Introduction

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In the case of a single M-brane the full supersymmetric dynamics are known and have powerful, seemingly magical properties:

- M5 - [Howe, Sezgin, West], [Schwarz, Perry] (1996)

But the dynamics of multiple M-branes has proved to be elusive:

- No dilaton to enable a weakly coupled limit
- M2 - \( n^{3/2} \) degrees of freedom
- M5 - \( n^3 \) degrees of freedom
- No known Lagrangian description
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- M5 - $n^3$ degrees of freedom
- No known Lagrangian description,..., or maybe not.
A stack of M2-branes has 8 scalars $X^I$ and their fermionic superpartners $\Psi$, $\Gamma_{012}\Psi = -\Psi$.

We assume that these take values in some vector space $\mathcal{A}$.

A natural ansatz for the susy algebra is

\[
\delta X^I = i\bar{\epsilon}\Gamma^I\Psi \\
\delta \Psi = \partial_\mu X^I \Gamma^I \Gamma^I \epsilon + [X^I, X^J, X^K] \Gamma^{JK} \epsilon,
\]

where $[A, B, C]$ is totally anti-symmetric triple product on $\mathcal{A}$.

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- So $\mathcal{A}$ needs a triple product: 3-algebra.

For an M2 ending on an M5-brane this gives the BPS equation

$$
\frac{dX^I}{d(x^2)} = \epsilon^{IJKL}[X^J, X^K, X^L] \quad [\text{Basu, Harvey}]
$$

Closure of the algebra implies a gauge symmetry:

$$
[\delta_1, \delta_2]X^I = 2i\bar{\epsilon}_1 \Gamma^\mu \epsilon_2 \partial_\mu X^I + 2i\bar{\epsilon}_1 \Gamma^{JK} \epsilon_2 [X^J, X^K, X^I]
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This must be dealt with to realize the full superalgebra.

We will proceed by introducing a basis $T^a$ for $\mathcal{A}$

$$[T^a, T^b, T^c] = f^{abc} d T^d, \quad f^{abc} d = f[abc]_d$$

so we also introduce a gauge field

$$\delta X^I_d = \Lambda_{ab} f^{abc} d X^I_c \quad D_\mu X^I_c = \partial_\mu X^I_c - \tilde{A}_\mu^c d X^I_d$$

- Independently closed using a different, but equivalent algebraic approach [Gustavsson].

Full superalgebra takes the form

$$\begin{align*}
\delta X^I_d &= i\bar{\epsilon} \Gamma^I \Psi_d \\
\delta \Psi_d &= D_\mu X^I d \Gamma^I \epsilon - \frac{1}{6} X^I_a X^J_b X^K_c f^{abc} d \Gamma^{JK} \epsilon \\
\delta \tilde{A}_\mu^c d &= i\bar{\epsilon} \Gamma_\mu \Gamma^I \epsilon X^I_a \Psi_b f^{abc} d
\end{align*}$$

Indeed this closes (on-shell) if $f^{abcd}$ satisfies the fundamental identity:

$$f^{efg} b f^{cba} d + f^{fea} b f^{cbg} d + f^{gaf} b f^{ceb} d + f^{age} b f^{cfb} d = 0.$$
Full superalgebra takes the form

\[ \delta X^I_d = i \bar{\epsilon} \Gamma^I \Psi_d \]
\[ \delta \Psi_d = D_\mu X^I_d \Gamma^\mu \Gamma^I \bar{\epsilon} - \frac{1}{6} \chi^I_a X^J_b X^K_c f^{abc}_d d^IJK \bar{\epsilon} \]
\[ \delta \tilde{A}_\mu \epsilon_d = i \bar{\epsilon} \Gamma_\mu \Gamma_I X^I_a \Psi_b f^{abc}_d, \]

Indeed this closes (on-shell) if \( f^{abcd} \) satisfies the fundamental identity:

\[ f^{efg}_b f^{cba}_d + f^{fe}_a f^{cbg}_d + f^{gaf}_b f^{ceb}_d + f^{age}_b f^{cfa}_d = 0. \]

This ensures that the gauge symmetries generated by the triple product are those of a Lie-algebra.

The invariant Lagrangian is a Chern-Simons theory:

\[ \mathcal{L} = -\frac{1}{2} (D_\mu X^a_l) (D^\mu X^a_l) + \frac{i}{2} \bar{\Psi}^a \Gamma_\mu D_\mu \psi_a + \frac{i}{4} \bar{\Psi}_b \Gamma I J X^I_d X^J_d \psi_a f^{abcd} + \frac{1}{2} \epsilon^{\mu \nu \lambda} (f^{abcd} A_{\mu \lambda \nu \lambda} \partial_\mu A_{\nu \lambda} + \frac{2}{3} f^{cda}_g f^{efg}_b A_{\mu \lambda \nu \lambda} A_{\nu \lambda} + \frac{1}{12} \text{Tr}([X^I_l, X^J_l, X^K_l])^2 \]

- \( \text{Tr} \) is an invariant trace (or inner-product) on \( \mathcal{A} \)
- gauge invariance implies \( f^{abcd} = f^{[abcd]} \)
- \( \tilde{A}_\mu \epsilon_d = f^{abc}_d A_{\mu \lambda} \)

This Lagrangian has all the expected symmetries of multiple M2-branes: \( \mathcal{N} = 8 \) supersymmetry, \( SO(8) \) R-symmetry and Parity.
However, if $\text{Tr}$ is positive definite then there is only one, finite dimensional, solution [Nagy],[Gauntlett, Gutowski],[Papadopoulou]:

\[
f^{abcd} = \frac{2\pi}{k} \varepsilon^{abcd}
\]

In this case the Lagrangian is that of an $SU(2) \times SU(2)$ Chern-Simons theory coupled to matter in the bi-fundamental.

- quantization condition implies $k \in \mathbb{Z}$

Infinite dimensional examples can be constructed by considering the Nambu bracket on an ‘auxiliary' three-manifold $\Sigma$.

- More recently been identified with a single M5 on $\mathbb{R}^3 \times \Sigma$ [Ho,Matsuo] [Banados, Townsend]
What is the relation of this to M-theory (see also [Distler, Mukhi, Papageorgakis, van Raamsdonk]):

Vacuum moduli space \( (\mathbb{R}^8 \times \mathbb{R}^8)/D_{2k} \)
- \( k = 1 \) \( \mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{R}^8/\mathbb{Z}_2 \)
  - moduli space of an \( SO(4) \) gauge theory
- \( k = 2 \) \( (\mathbb{R}^8/\mathbb{Z}_2 \times \mathbb{R}^8/\mathbb{Z}_2)/\mathbb{Z}_2 \)
  - moduli space of an \( SO(5) \) gauge theory
- 2 objects on \( \mathbb{R}^8/\mathbb{Z}_2 \)
There are two maximally-susy $\mathbb{R}^8/\mathbb{Z}_2$ orbifolds of two M2’s with gauge groups $O(4)$ and $SO(5)$ (with and without discrete torsion) [Sethi],[Berkooz,Kapustin]

- $k = 1$ case differs from orbifold in that $SO(4)$ should be $O(4)$
- Good agreement for $k = 2$
- No clear picture for $k > 2$
  - The orbifold action moves the branes and doesn’t preserve the distances between them
  - N.B. for $k = 3$ $(\mathbb{R}^8 \times \mathbb{R}^3)/D_6$ is the moduli space of a $G_2$ gauge theory

It is interesting to note that on the Coulomb branch one finds the classical mass formula

$$M = \frac{2\pi}{k} \text{(area of a triangle with vertices on an M2)}$$

- Tempting clue to microscopic states in M-theory analogous to stretched open strings
  - N.B. orbifold action preserves $M$
- hints towards an origin of $N^3$.
- note that there is an enhanced gauge symmetry whenever the M2s are collinear
  - strongly coupled (c.f. origin of $\mathbb{R}^7$ Coulomb branch in D2-branes)
So it is of great interest to generalize the Lagrangian construction:

The equations of motion don’t require that $f^{abcd}$ is totally anti-symmetric: infinitely many examples [Gran, Nilsson, Petersson]

- But no gauge invariant trace, so no observables
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Infinitely many models with $h^{ab}$ having a Lorentzian signature have been proposed by [Gomis, Milanesi, Russo], [Benvenuti, Rodriguez-Gomez, Tonni, Verlinde] and [Ho, Imamura, Matuso]

- Despite the negative norm scalars the quantum theory appears to be unitary.
- Their status is not clear

Look for less supersymmetry:

[Aharony, Bergman, Jafferis, Maldacena] proposed models with $\mathcal{N} = 6$ and an $SU(4) \times U(1)$ R-symmetry.

- Chern-Simons matter theory with gauge group $U(N) \times U(N)$ for any $N$ and level $k$
- Proposed to describe $N$ M2’s on $\mathbb{R}^8/\mathbb{Z}_k$
  - Including $k = 1$!
- Large $N$ and $k$ limit: dual to $adS_4 \times S^7/\mathbb{Z}_k$ (and $adS_4 \times \mathbb{CP}^3$ by compactification to type IIA)
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More recently [Aharony, Bergman, Jafferis] generalized to include discrete torsion by considering gauge group $U(N_1) \times U(N_2)$

- Includes $\mathbb{R}^8/\mathbb{Z}_2$ $SO(5)$ orbifold but without manifest $\mathcal{N} = 8$

So let us derive the most general scale-invariant Lagrangian with $\mathcal{N} = 6$ and an $SU(4) \times U(1)$ R-symmetry:

- scalars $Z^A_a \in \mathbf{4}_1$ of $SU(4) \times U(1)$
- fermions $\psi_{Aa} \in \bar{\mathbf{4}}_1$ of $SU(4) \times U(1)$
- susys $\epsilon_{AB} \in \mathbf{6}_0$ of $SU(4) \times U(1)$
  - $(\epsilon_{AB})^* = \epsilon^{AB} = \frac{1}{2} \epsilon^{ABCD} \epsilon_{CD}$

Complex conjugation raises/lowers and $A$-index and flips the $U(1)$ charge
Starting from the most general form for the susy's one finds

\[
\begin{align*}
\delta Z_d^A &= i \epsilon^{AB} \psi_{Bd} \\
\delta \psi_{Bd} &= \gamma^\mu D_\mu Z_d^A \epsilon_{AB} + f^{abc}_d Z_a^C Z_b^A Z_c^A \epsilon_{AB} + f^{abc}_d Z_a^C Z_b^D Z_c^\epsilon_{CD} \\
\delta \tilde{A}_{\mu c}^d &= -i \tilde{e}_{AB} \gamma^\mu Z_a^A \tilde{\psi}_B f^{cab}_d + i \epsilon^{AB} \gamma_{\mu} Z_{Ab} \psi_{Ba} f^{cba}_d
\end{align*}
\]

Provided that

\[
f_{efg}_b f^{cba}_d + f_{feab}_b f^{cgb}_d + f_{gaf}^* f_{cbg}_d + f_{age}^* f_{cfb}_d = 0
\]

and

\[
f_{abcd} = - f_{bacd} = - f_{abdc} = f^{*cdab}.
\]

i.e. \( f_{abcd} \) need not be real nor totally anti-symmetric.

► Everything is determined by specifying a triple product \([X, Y; Z] \) on \( A \)

The Lagrangian is

\[
\mathcal{L} = -D_\mu Z_a^A D^\mu Z_a^A - i \tilde{\psi}_a^A \gamma^\mu D_\mu \psi_a^A - i f^{abcd} \tilde{\psi}_d^A \psi_a^B Z_b^Z_{Zc} + 2 i f^{abcd} \tilde{\psi}_d^A \psi_b^B Z_b^B Z_c^c - \frac{i}{2} \epsilon_{ABCD} f^{cdab} \tilde{\psi}_c^A \psi_b^B Z_c^a Z_d^d - \frac{1}{2} \epsilon^{\mu\nu\lambda} (f_{abcd} A_{\mu c b d} \partial_\nu A_{\lambda d a} + \frac{2}{3} f^{acd}_g f_{g e b} A_{\mu a b} A_{\nu d c} A_{\lambda e}) - \frac{2}{3} \gamma_{Bd}^{CD} \gamma_{Bd}^{CD}.
\]

with

\[
\gamma_{Bd}^{CD} = Z_a^C Z_b^D Z_{Bc} f^{abc}_d - \frac{1}{2} \delta_B^C Z_a^E Z_b^D Z_{E c} f^{abcd}_d + \frac{1}{2} \delta_B^D Z_a^E Z_b^C Z_{E c} f^{abcd}_d
\]
An infinite class of 3-algebras can be constructed as follows:

- $\mathcal{A}$ = linear maps between two complex vector spaces $V_1$ and $V_2$ with dimensions $N_1$ and $N_2$.
- $[X, Y; Z] = XZ^\dagger Y - YZ^\dagger X$

The gauge symmetry generated by the triple product is

$$\delta X = XM_1 - M_2 X$$

with $M_1 \in u(N_1)$ and $M_2 \in u(N_2)$.

The fundamental identity ensures that

$$\delta[X, Y; Z] = [X, Y; Z]M_1 - M_2[X, Y; Z]$$
The Lagrangian reduces to the $\mathcal{N}=6$, $U(N) \times U(N)$ Chern-Simons model of ABJM for $N_1 = N_2$ and also the more general $U(N_1) \times U(N_2)$ models of ABJ.

There are also other possibilities:

- $SU(N_1) \times SU(N_2)$
- $Sp(2N) \times O(2)$ [Hosomichi, 3-Lee, Park]
- Classified by [Schnabl and Tachikawa]
- see also papers by [Bandres, Lipstein, Schwarz], [Bergshoeff, Holm, Roest, Samtleben, Serzgin], [Cherkis, Saemann][Nilsson, Palmkvist]
We constructed a (rather unique!) $\mathcal{N} = 8$ 3D Lagrangian field theory with an SO(8) R-symmetry and Parity

- New maximally superconformal Chern-Simons gauge theory that is not Yang-Mills
- Identified with 2 M2-branes on $\mathbb{R}^8/\mathbb{Z}_2$ (at least for $k = 2$)
- Infinitely many theories with $\mathcal{N} = 6$ and an $SU(4) \times U(1)$ R-symmetry
  - Proposed by ABJM and ABJ as $N$ M2’s on $\mathbb{R}^8/\mathbb{Z}_k$, for $k = 1, 2, ...$

- Gained some insight into the degrees of freedom of multiple M2-branes?
  - Classically the massive states associated to triangles with vertices on the M2’s
There are still many issues to understand:

- What is the role of the $\mathcal{N} = 8$ theory?
- Understand the enhancement of $k = 1, 2$ ABJM to $\mathcal{N} = 8$
- What is the role of the Lorentzian theories?
  - Equivalent to $\mathcal{N} = 8$ super-Yang-Mills? [Gomis, Rodriguez-Gomez, van Raamsdonk, Verlinde], [Bandres, Lipstein, Schwarz]
  - M2/D2's on a cylinder [Banerjee, Sen]
  - scaling limit of ABJM [Honma, Iso, Sumitomo, Umetsu, Zhang]
- Can one see the $n^3$?
  - e.g. see [de Madeiros, Figueroa-O’Farrill, Mendez-Escobar] and [Chu, Hi, Matsuo, Shiba]

We have tried to stress the central role of 3-algebras. Why?
We have tried to stress the central role of 3-algebras. Why?

- They naturally encode all the information of the theory
  - Classification of $\mathcal{N} \geq 6$ theories is a classification of 3-algebras.
- The dynamics of M2-branes is primarily determined by the scalars and fermions and these don’t directly see a Lie-bracket.
- Hopefully they are interesting in their own right and a clue to the microscopic degrees of freedom in M-theory.