Unveiling the structure of amplitudes in gauge theory and gravity

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Outline

• Introduction
• Some tools
• Applications
  – QCD for LHC
  – N=4 super-Yang-Mills and AdS/CFT
  – N=8 supergravity – might it be finite?
• Conclusions

see talks by
Alday
Berkovits
Sokatchev
Green
Cachazo
Kallosh
Introduction

• Scattering amplitudes are basic to almost everything we learn directly from collider experiments.
• They also have proved extremely useful over the ages as gedanken experiments – probes of properties of theories
• Their on-shell nature makes them (and all information extracted from them) physical:
  – independent of field redefinitions
  – independent of gauge choice

Unveiling structures

Sometimes hidden structures are found in amplitudes, which are only fully understood later

$$A_n = \frac{\langle i j \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \cdots \langle n 1 \rangle}$$

Parke-Taylor; Mangano, Parke, Xu; Berends, Giele (1986-1987)

$$\Rightarrow$$ twistor string theory Witten (2003)
A technical issue…

Beyond tree-level, scattering amplitudes afflicted with infrared (soft & collinear) divergences

Fortunately we know how to deal with them in gauge theories: Use dimensional regulation, \( D = 4-2\epsilon \).

Exploit years of experience in QED and QCD:
- soft/collinear factorization & exponentiation
- virtual/real cancellation in infrared-safe cross sections

Amplitudes are “plastic”

**Factorization:** How they “fall apart” into simpler ones in special limits

**Unitarity:** Discontinuities across branch cuts

Exploit these properties to determine them
Simplest in helicity formalism

Many helicity amplitudes either vanish or are very short

![Helicity Amplitudes Diagram]

\[
\begin{align*}
\text{right-handed} & \quad h = +1 \\
\text{left-handed} & \quad h = -1 \\
A_n & = 0
\end{align*}
\]

\[
A_n = \frac{\langle i j \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \cdots \langle n 1 \rangle}
\]

The tail of the mantis shrimp

- Reflects left and right circularly polarized light differently
- Led biologists to discover that its eyes have differential sensitivity
- It communicates via the helicity formalism

“\text{It's the most private communication system imaginable. No other animal can see it.}”

- Roy Caldwell (U.C. Berkeley)
What the biologists didn’t know

Particle theorists have also evolved capability to communicate results via **helicity formalism**

**LHC experimentalists are blind to it**

any final-state polarization effects washed out by fragmentation

must sum over all helicity configurations

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Spinor helicity formalism

**Use Weyl spinors** $u_\alpha(k)$ (spin $\frac{1}{2}$)

right-handed: $(\lambda_i)_\alpha = u_+(k_i)$

$h = +1/2$

left-handed: $(\bar{\lambda}_i)_\dot{\alpha} = u_-(k_i)$

$h = -1/2$

Instead of Lorentz products:

Use spinor products:

Always obey

$$\langle i \ j \rangle \ [j \ i \rangle = s_{ij}$$

If momenta are real, they are complex square roots of Lorentz products:

$$\langle i \ j \rangle = \sqrt{s_{ij}} e^{i\phi_{ij}}$$

$$[j \ i \rangle = \sqrt{s_{ij}} e^{-i\phi_{ij}}$$
Virtues of complex momenta

- Makes sense of basic process with 3 lightlike (massless) particles

\[ s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2 = 0 \quad \forall \ i, j \]

real (singular)

\[ \langle i,j \rangle = [ i,j ] = s_{ij} = 0 \quad \forall \ i, j \]

complex (nonsingular)

\[ [ i,j ] = 0 \text{ but } \langle i,j \rangle \neq 0 \]

3-gluon amplitude

\[ \langle i,j \rangle^4 \]

\[ \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \]

makes sense

Exploit analyticity at tree level

Inject complex momentum at leg 1, remove it at leg n.

\[ \lambda_1 = \lambda_1 + z \lambda_n \quad \hat{\lambda}_1 = \hat{\lambda}_1 \]
\[ \lambda_n = \lambda_n \quad \hat{\lambda}_n = \hat{\lambda}_n - z \hat{\lambda}_1 \]

\[ k_1(z) + k_n(z) = k_1 + k_n \]
\[ k_1^2(z) = k_n^2(z) = 0 \]

\[ A(0) \rightarrow A(z) \quad \text{factorization limits} \leftrightarrow \text{poles in } z \]

Cauchy: If \( A(\infty) = 0 \) then

\[ 0 = \frac{1}{2\pi i} \int dz \frac{A(z)}{z} = A(0) + \sum_k \operatorname{Res} \left[ \frac{A(z)}{z} \right]_{z=z_k} \]

residue at \( z_k = [k^{th} \text{ factorization limit}] = \]
**BCFW (on-shell) recursion relations**

Britto, Cachazo, Feng, hep-th/0412308

\[ A_n = \sum_{h,k} A_{n-k+1} + A_{k+1} \]

\( A_{k+1} \) and \( A_{n-k+1} \) are on-shell tree amplitudes with fewer legs, and with momenta shifted by a fixed complex amount.

**Trees recycled into trees**

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**A 6-gluon example**

220 Feynman diagrams for \( gggggg \)

Consider \( A_6(1^+, 2^+, 3^+, 4^-, 5^-, 6^-) \)

(the other nontrivial helicity assignment is similar)

2 nonvanishing recursive diagrams, related by a symmetry.
Yields simplest final form

\[
-iA_6(1^+, 2^+, 3^+, 4^-, 5^-, 6^-) = \frac{\langle 6^-| (1 + 2)|3^- \rangle^3}{\langle 6 \ 1 \ 12 \ 134 \ 45 \ s_{612} \ 2^-| (6 + 1)|5^- \rangle} + \frac{\langle 4^-| (5 + 6)|1^- \rangle^3}{\langle 2 \ 3 \ 34 \ 56 \ 61 \ s_{561} \ 2^-| (6 + 1)|5^- \rangle} \\
\langle a^-|(b + c)|d^- \rangle \equiv \langle a \ b| [b \ d] + \langle a \ c| [c \ d] \]
\]

- All symmetries and physical pole behavior manifest
- Only one “spurious singularity”

Degrees of difficulty

[Image of divers in Beijing]
Why multi-parton loop amplitudes?
Because LHC is a QCD machine

- **Backgrounds to new physics** require detailed understanding of scattering amplitudes for many ~ massless particles – especially quarks and gluons of QCD.

- Depending on how big the signal is, leading-order QCD (tree amplitudes) may not be precise enough → need next-to-leading order (NLO) QCD corrections, which include loop amplitudes, as well as real radiation.
**Example: search for supersymmetry**

- **Cascade from gluino to neutralino**
  - (dark matter, escapes detector)

- **Signal: missing energy + 4 jets**

- **SM background** from $Z + 4$ jets, $Z \rightarrow$ neutrinos

- **ALPGEN** based on LO tree amplitudes, normalization still quite uncertain

$$\Rightarrow pp \rightarrow Z + 4 \text{ jets at NLO}$$

2 legs beyond state-of-art

**Strong growth in complexity**

with number of external legs

<table>
<thead>
<tr>
<th># of legs</th>
<th># of 1-loop Feynman diagrams (gluons only)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>810</td>
</tr>
<tr>
<td>6</td>
<td>10,86C</td>
</tr>
<tr>
<td>7</td>
<td>168,92E</td>
</tr>
<tr>
<td>8</td>
<td>3,017,490</td>
</tr>
</tbody>
</table>

Motivation for exploiting analyticity of amplitudes at one loop
One-loop amplitude decomposition

For external momenta in $D=4$, loop momenta in $D=4-2\epsilon$:

\[ A^{1-\text{loop}} = C + R + \mathcal{O}(\epsilon) \]

- cut part
- rational part

Known scalar one-loop integrals, same for all amplitudes

Coefficients of boxes, triangle and bubble integrals – all rational functions – get using generalized unitarity

Generalized unitarity at one loop

**Ordinary unitarity:**
- put 2 particles on shell

**Generalized unitarity:**
- put 3 or 4 particles on shell

Cut conditions require complex loop momenta
Trees get simpler
Gen. unitarity for box coefficients $d_i$

Britto, Cachazo, Feng, hep-th/0412308

\[ \int d^4 \ell \, \delta(\ell_1^2 - m_1^2) \delta(\ell_2^2 - m_2^2) \times \delta(\ell_3^2 - m_3^2) \delta(\ell_4^2 - m_4^2) \times A^{1\text{-loop}}(\ell_i) = A_1^{\text{tree}}(\ell_0) A_2^{\text{tree}}(\ell_0) A_3^{\text{tree}}(\ell_0) A_4^{\text{tree}}(\ell_0) = d_i \]

[$\# \text{ of dimensions} = 4 = \# \text{ of constraints}$ $\rightarrow$ discrete solutions

• unique box selected

• no $L_5^2 = 0$ solution $\rightarrow$ no pentagons

Generalized unitarity (cont.)

With a 4-ple cut we select one coefficient

\[ \begin{array}{c}
\begin{array}{c}
\text{triangle and bubble coefficients are more complicated since a}
\text{double or triple cut does not isolate a single coefficient.}
\end{array}
\end{array} \]

Also, solutions to cut constraints are now continuous, so there are multiple ways to solve and eliminate $d_i$, etc.

Britto et al. (2005,2006); Mastrolia (2006); Ossola, Papadopoulos, Pittau (2007); Forde, 0704.1835; Ellis, Giele, Kunszt, 0708.2398; …
Rational function $R$

No cuts in $D=4$ – can’t get from $D=4$ unitarity
However, can get using $D=4-2\varepsilon$ unitarity:

$$\int d^{D-2\varepsilon} x \quad \Rightarrow \quad R(s_{ij}) \rightarrow R(s_{ij}) (-s_{12})^{-\varepsilon} = R(s_{ij}) [1 - \varepsilon \ln(-s_{ij})]$$

Bern, Morgan (1996); Bern, LD, Kosower (1996); Brandhuber, McNamara, Spence, Travaglini hep-th/0506068; Anastasiou et al., hep-th/0609191, hep-th/0612277; Britto, Feng, hep-ph/0612089, 0711.4284; Giele, Kunszt, Melnikov, 0801.2237; Britto, Feng, Mastrolia, 0803.1989; Britto, Feng, Yang, 0803.3147; Ossola, Papadopolous, Pittau, 0802.1876; Mastrolia, Ossola, Papadopolous, Pittau, 0803.3964; Giele, Kunszt, Melnikov (2008); Giele, Zanderighi, 0805.2152; Ellis, Giele, Kunszt, Melnikov, 0806.3467; Feng, Yang, 0806.4106; Badger, 0806.4600

Or: $R$ from 1-loop recursion


Use same complex momentum shift as at tree level

$A^{\text{1-loop}}(z)$ has cuts as well as poles in $z$-plane:

$R(z)$ is rational; has only poles
– but some of them are spurious,
cancelling against $C(z)$ in full amplitude

Physical poles generate recursive diagrams a la BCFW

Trees + loops recycled into loops!
Ideas now being implemented numerically and automatically

**CutTools:** Ossola, Papadopolous, Pittau, 0711.3596  
NLO production of WWW  
Binoth+OPP, 0804.0350

**Rocket:** Giele, Zanderighi, 0805.2152  
One-loop n-gluon amplitudes for n up to 20

**Blackhat:** Berger, Bern, LD, Febres Cordero, Forde, H. Ita,  
D. Kosower, D. Maître, 0803.4180, 0808.0941  
One-loop n-gluon amplitudes for n up to 7; some amplitudes needed for NLO production of $Z + 3\text{ jets}$

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Multi-leg **and** multi-loop

- To really push this frontier we need the aid of maximal supersymmetry:
  - $N=4$ for gauge theory
  - $N=8$ for supergravity
- Of significant interest due to:
  - AdS/CFT correspondence
  - Finiteness questions
- Progress in $N=4$ leads to progress in $N=8$  
  thanks to KLT relations between tree amplitudes  
  Kawai, Lewellen, Tye (1986)
Multi-loop generalized unitarity

Ordinary cuts of multi-loop amplitudes contain loop amplitudes. But it is very convenient to work with tree amplitudes only.

For example, at 3 loops, one encounters the product of a 5-point tree and a 5-point one-loop amplitude:

 Cut 5-point loop amplitude further, into (4-point tree) x (5-point tree), in all 3 inequivalent ways:

But one can do better

Allowing for complex momenta everywhere, one can chop an amplitude entirely into 3-point trees → maximal cuts or ~ leading singularities

Advantage is that these cuts are maximally simple, yet give an excellent starting point for constructing the full answer.

For example, in planar (leading in $N_c$) $N=4$ SYM they find all terms in the complete answer for 1, 2 and 3 loops.
Finding missing terms

Maximal cut method:
Allowing one or two propagators to collapse from each maximal cut, one obtains near-maximal cuts

These near-maximal cuts are very useful for analyzing N=4 SYM (including nonplanar) and N=8 SUGRA at 3 loops

BCDJKR, BCJK (2007); BCDJR, to appear

Leading singularity method:
Uses consistent behavior with respect to “hidden singularities”

Cachazo, Skinner; Cachazo (2008)

Recent supersum advances for more complicated cuts
e.g. Elvang, Freedman, Kiermaier, 0808.1720

Planar N=4 SYM and AdS/CFT

recent review: Alday, Roiban, 0807.1889

• In the ’t Hooft limit, \[ N_c \rightarrow \infty \]
\[ \lambda = g^2 N_c \] fixed, planar diagrams dominate

• AdS/CFT duality

suggests that weak-coupling perturbation series in \( \lambda \) for large-\( N_c \) (planar) N=4 SYM should have hidden structure, because

large \( \lambda \) limit \( \leftrightarrow \) weakly-coupled gravity/string theory on AdS\(_5\) x S\(_5\)

Maldacena; Gubser, Klebanov, Polyakov; Witten
Three Hidden Structures Recently Unveiled in Planar N=4 SYM

- Exponentiation of finite terms in the amplitude (for 4 and 5 gluons)
- Dual (super)conformal invariance
- Equivalence between (MHV) amplitudes and Wilson lines

Inspired by IR structure of QCD, Mueller, Collins, Sen, Magnea, Sterman,… based on evidence collected at 2 and 3 loops for \( n=4,5 \) using generalized unitarity and factorization, ansatz proposed:

\[
\frac{A_n}{A_n^{\text{tree}}} = M_n = \exp \left[ \sum_{l=1}^{\infty} \left( \frac{\lambda}{8\pi^2} \right)^l \left( f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + O(\epsilon) \right) \right].
\]

all kinematic dependence in known 1-loop amplitude (normalized by tree)

\( n=4 \quad \Rightarrow \quad M_4|_{\text{finite}} = \exp \left[ \frac{1}{8} \gamma_E(\lambda) \ln^2 \left( \frac{s}{t} \right) + \text{const.} \right] \)

Confirmed at strong coupling using AdS/CFT, directly at \( n=4 \), indirectly at \( n=5 \). (But: fails for \( n > 5 \).)
The constants

\[ f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)} \]

\[ f_0^{(l)} = \frac{1}{4} \gamma_K^{(l)} \quad f_1^{(l)} = \frac{1}{2} \mathcal{G}_0^{(l)} \quad f_2^{(l)} = (???) \quad C^{(l)} = (???) \]

collects 3 series of constants:

- \( \gamma_K^{(l)} \) and \( \mathcal{G}_0^{(l)} \) are \( l \)-loop coefficients of:
  - cusp anomalous dimension \( \gamma_K^{(l)}(\lambda) \) (source term for differential equation for Sudakov form factor)
  - “collinear” anomalous dimension \( \mathcal{G}_0^{(l)}(\lambda) = \mathcal{G}(-1, \lambda, \epsilon = 0) \) (integration constant for differential equation)

\[ \gamma_K^{(l)} \] are \( l \)-loop coefficients of

- cusp anomalous dimension \( \gamma_K^{(l)}(\lambda) \)
  - Beisert, Eden, Staudacher, hep-th/0610251

- “collinear” anomalous dimension \( \mathcal{G}_0^{(l)}(\lambda) = \mathcal{G}(-1, \lambda, \epsilon = 0) \)
  - Beisert, Eden, Staudacher, hep-th/0610251

Dual Conformal Invariance

A conformal symmetry acting in momentum space, on dual or sector variables \( x_i \)

First seen in N=4 SYM planar amplitudes in the loop integrals

\[ I = \int d^4 k \frac{(k_1 + k_2)^2 (k_2 + k_3)^2}{k^2 (k - k_1)^2 (k - k_1 - k_2)^2 (k + k_4)^2} \]

\[ I = \int d^4 x_5 \frac{x_2^2 x_3^2 x_4^2}{x_1^2 x_2^2 x_3^2 x_4^2 x_5^2} \]

invariant under inversion:

\[ x_i \rightarrow x_i^{-2} \]
Dual conformal invariance (cont.)

- Simple graphical rules:
  - 4 (net) lines into inner $x_i$
  - 1 (net) line into outer $x_i$
- Dotted lines are for numerator factors

4 loop planar integrals all of this form

also true at 5 loops
Bern, Carrasco, Johansson, Kosower, 0705.1864

Insight from string theory

- As a property of full amplitudes, rather than integrals, dual conformal invariance follows, at strong coupling, from bosonic T duality symmetry of $\text{AdS}_5 \times \text{S}^5$.
- Also, strong-coupling calculation ~ equivalent to computation of Wilson line for n-sided polygon with vertices at $x_i$.
- Wilson line blind to helicity formalism – doesn’t know MHV from non-MHV.
Dual (super)conformal invariance

- Surprisingly, dual conformal invariance and Wilson line equivalence both persist to weak coupling for MHV amp’s
  - Drummond, Korchemsky, Sokatchev, 0707.0243
  - Brandhuber, Heslop, Travaglini, 0707.1153
  - Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223

- Can embed dual conformal invariance into a richer dual superconformal invariance (needed to understand structure of non-MHV amplitudes)
  - DHKS, 0807.1095, 0808.0491; Sokatchev talk

Whole structure now explained better (at weak coupling too) by a combined bosonic and fermionic T duality symmetry leaving dilaton fixed
  - Berkovits, Maldacena, 0807.3196; Berkovits talk

More than 4 gluons

- Ansatz known to work for $n = 5$ (all MHV) two loops
  - Cachazo, Spradlin, Volovich, hep-th/0602228
  - Bern, Czakon, Kosower, Roiban, Smirnov, hep-th/0604074

- Should work for $n = 5$ to all loops, assuming dual conformal invariance.
- $n = 6$ is first place it does not fix form of amplitude, due to cross ratios such as $u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}$

- There were indications of a failure looming for $n = 6$,
  - A large $n$, strong-coupling limit Alday, Maldacena, 0710.1060
  - A Wilson line calculation Drummond, Henn, Korchemsky, Sokatchev, 0712.4138
  - A high-energy/Regge limit Bartels, Lipatov, Sabio Vera, 0802.2065
Tested ansatz at 2 loops, 6 gluons, MHV

Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465 [even terms]

Cachazo, Spradlin, Volovich, 0805.4832 [odd terms]

Not organic chemistry but dual conformal integrals

Ansatz definitely breaks down at n=6.
Yet the following properties persist:
1) dual conformal invariance
2) equivalence to Wilson loops

Drummond, Henn, Korchemsky, Sokatchev, 0712.4138; 0803.1466

More structure still to be unveiled, for MHV and non-MHV

Hidden cancellations in $N=8$ Supergravity

1. Multi-leg

- Can construct recursive representations of tree amplitudes, using analyses of large $z$ behavior
  Bedford, Brandhuber, Spence, Travaligni, hep-th/0502146;
  Cachazo, Srvcék, hep-th/0502160;
  Benincasa, Boucher-Veronneau, Cachazo, hep-th/0702032
  Arkani-Hamed, Kaplan, 0801.2385

- Use unitarity to study $n$-graviton one loop amplitudes, relating bubble and triangle coefficients to large $z$ behavior in 2- and 3-particle cuts
  Bern, Carrasco, Forde, Ita, Johansson, 0707.1035

- For $N=8$ SUGRA → proof of no-triangle hypothesis
  Bern, Dunbar et al. formulated & provided much evidence;
  Arkani-Hamed, Cachazo, Kaplan, 0808.1446

- Proved earlier for $n$-graviton states using superstring-based rules
  Bjerrum-Bohr, Vanhove, 0805.3682
No-triangle property

Recall the one-loop decomposition

\[ A^{1-\text{loop}} = C + \sum_i d_i + \sum_i c_i + \sum_i b_i + O(\epsilon) \]

States that \( c_i = b_i = R = 0 \)

just as in N=4 SYM!

For a large number of gravitons, samples large momentum behavior of many high-spin vertices (but only at one loop)

\[ \sum h_1 h_2 \ldots \sum \ell_{H1} \ell_{H2} \eta_{\nu1\rho1} \eta_{\nu2\rho2} + \ldots \]

Hidden cancellations in N=8 Supergravity

2. Multi-loop

• Use Kawai-Lewellen-Tye relations, in conjunction with generalized unitarity & N=4 SYM amplitudes, to compute N=8 SUGRA amplitudes
• Examine their UV behavior
• Results to date, through 3 loops, show that N=8 SUGRA is just as well behaved in UV as N=4 SYM

Bern, Carrasco, LD, Johansson, Kosower, Roiban, hep-th/0702112; BCDJR to appear
Kawai-Lewellen-Tye relations

Derive from relation between open & closed string amplitudes.

Low-energy limit gives N=8 supergravity amplitudes as quadratic combinations of N=4 SYM amplitudes consistent with product structure of Fock space, \( 256 = 16 \times 16 \)

\[ [\mathcal{N} = 8] = [\mathcal{N} = 4] \otimes [\mathcal{N} = 4] \]

\[
M_4^{\text{tree}}(1, 2, 3, 4) = -i s_{12} A_4^{\text{tree}}(1, 2, 3, 4) A_4^{\text{tree}}(1, 2, 4, 3) \\
M_5^{\text{tree}}(1, 2, 3, 4, 5) = i s_{12} s_{34} A_5^{\text{tree}}(1, 2, 3, 4, 5) A_5^{\text{tree}}(2, 1, 4, 3, 5) \\
+ (2 \leftrightarrow 3) \\
M_6^{\text{tree}}(1, 2, 3, 4, 5, 6) = \ldots
\]

Multi-loop “KLT copying”

Suppose we know an N=4 SYM amplitude at some loop order – both planar and nonplanar terms. Then we have “simple” forms for all of its generalized cuts, i.e. products of N=4 SYM trees, already summed over all internal states. The KLT relations let us write the N=8 SUGRA cuts, which are products of N=8 SUGRA trees, summed over all internal states, very simply in terms of sums of products of two copies of the N=4 SYM cuts.
Example of KLT copying at 3 loops

Using

\[ M_4^{\text{tree}}(1, 2, 3, 4) = -i \frac{s_t}{u} [A_4^{\text{tree}}(1, 2, 3, 4)]^2 \]
\[ M_5^{\text{tree}}(1, 2, 3, 4, 5) = -i s_{51} s_{23} A_5^{\text{tree}}(1, 2, 3, 4, 5) A_5^{\text{tree}}(1, 4, 2, 3, 5) + (1 \leftrightarrow 2) \]

it is easy to see that

\[ \text{N=8 SUGRA} \]
\[ \begin{array}{ccc}
2 & \bullet & 3 \\
1 & | & 4 \\
\end{array} \]
\[ = \]
\[ \text{N=4 SYM} \]
\[ \begin{array}{ccc}
2 & \bullet & 3 \\
1 & | & 4 \\
\end{array} \]
\[ \times \]
\[ \text{N=4 SYM} \]
\[ \begin{array}{ccc}
2 & \bullet & 3 \\
1 & | & 4 \\
\end{array} \]
\[ + \text{ permutations} \]
\[ (1 \leftrightarrow 2, 3 \leftrightarrow 4) \]

3 loop N=8 SUGRA amplitude

Nine basic integral topologies

Seven (a-g) were already known
(2-particle cuts \( \rightarrow \) rung rule)

BDDPR (1998)

Two new ones (h,i)
have no 2-particle cuts

BCDJKR (2007)
Numerator factors for the 3 loop integrals

\[ \text{Integral } I^{(e)} \quad \text{old numerator factors} \quad N^{(e)} \text{ for } \mathcal{N} = 8 \text{ Supergravity} \quad \text{BCDJR (2007)} \]

<table>
<thead>
<tr>
<th>Integral</th>
<th>( \text{old numerator factors} )</th>
<th>( N^{(e)} \text{ for } \mathcal{N} = 8 \text{ Supergravity} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)–(d)</td>
<td>([s_{12}^2]^2)</td>
<td>BCDJR to appear</td>
</tr>
<tr>
<td>(e)–(g)</td>
<td>([s_{12} s_{89}]^2)</td>
<td></td>
</tr>
<tr>
<td>(h)</td>
<td>((s_{12} s_{89} + s_{14} s_{11,14} - s_{12} s_{14})^2 - s_{12}^2(2(s_{89} - s_{14}) + l_M^2) - s_{14}^2(2(s_{11,14} - s_{12}) + l_M^2) - s_{12}^2(2l_M^2) + 2l_M^2 + l_M^2 + l_M^2) - s_{14}^2(2l_M^2 - 2l_M^2 + l_M^2 + l_M^2) + 2s_{12} s_{14} l_M^2))</td>
<td>quartic in ( l_M )</td>
</tr>
<tr>
<td>(i)</td>
<td>((s_{12} s_{15} - s_{14} s_{90})^2 - (s_{12}^2 s_{89} + s_{14}^2 s_{89} + \frac{1}{2} s_{12} s_{13} s_{14} l_M^2))</td>
<td></td>
</tr>
</tbody>
</table>

\[ s_i M = (k_i + l_M)^2 \]

Three loop N=8 SUGRA (cont.)

- Reshuffling makes manifest cancellations that lead 3-loop N=8 SUGRA to be just as well behaved in UV as N=4 SYM (also quadratic at 3 loops).
- Same feature encountered at 1 loop, for \( n \) legs: no-triangle property. B-BV; A-HCK (2008)
- Are there any further 3-loop cancellations? No.
- New form makes it easier to compute the log divergence at \( D=6 \), show that it is nonvanishing BCDJR to appear
### Beyond three loops

- Higher-loop cancellations inferred from 1-loop n-point (no triangle hypothesis)
- Inferred from 2- and 3-loop 4-point, via cuts
- Same UV behavior as $N=4$ super-Yang-Mills
- UV behavior unknown
- 4 loop calculation in progress: 52 cubic topologies

### Could N=8 SUGRA be finite to all orders?

- If it continues to be **just as well behaved** in UV as $N=4$ SYM then obviously the answer is yes.
- Also recent very interesting conjecture that if $N=8$ amplitudes follow from leading singularities, they must be finite to all orders. **A-HCK (2008); Cachazo talk**
- Explicit results through 3 loops are consistent with two more indirect arguments:
  - Superstring calculations in pure spinor formalism
    - Berkovits, hep-th/0609006; Green, Russo, Vanhove, hep-th/0611273
  - M theory dualities
    - Green, Russo, Vanhove, hep-th/0610299, 0807.0389; Green talk
- For the final answer, stay tuned (maybe for a while)!
Conclusions

- Many computations of gauge theory scattering amplitudes exploit the helicity formalism, complex momenta, generalized unitarity and factorization.
- **Multi-leg one-loop QCD amplitudes** needed for LHC applications are now falling to these techniques – first analytically, and now numerically (**CutTools, Rocket, BlackHat**)
- More phenomenologically important processes now under construction.
- Same techniques can be applied to **N=4 super-Yang-Mills theory** and **N=8 supergravity**, leading to multi-loop amplitudes unveiling hidden structures:
  - exact exponentiation (n=4,5); dual (super)conformal invariance; (MHV) equivalence to Wilson lines; more to come
  - cancellations indicating possible all-orders perturbative finiteness