12}^2\,.}$$

The S-matrix for AdS/CFT has the following symmetry structure [Beisert '05]

$$\boxed{S_{12}=\left(S_{12}^{\mathfrak{su}(2|2)_L}\otimes S_{12}^{\mathfrak{su}(2|2)_R}\right)\sigma_{12}^2\,,}$$

It was first motivated from the gauge theory spin chain, and subsequently also using string theory arguments. [Arutyunov, Frolov, Plefka, Zamaklar '06]

The Asymptotic All-Loop Bethe Equations

[Beisert, MS '05]

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}}\,,$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}}\,,$$

$$1 = \left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^L \prod_{\substack{j=1\\ j\neq k}}^{K_4} \left(\frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\,\sigma^2(x_{4,k},x_{4,j})\right) \prod_{j=1}^{K_3} \frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}} \prod_{j=1}^{K_5} \frac{x^-_{4,k}-x_{5,j}}{x^+_{4,k}-x_{5,j}}\,,$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k}-x^+_{4,j}}{x_{5,k}-x^-_{4,j}}\,,$$

$$1 = \prod_{\substack{j=1\\ j\neq k}}^{K_6} \frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i} \prod_{j=1}^{K_5} \frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}}\,,$$

$$E(g) = 2\sum_{j=1}^{K_4}\left(\frac{i}{x^+_{4,j}}-\frac{i}{x^-_{4,j}}\right) = \frac{1}{g^2}\sum_{j=1}^{K_4}\left(\sqrt{1+16\,g^2\,\sin\frac{p_j}{2}}-1\right), \qquad \Delta = \Delta_0 + g^2\,E(g)\,, \qquad K_4 = M\,.$$

$$1 = \prod_{j=1}^{K_4}\left(\frac{x^+_{4,j}}{x^-_{4,j}}\right) = \prod_{j=1}^{K_4} e^{ip_j}\,, \qquad u_k = x_k + \frac{g^2}{x_k}\,, \qquad u_k \pm \tfrac{i}{2} = x^\pm_k + \frac{g^2}{x^\pm_k}\,.$$

The Hubbard Connection

The AdS/CFT system is mysteriously related to the Hubbard model:

- Identical asymptotic dispersion law. Hubbard Hamiltonian is identical to the "rational part" of the $\mathfrak{su}(2)$ sector of the dilatation generator.

  [Rej, Serban, Staudacher '05]

- S-matrix factors into two of Shastry's Hubbard R-matrices

  [MS conjecture (unpublished) '05; Beisert '06]

- Dressing phase constants look like commuting charge expectation values in a half-filled "bosonic" Hubbard model [Beisert, Eden, Staudacher '06]

The Interpolating Scaling Function

The scaling dimension of operators of low twist J_3 behaves in a very interesting logarithmic way at large spin $S_1 \to \infty$:

$$\boxed{\Delta - S_1 - J_3 \;=\; f(g)\;\log S_1 \;+\; O(S_1^0)\,.}$$

$f(g)$ is the universal scaling function, where $g^2 = \lambda/16\,\pi^2$.

Also appears in the structure of MHV-amplitudes und in lightcone segmented Wilson loops $\mathcal{W}$! Gluon 4-point function in $4-2\,\epsilon$ dimensions:

[Bern, Dixon, Smirnov]

$$\boxed{\mathcal{M}_4^{\mathrm{All-Loop}} \simeq \exp\left[f(g)\,\mathcal{M}_4^{\mathrm{One-Loop}}\right]\,, \qquad \mathcal{M}_4^{\mathrm{All-Loop}} \simeq \langle\mathcal{W}\rangle\,.}$$

The Interpolating Integral Equation

The non-linear asymptotic Bethe equations reduce in the limit $S_1 \to \infty$, where $L \to \infty$ with $L << \log S_1$, to a linear integral equation for the density $\hat{\sigma}$ of Bethe roots. These describe the one-dimensional "motion" of the covariant derivatives of the twist operators: [Beisert, Eden, MS '06]

$$\boxed{\hat{\sigma}(t) \;=\; \frac{t}{e^{\,t}-1}\left[\;\hat{K}(2\,g\,t,0) - 4\,g^2\int_0^\infty dt'\;\hat{K}(2\,g\,t,2\,g\,t')\;\hat{\sigma}(t')\right].}$$

The universal scaling function $f(g)$ is then given by

$$\boxed{f(g) \;=\; 16\,g^2\,\hat{\sigma}(0)\,.}$$

The kernel $\hat{K}$ is of a rather involved structure, it will not be written here.

Gauge Theory Meets String Theory

This equation is analytic at small g, and therefore valid for arbitrary values of the coupling constant g !

At weak coupling the equation was (numerically) tested up to four loop order in gauge theory: [Bern, Czakon, Dixon, Kosower, Smirnov, '06]:

$$\boxed{f(g) = 8\,g^2 - \tfrac{8}{3}\,\pi^2 g^4 + \tfrac{88}{45}\,\pi^4 g^6 - 16\left(\tfrac{73}{630}\,\pi^6 + 4\,\zeta(3)^2\right) g^8 \pm \ldots .}$$

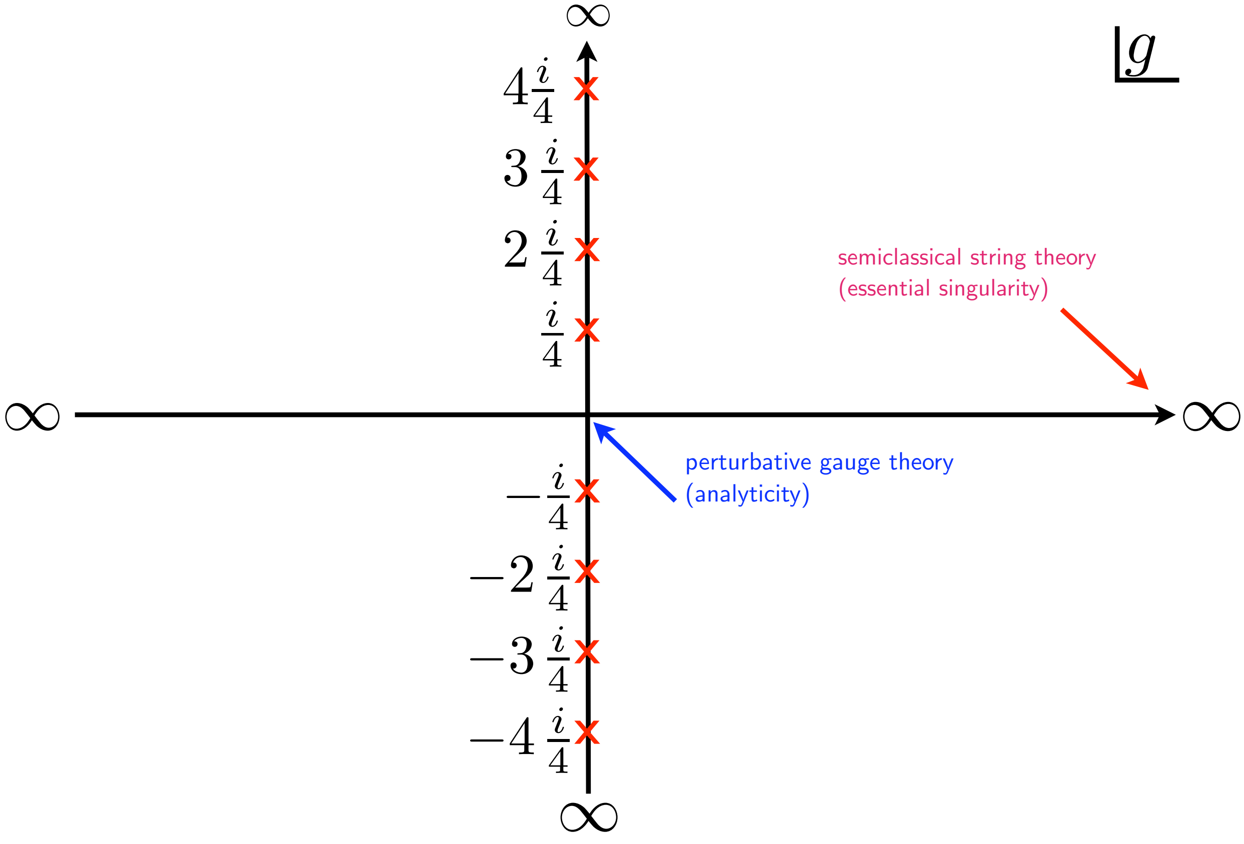
Improved to 0.001% by [Cachazo, Spradlin, Volovich '06].

At strong coupling the scaling function agrees with string theory to the three known orders [Gubser, Klebanov, Polyakov '02], [Frolov, Tseytlin '02], [Roiban, Tirziu, Tseytlin '07; Roiban, Tseytlin '07] as was recently shown analytically from the equation [Basso, Korchemsky, Kotański '07] (see also [Kostov, Serban, Volin '08]):

$$\boxed{f(g) = 4\,g - \frac{3\log 2}{\pi} - \frac{\mathrm{K}}{4\pi^2}\,\frac{1}{g} - \ldots}$$

→ The AdS/CFT correspondence is exactly true !

Analytic Structure in the Coupling Constant Plane

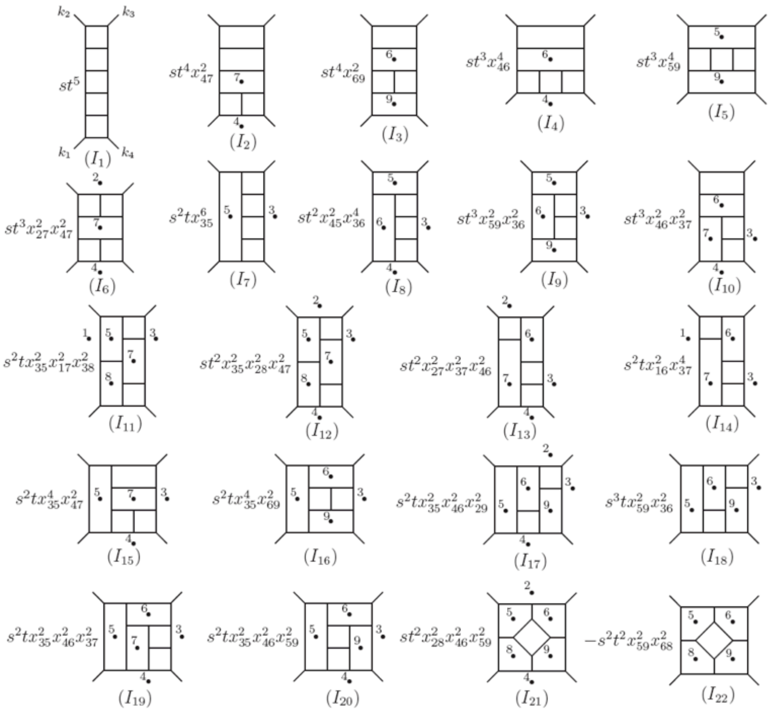


semiclassical string theory
(essential singularity)

perturbative gauge theory
(analyticity)

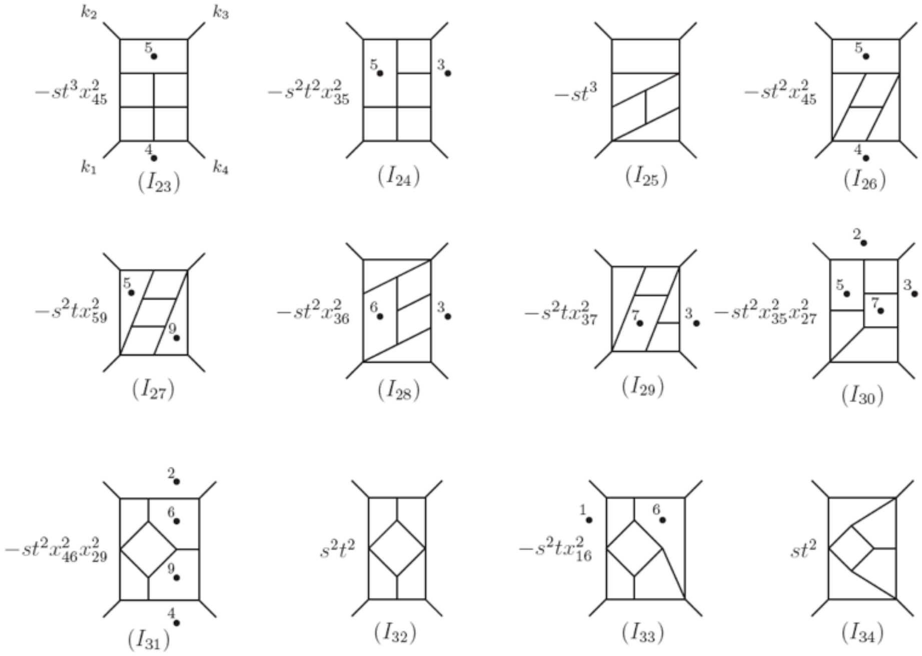
Challenge I: Compute 5-Loop diagrams ...

[Z. Bern, J. J. M. Carrasco, H. Johansson and D. A. Kosower ‘07]



... and ...

[Z. Bern, J. J. M. Carrasco, H. Johansson and D. A. Kosower ‘07]



Challenge II: Compute 3-Loop String Corrections ...

... and compare to the 3-loop prediction

[Beisert, Eden, MS '06, Basso, Korchemsky, Kotański '07]

$$\boxed{f\left(g + \tfrac{3\log 2}{4\,\pi}\right) = 4\,g\; - \tfrac{\mathrm{K}}{4\pi^2}\,\tfrac{1}{g} - \tfrac{27\,\zeta(3)}{2^9\,\pi^3}\,\tfrac{1}{g^2} - \ldots}$$

Tough ... are there other ways to test the capacity of the asymptotic Bethe ansatz to interpolate between gauge and string theory?

... and compare to the 5-Loop Prediction

[Beisert, Eden, MS '06]

$$\boxed{\begin{aligned} f(g) = \quad & 8\,g^2 - \tfrac{8}{3}\,\pi^2 g^4 + \tfrac{88}{45}\,\pi^4 g^6 - 16\left(\tfrac{73}{630}\,\pi^6 + 4\,\zeta(3)^2\right) g^8 \\ & +32\left(\tfrac{887}{14175}\,\pi^8 + \tfrac{4}{3}\,\pi^2 \zeta(3)^2 + 40\,\zeta(3)\,\zeta(5)\right) g^{10} \mp \ldots . \end{aligned}}$$

Challenge II: Compute 3-Loop String Corrections ...

... and compare to the 3-loop prediction

[Beisert, Eden, MS '06, Basso, Korchemsky, Kotański '07]

$$\boxed{f\left(g+\frac{3\log 2}{4\,\pi}\right) = 4\,g - \frac{\mathrm{K}}{4\pi^2}\,\frac{1}{g} - \frac{27\,\zeta(3)}{2^9\,\pi^3}\,\frac{1}{g^2} - \dots}$$

Tough ... are there other ways to test the capacity of the asymptotic Bethe ansatz to interpolate between gauge and string theory?

Bethe Ansatz: Hidden "Hole" Rapidities

[Belitsky, Gorsky, Korchemsky '06]

There are $S_1 = s_1 + \ldots + s_J$ $\mathcal{D}$-particles in the background of the J_3 $\mathcal{Z}$'s:

$$\boxed{\mathrm{Tr}\Big((\mathcal{D}^{s_1}\mathcal{Z})(\mathcal{D}^{s_2}\mathcal{Z})\ldots(\mathcal{D}^{s_{J-1}}\mathcal{Z})(\mathcal{D}^{s_J}\mathcal{Z})\Big)\,,}$$

whose motion is described by S_1 rapidities u_k. Drop the index k :

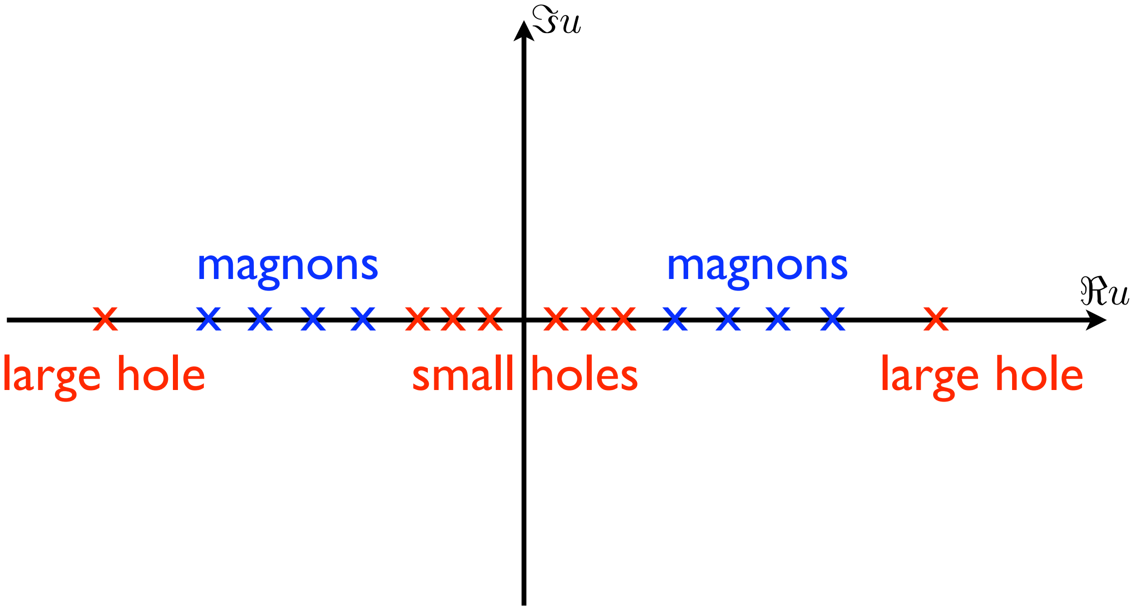
$$\boxed{\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^{J_3} = \prod_{\substack{j=1\\ j\neq k}}^{S_1} \frac{u_k-u_j-i}{u_k-u_j+i} \quad\Longrightarrow\quad \left(\frac{\tilde{u}+\frac{i}{2}}{\tilde{u}-\frac{i}{2}}\right)^{J_3} = \prod_{\substack{j=1\\ j\neq k}}^{S_1} \frac{\tilde{u}-u_j-i}{\tilde{u}-u_j+i}}$$

Then there are J_3 further hidden solutions $\tilde{u}$ of the Bethe equations. These are the rapidities of the $\mathcal{Z}$-particles in the background of the $\mathcal{D}$'s:

$$\boxed{\mathrm{Tr}\Big((\mathcal{D}^{s_1}\mathcal{Z})(\mathcal{D}^{s_2}\mathcal{Z})\ldots(\mathcal{D}^{s_{J-1}}\mathcal{Z})(\mathcal{D}^{s_J}\mathcal{Z})\Big)\,.}$$

Magnon and Hole Root Distributions

For the lowest state in the $\mathfrak{sl}(2)$ twist operator sector we have



J_3=8 S_1=8

One-Loop Bethe Ansätze for Magnons and Holes

The S_1 one-loop Bethe equations for the S_1 magnon rapidities u_k (or Bethe roots) read

$$\boxed{\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^{J_3}=\prod_{\substack{j=1\\j\neq k}}^{S_1}\frac{u_k-u_j-i}{u_k-u_j+i}\,.}$$

For the J_3 hole roots $\tilde{u}_k$ one needs to write a non-linear integral equation (NLIE, or Destri-DeVega equation). However, for large S_1 we have

$$\boxed{2^{2i\tilde{u}_k}\left(\frac{\Gamma(\frac{1}{2}-i\tilde{u}_k)}{\Gamma(\frac{1}{2}+i\tilde{u}_k)}\right)^{J_3}\simeq\prod_{\substack{j=1\\j\neq k}}^{J_3}\frac{\Gamma(+i(\tilde{u}_k-\tilde{u}_j))}{\Gamma(-i(\tilde{u}_k-\tilde{u}_j))}\,.}$$

One-Loop Dispersion Laws for Magnons and Holes

The additive dispersion law for magnons reads

$$\boxed{\gamma_1 = \sum_{k=1}^{S_1} \frac{2}{u_k^2 + \frac{1}{4}}\,.}$$

For the holes the dispersion law is not quite additive due to vacuum polarization effects. However, for large S_1 we have

$$\boxed{\gamma_1 \simeq 2 \sum_{k=1}^{J_3} \left(\psi\left(\frac{1}{2} + i\tilde{u}_k\right) + \psi\left(\frac{1}{2} - i\tilde{u}_k\right) - 2\,\psi(1) \right) + 8\,\log 2\,.}$$

The two large holes scale as $\tilde{u} \simeq \pm S_1$, giving immediately $\gamma_1 \sim 8\,\log S_1$. For exactly two or three holes the system is hyperintegrable. Can derive exact higher loop anomalous dimensions for arbitrary S_1 [Kotikov, Rej, Zieme, to appear].

Fine Structure of the One-Loop Anomalous Dimension

Consider the limit

$$\boxed{S_1 \to \infty\,, J_3 \to \infty\,, \qquad \text{with} \qquad j := \frac{J_3}{\log S_1} = \text{fixed}\,.}$$

The one-loop anomalous dimension becomes

$$\boxed{\frac{\gamma_1(j)}{\log S_1} = 8 + 2\int_{-1}^{1} d\bar{u}\,\bar{\rho}_h(\bar{u})\,\Big(\psi(\tfrac{1}{2} + ia\bar{u}) + \psi(\tfrac{1}{2} - ia\bar{u}) - 2\,\psi(1)\Big).}$$

This leads to the expansion

$$\boxed{\frac{\gamma_1(j)}{\log S_1} = 8 - 8j\log 2 + \frac{7}{12}j^3\pi^2\zeta(3) - \frac{7}{6}j^4\pi^2\log 2\,\zeta(3) + \mathcal{O}(j^5)}$$

Apparently a convergent series in j!

A Generalized Scaling Function for AdS/CFT

[Freyhult, Rej, MS, '07]

This suggests that the anomalous dimension γ of twist J_3 ops generates a generalized, two-parameter, bi-analytic scaling function $f(g,j)$

$$\boxed{\Delta - S_1 - J_3 \;=\; \gamma \;=\; f(g,j)\ \log S_1 \;+\; O(S_1^0)\,,}$$

in the limit

$$\boxed{S_1 \rightarrow \infty\,, J_3 \rightarrow \infty\,, \qquad \text{with} \qquad j := \frac{J_3}{\log S_1} = \text{fixed}\,.}$$

Indeed true, as may be shown from the all-loop Bethe ansatz!

Interestingly, it does not appear to be possible to decouple magnons and holes beyond one-loop order.

Integral Equation for the Generalized Scaling Function

[Freyhult, Rej, MS, '07]

We find the following generalization of the linear integral equation for the universal scaling function

$$\boxed{\hat{\sigma}(t) \;=\; \frac{t}{e^{\,t}-1}\left[\;\hat{K}(t,0) - \frac{j}{8}\frac{J_0(2\,g\,t)}{t} - 4\int_0^\infty dt'\ \hat{K}(t,t')\ \hat{\sigma}(t')\right]}$$

The generalized universal scaling function $f(g,j)$ is then given by

$$\boxed{f(g,j) \;=\; 16\left(\hat{\sigma}(0) + \frac{j}{16}\right)\,.}$$

The generalized kernel $\hat{K}$ is even more involved as before.

Being bi-analytic, the equation should be exact in both g and j!

The O(6) Sigma-Model from Planar $\mathcal{N} = 4$ SYM

The corresponding limit was also studied to one- and two-loop order on the string side. [Frolov, Tirziu, Tseytlin '06; Roiban, Tseytlin '07]

It was then suggested that $f(g, j)$ may be exactly determined at strong coupling. [Alday, Maldacena '06]

This was done by reducing the full sigma model to an integrable O(6) sigma model. Its free energy is known from the thermodynamic Bethe ansatz (TBA), and was conjectured to be given, for $j << g$, by

$$\mathrm{O}(6)\,\sigma\mathrm{-model\ free\ energy} = f(g, j) - f(g, 0)$$

Very recently, [Basso, Korchemsky '08] this was proven by extracting the O(6) TBA equations, including the exact expression for the mass gap, from the strong coupling limit of our above integral equation. [Freyhult, Rej, MS, '07]

See also [Fioravanti, Grinza, Rossi, Buccheri '08].

AdS/CFT Interpolation Works

- This is the second example, after the cusp anomalous dimension, of a non-trivial quantity which smoothly interpolates between perturbative gauge theory and quantized string theory.

- It is fascinating to see how the "knowledge" of the O(6) symmetry is restored when tracking the anomalous dimension of a twist operator $\mathcal{O} = \mathrm{Tr}\left(\mathcal{D}^{S_1}\mathcal{Z}^{J_3}\right)$ in a "closed sector" from weak to strong coupling!

Wrapping Restrictions

$\mathfrak{sl}(2)$ length	1	2	3	4	5	6	7... loops
⋮							
6	✓	✓	✓	✓	✓	✓	✓
5	✓	✓	✓	✓	✓	✓	?
4	✓	✓	✓	✓	✓	?	?
3	✓	✓	✓	✓	?	?	?
2	✓	✓	✓	See Janik‘s talk !	?	?	?

TBA

- So in the sense of the table just shown it appears that “exactly half” of the perturbative spectrum of $\mathcal{N} = 4$ gauge theory is now known.

- However, it has been known for some time that the asymptotic Bethe ansatz indeed does not properly include finite size effects. This was shown on the string side in [Schäfer-Nameki, Zamaklar, Zarembo ‘06] and in [Arutyunov, Frolov, Zamaklar, ‘06], [Astolfi, Forini, Grignani, Semenoff ‘07] and on the gauge side in [Kotikov, Lipatov, Rej, MS, Velizhanin, ‘07].

- For concrete ideas on the lower half of the table, see Janik’s talk later today.

Crucial Open Problems

- Actually, what exactly is this system we are solving?
  How can we define it, and prove its integrability?
- In other words, what is it we have been/currently are diagonalizing?
- How can we derive this system from the planar $\mathcal{N} = 4$ gauge theory?
- And the same question remains open for the σ-model on $AdS_5 \times S^5$.

Solvable Structures in the (Planar) AdS/CFT System

- Spectral Problem
- Gluon Amplitudes
- Wilson Loops
- High Energy Scattering (BFKL)

These are all related!

- Olive-Montonen Duality → Solvability beyond the planar limit?

Integrability beyond the Spectral Problem

Integrability in planar gauge theories actually first appeared in the high-energy scattering context. [Lipatov '93; Faddeev, Korchemsky '94]

Evidence is accumulating that it also rules the (planar) space-time scattering processes in $\mathcal{N}=4$ gauge theory. Several talks on this at this conference. Is there a "spin chain" for gluon amplitudes?

Lipatov showed very recently that an integrable open spin chain appears in the Regge limit of gluon amplitudes. [Lipatov, to appear]

An exciting new example for integrable AdS/CFT?

The planar $\text{AdS}_4/\text{CFT}_3$ system. See Juan Maldacena's talk.

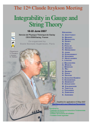
Has evolved very fast, but there is much less "data" than for $\text{AdS}_5/\text{CFT}_4$.

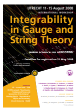
- Prove (or disprove?) full one-loop integrability, extending [Minahan, Zarembo '08]
- Is the CFT really integrable beyond one loop?
- The Lax-pair for the string σ-model was found [Arutyunov, Frolov '08]. But is the σ-model really quantum integrable?
- Are the Gromov-Vieira Bethe equations correct as is? [Gromov, Vieira '08]
- What is this $h(\lambda)$ function in the dispersion law?
- Finally, does this new model use the same "trick" to be integrable?

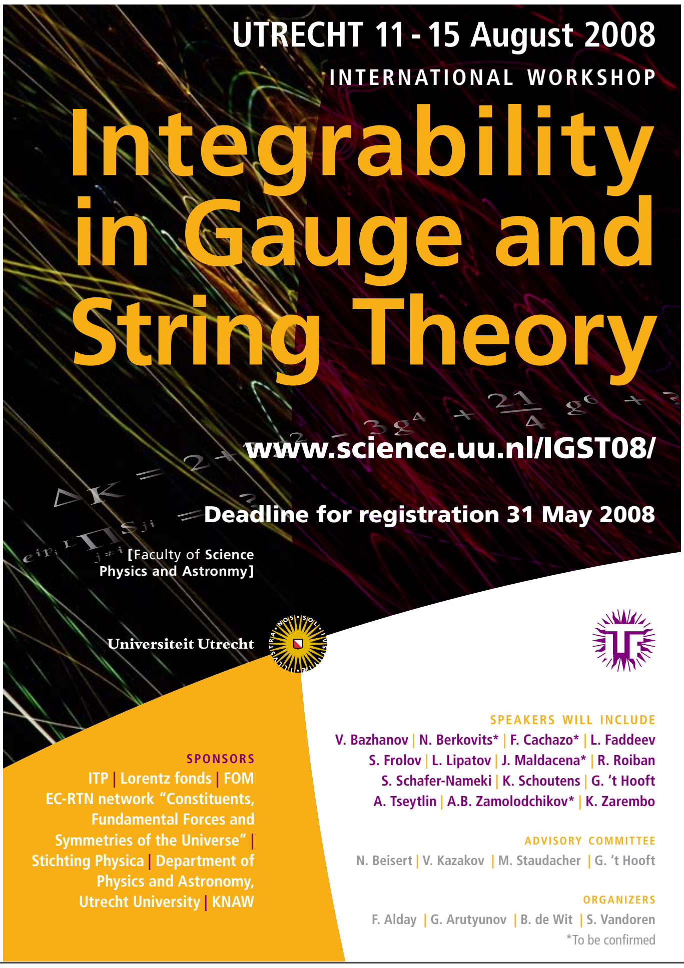
Conference Series:
Integrability in Gauge and String Theory

This is becoming an exciting annual event:

- 2005: Paris (ENS Summer Institute)
- 2006: AEI Potsdam
- 2007: Paris (Itzykson Conference, Saclay and ENS)
- 2008: Utrecht







UTRECHT 11 - 15 August 2008

INTERNATIONAL WORKSHOP

Integrability in Gauge and String Theory

www.science.uu.nl/IGST08/

Deadline for registration 31 May 2008

[Faculty of Science
Physics and Astronomy]

Universiteit Utrecht

SPONSORS

ITP | Lorentz fonds | FOM
EC-RTN network "Constituents, Fundamental Forces and Symmetries of the Universe" | Stichting Physica | Department of Physics and Astronomy, Utrecht University | KNAW

SPEAKERS WILL INCLUDE

V. Bazhanov | N. Berkovits* | F. Cachazo* | L. Faddeev
S. Frolov | L. Lipatov | J. Maldacena* | R. Roiban
S. Schafer-Nameki | K. Schoutens | G. 't Hooft
A. Tseytlin | A.B. Zamolodchikov* | K. Zarembo

ADVISORY COMMITTEE

N. Beisert | V. Kazakov | M. Staudacher | G. 't Hooft

ORGANIZERS

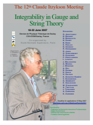
F. Alday | G. Arutyunov | B. de Wit | S. Vandoren

*To be confirmed

Conference Series:
Integrability in Gauge and String Theory

This is becoming an exciting annual event:

- 2005: Paris (ENS Summer Institute)
- 2006: AEI Potsdam
- 2007: Paris (Itzykson Conference, Saclay and ENS)
- 2008: Utrecht
- 2009: AEI Potsdam (probably the week after Strings ‘09)



The 12th Claude Itzykson Meeting

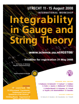
Integrability in Gauge and String Theory

18-22 June 2007

Service de Physique Théorique de Saclay
CEA-DSM Saclay, France

Co-organized by
Ecole Normale Supérieure, Paris

Deadline for applications 15 May 2007



UTRECHT 11 - 15 August 2008

INTERNATIONAL WORKSHOP

Integrability in Gauge and String Theory

www.science.uu.nl/IGST08/

Deadline for registration 21 May 2008

Outlook

Unique chance to participate in the first exact solution of a four-dimensional Yang-Mills theory!