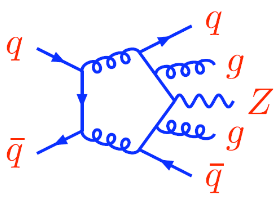
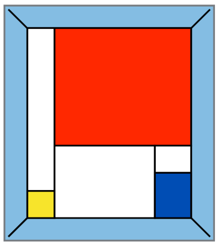
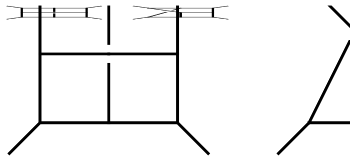
Unveiling the structure of amplitudes in gauge theory and gravity





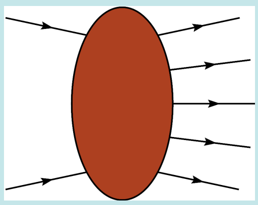


Lance Dixon, SLAC

STRINGS 2008, CERN

August 20

Outline



- Introduction
- Some tools
- Applications
  - QCD for LHC
  - N=4 super-Yang-Mills and AdS/CFT
  - N=8 supergravity – might it be finite?
- Conclusions

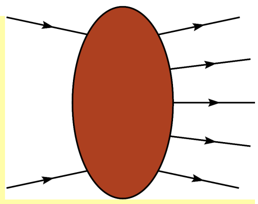
see talks by

Alday
Berkovits
Sokatchev

Green
Cachazo
Kallosh

Introduction

- **Scattering amplitudes** are basic to almost everything we learn directly from collider experiments.

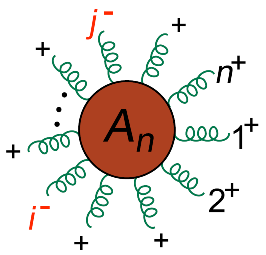


- They also have proved extremely useful over the ages as gedanken experiments – probes of properties of theories
- Their on-shell nature makes them (and all information extracted from them) physical:
  - independent of field redefinitions
  - independent of gauge choice

Unveiling structures

Sometimes hidden structures are found in amplitudes, which are only fully understood later

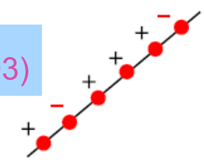
maximally helicity violating tree amplitudes



$$= \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle\cdots\langle n\, 1\rangle}$$

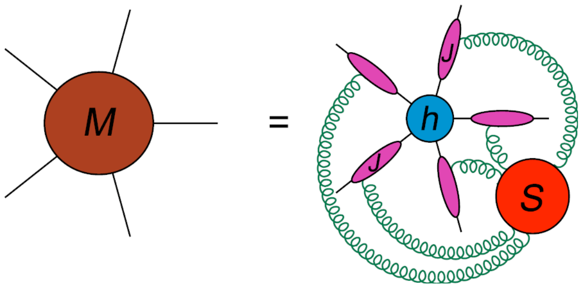
Parke-Taylor; Mangano, Parke, Xu; Berends, Giele (1986-1987)

$\Rightarrow$ twistor string theory Witten (2003)



A technical issue…

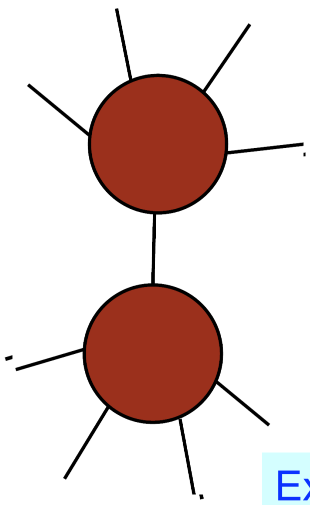
Beyond tree-level, scattering amplitudes afflicted with infrared (soft & collinear) divergences



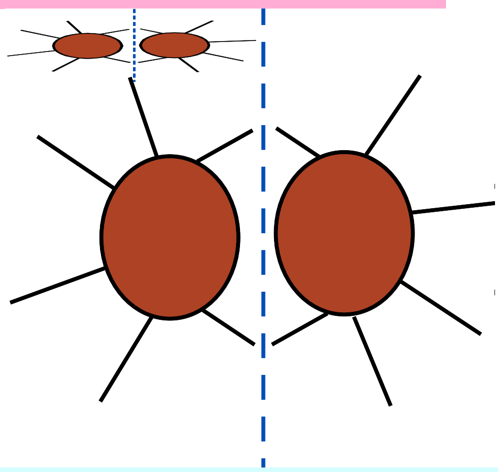
Fortunately we know how to deal with them in gauge theories: Use dimensional regulation, $D = 4-2\varepsilon$.
Exploit years of experience in QED and QCD:
- soft/collinear factorization & exponentiation
- virtual/real cancellation in infrared-safe cross sections

Amplitudes are "plastic"

Factorization: How they "fall apart" into simpler ones in special limits



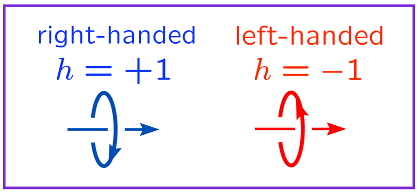
Unitarity: Discontinuities across branch cuts

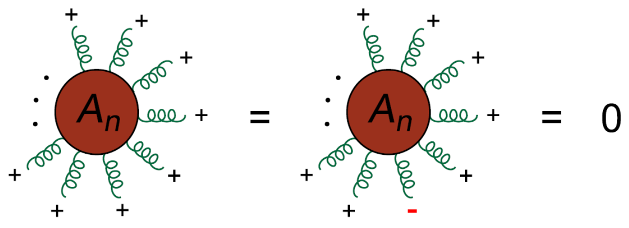


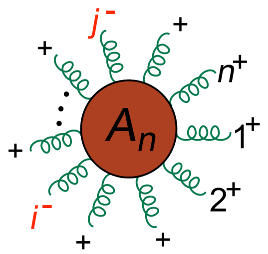
Exploit these properties to determine them

Simplest in helicity formalism

Many helicity amplitudes either vanish or are very short







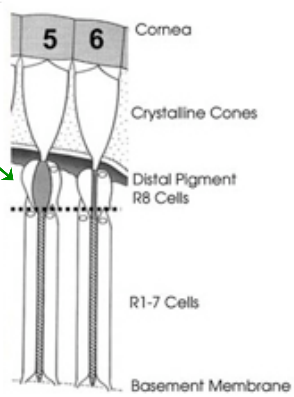
$$= \frac{\langle i\, j \rangle^4}{\langle 1\,2\rangle \langle 2\,3\rangle \cdots \langle n\,1\rangle}$$

The tail of the mantis shrimp

• Reflects left and right circularly polarized light differently

• Led biologists to discover that its eyes have differential sensitivity
• It communicates via the helicity formalism

$\lambda/4$ plate



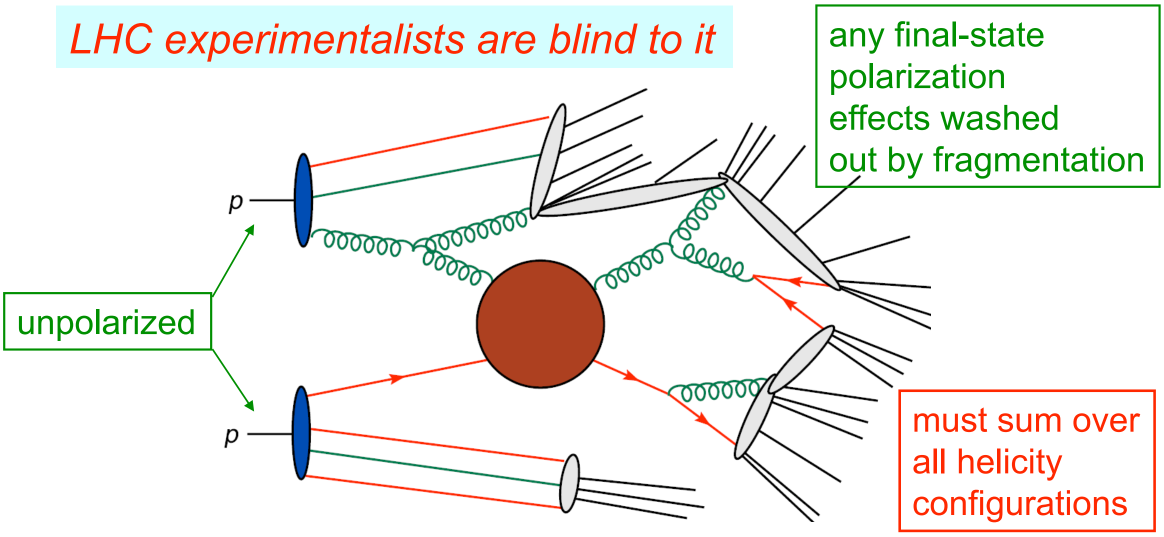
"It's the most private communication system imaginable. No other animal can see it."

- Roy Caldwell (U.C. Berkeley)

What the biologists didn't know

Particle theorists have also evolved capability to communicate results via helicity formalism

LHC experimentalists are blind to it



Spinor helicity formalism

Use Weyl spinors $u_\alpha(k_i)$ (spin ½)

right-handed: $(\lambda_i)_\alpha = u_+(k_i)$

$h = +1/2$

left-handed: $(\tilde{\lambda}_i)_{\dot\alpha} = u_-(k_i)$

$h = -1/2$

Instead of Lorentz products: $s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$

Use spinor products:

$$\varepsilon^{\alpha\beta}(\lambda_i)_\alpha(\lambda_j)_\beta = \langle i\, j\rangle$$

$$\varepsilon^{\dot\alpha\dot\beta}(\tilde\lambda_i)_{\dot\alpha}(\tilde\lambda_j)_{\dot\beta} = [i\, j]$$

Always obey

$$\langle i\, j\rangle\,[j\, i] = s_{ij}$$

If momenta are real, they are complex square roots of Lorentz products:

$$\langle i\, j\rangle = \sqrt{s_{ij}}\, e^{i\phi_{ij}}$$

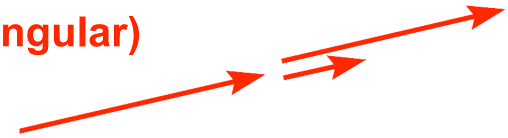
$$[j\, i] = \sqrt{s_{ij}}\, e^{-i\phi_{ij}}$$

Virtues of complex momenta

• Makes sense of basic process with 3 lightlike (massless) particles

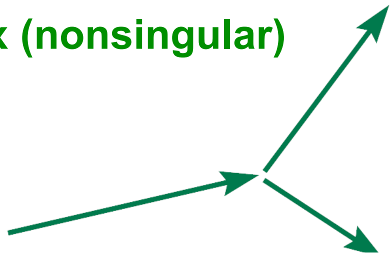
$$s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2 = 0 \quad \forall\ i,j$$

real (singular)



$$\langle i\,j \rangle = [i\,j] = s_{ij} = 0 \qquad \forall\ i,j$$

complex (nonsingular)



$$[i\,j] = 0 \quad \text{but} \quad \langle i\,j \rangle \neq 0$$

3-gluon amplitude

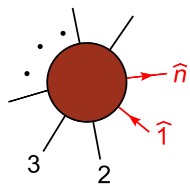
$$\frac{\langle i\,j \rangle^4}{\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,1 \rangle}$$

makes sense

Exploit analyticity at tree level

Britto, Cachazo, Feng, Witten, hep-th/0501052

Inject complex momentum at leg 1, remove it at leg *n*.



$$\hat{\tilde\lambda}_1 = \lambda_1 + z\lambda_n \qquad \hat{\tilde\lambda}_1 = \tilde\lambda_1$$

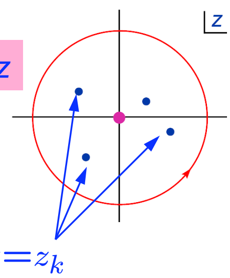
$$\hat\lambda_n = \lambda_n \qquad \hat{\tilde\lambda}_n = \tilde\lambda_n - z\tilde\lambda_1$$

$$k_1(z) + k_n(z) = k_1 + k_n$$

$$k_1^2(z) = k_n^2(z) = 0$$

$$\Rightarrow \quad A(0) \to A(z)$$

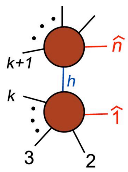
factorization limits $\Leftrightarrow$ poles in z



Cauchy: If $A(\infty) = 0$ then

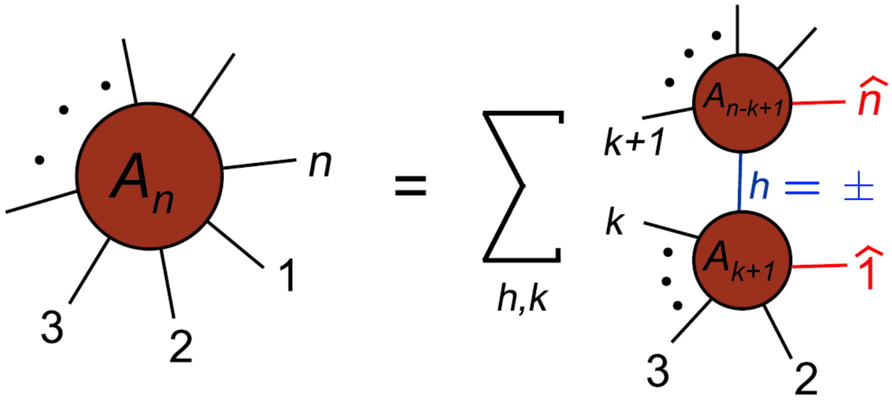
$$0 = \frac{1}{2\pi i} \oint dz \frac{A(z)}{z} = A(0) + \sum_k \mathrm{Res}\left[\frac{A(z)}{z}\right]\Big|_{z=z_k}$$

residue at z_k = [k^{th} factorization limit] =



BCFW (on-shell) recursion relations

Britto, Cachazo, Feng, hep-th/0412308



A_{k+1} and A_{n-k+1} are **on-shell** tree amplitudes with **fewer** legs, and with momenta shifted by a fixed **complex** amount

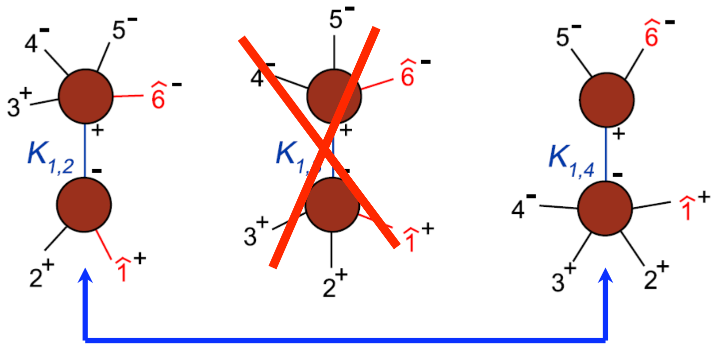
Trees recycled into trees

A 6-gluon example

220 Feynman diagrams for *gggggg*

Consider $A_6(1^+,2^+,3^+,4^-,5^-,6^-)$
(the other nontrivial helicity assignment is similar)

2 nonvanishing recursive diagrams, related by a symmetry



Yields simplest final form

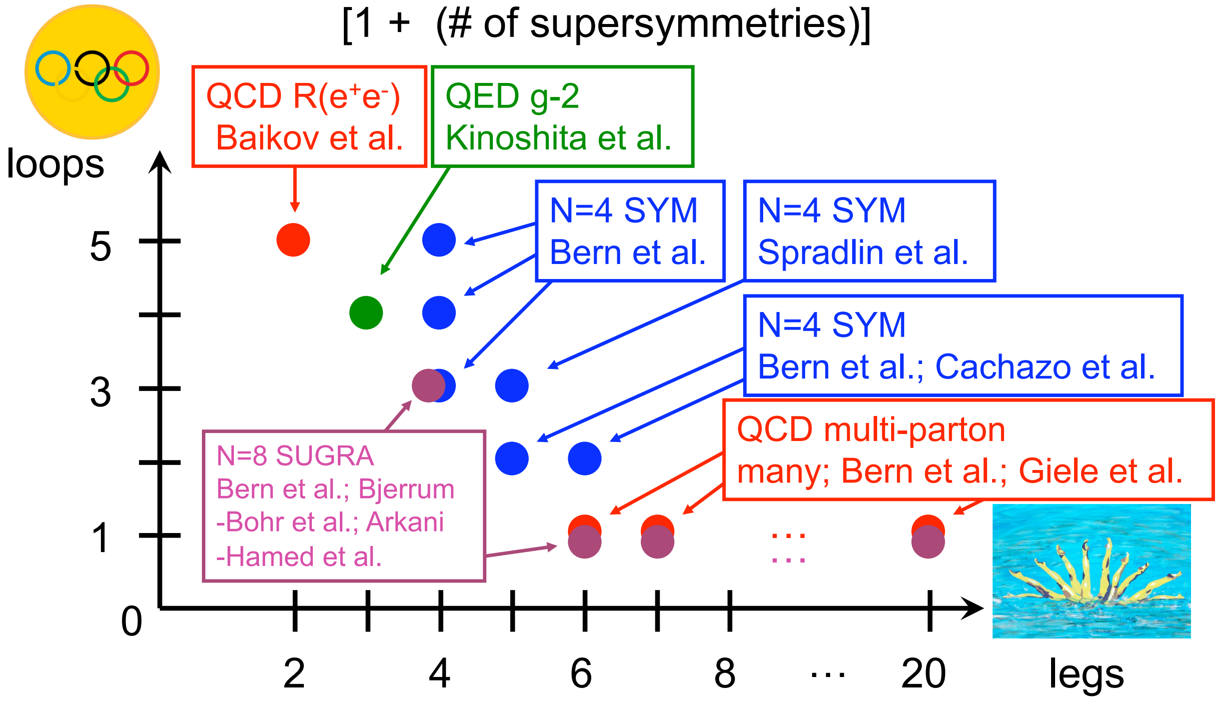
$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\langle 1\,2\rangle[3\,4][4\,5]\,s_{612}\langle 2^-|(6+1)|5^-\rangle} + \frac{\langle 4^-|(5+6)|1^-\rangle^3}{\langle 2\,3\rangle\langle 3\,4\rangle[5\,6][6\,1]\,s_{561}\langle 2^-|(6+1)|5^-\rangle}$$

$$\langle a^-|(b+c)|d^-\rangle \equiv \langle a\,b\rangle[b\,d] + \langle a\,c\rangle[c\,d]$$

- All symmetries and physical pole behavior manifest
- Only one “spurious singularity”

Degrees of difficulty

$$\text{Difficulty} = \frac{[\#\text{ of loops}] \times [\#\text{ of legs}]}{[1 + (\#\text{ of supersymmetries})]} \times [\text{style factor}]$$



Why multi-parton loop amplitudes?

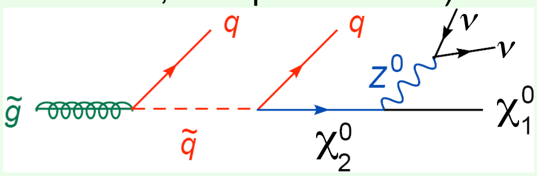
Because LHC is a QCD machine

- Backgrounds to new physics require detailed understanding of scattering amplitudes for many ~ massless particles – especially quarks and gluons of QCD.

- Depending on how big the signal is, leading-order QCD (tree amplitudes) may not be precise enough
  → need next-to-leading order (NLO) QCD corrections, which include loop amplitudes, as well as real radiation.

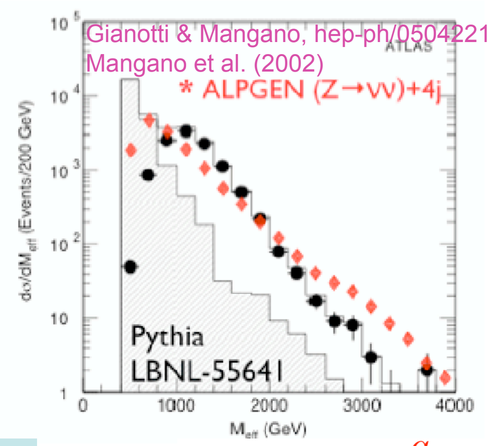
Example: search for supersymmetry

• **Cascade from gluino to neutralino** (dark matter, escapes detector)

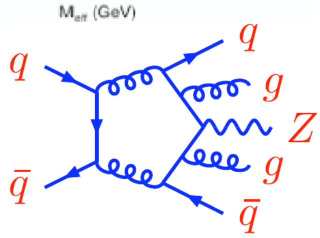


• **Signal: missing energy + 4 jets**

• SM background from Z + 4 jets,
Z → neutrinos



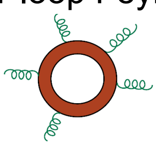
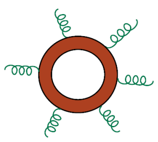
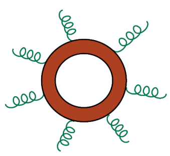
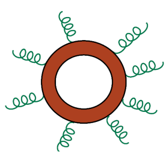
• ALPGEN based on **LO tree** amplitudes, normalization still quite uncertain



$\Rightarrow pp \rightarrow Z + 4$ jets at NLO

2 legs beyond state-of-art

Strong growth in complexity with number of external legs

# of legs	# of 1-loop Feynman diagrams (gluons only)
5	 810
6	 10,860
7	 168,925
8	 3,017,490

Motivation for exploiting analyticity of amplitudes at one loop

One-loop amplitude decomposition

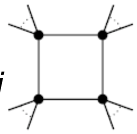
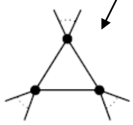
For external momenta in D=4, loop momenta in D=4-2ε:

cut part

rational part

$$A^{1-\text{loop}} = C + R + \mathcal{O}(\epsilon)$$

known **scalar** one-loop integrals, same for all amplitudes

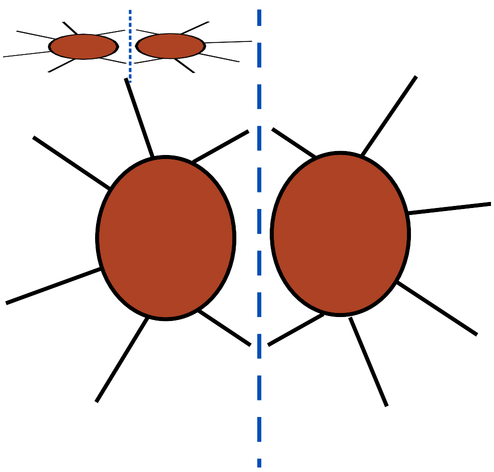
$C = \sum_i d_i$  $+ \sum_i c_i$  $+ \sum_i b_i$ 

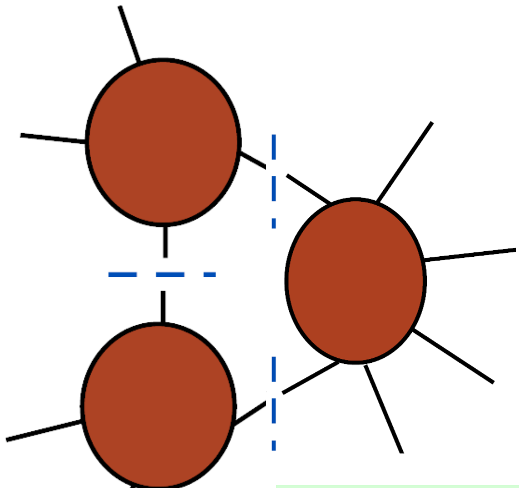
coefficients of boxes, triangle and bubble integrals
– all rational functions – get using generalized unitarity

Generalized unitarity at one loop

Ordinary unitarity:
put 2 particles on shell

Generalized unitarity:
put 3 or 4 particles on shell



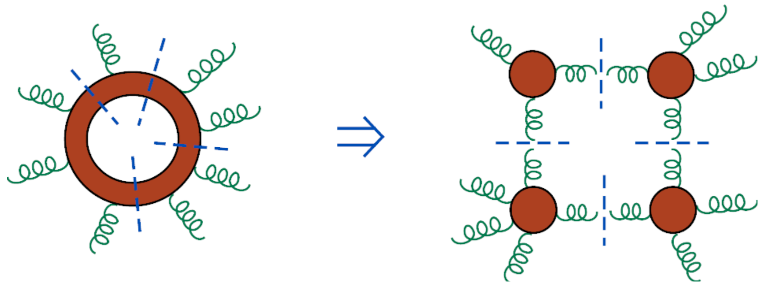


trees get simpler

cut conditions require complex loop momenta

Gen. unitarity for box coefficients d_i

Britto, Cachazo, Feng, hep-th/0412308



$$\int d^4\ell \; \delta(\ell_1^2 - m_1^2)\, \delta(\ell_2^2 - m_2^2) \\ \times \delta(\ell_3^2 - m_3^2)\, \delta(\ell_4^2 - m_4^2) \; \times A^{1-\text{loop}}(\ell_i) \\ = A_1^{\text{tree}}(\ell_0) A_2^{\text{tree}}(\ell_0) A_3^{\text{tree}}(\ell_0) A_4^{\text{tree}}(\ell_0) \\ = d_i$$

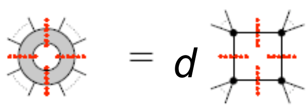
[# of dimensions] = 4 = [# of constraints] → discrete solutions

• unique box selected

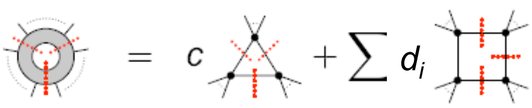
• no $l_5^2 = 0$ solution → no pentagons

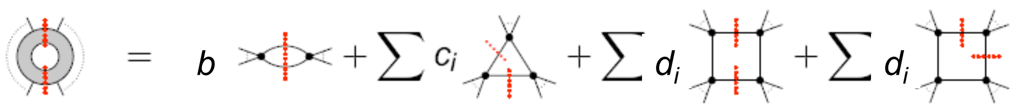
Generalized unitarity (cont.)

With a 4-ple cut we select one coefficient



Triangle and bubble coefficients are more complicated since a double or triple cut does not isolate a single coefficient.





Also, solutions to cut constraints are now continuous, so there are multiple ways to solve and eliminate d_i, etc.

Britto et al. (2005,2006); Mastrolia (2006); Ossola, Papadopoulos, Pittau (2007); Forde, 0704.1835; Ellis, Giele, Kunszt, 0708.2398; …

Rational function *R*

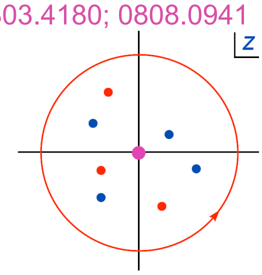
No cuts in D=4 – can't get from D=4 unitarity
However, can get using D=4-2ε unitarity:

$$\int d^{4-2\epsilon}\ell \quad \Rightarrow \quad R(s_{ij}) \;\rightarrow\; R(s_{ij})\,(-s_{12})^{-\epsilon} \;=\; R(s_{ij})\,[1-\epsilon\ln(-s_{ij})]$$

Bern, Morgan (1996); Bern, LD, Kosower (1996);
Brandhuber, McNamara, Spence, Travaglini hep-th/0506068;
Anastasiou et al., hep-th/0609191, hep-th/0612277;
Britto, Feng, hep-ph/0612089, 0711.4284;
Giele, Kunszt, Melnikov, 0801.2237;
Britto, Feng, Mastrolia, 0803.1989; Britto, Feng, Yang, 0803.3147;
Ossola, Papadopolous, Pittau, 0802.1876;
Mastrolia, Ossola, Papadopolous, Pittau, 0803.3964;
Giele, Kunszt, Melnikov (2008); Giele, Zanderighi, 0805.2152;
Ellis, Giele, Kunszt, Melnikov, 0806.3467;
Feng, Yang, 0806.4106; Badger, 0806.4600

Or: *R* from 1-loop recursion

Bern, LD, Kosower, hep-th/0501240, hep-ph/0505055, hep-ph/0507005;
Berger, et al., hep-ph/0604195, hep-ph/0607014, 0803.4180; 0808.0941



Use same complex momentum shift as at tree level

$A^{1-\mathrm{loop}}(z)$ has cuts as well as poles in z-plane:

$R(z)$ is rational; has only poles
– but some of them are spurious,
cancelling against $C(z)$ in full amplitude

Physical poles generate recursive diagrams a la BCFW

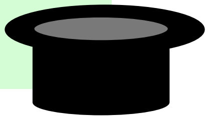
Trees + loops recycled into loops!

Ideas now being implemented numerically and automatically

CutTools: Ossola, Papadopolous, Pittau, 0711.3596
NLO production of *WWW* Binoth+OPP, 0804.0350

Rocket: Giele, Zanderighi, 0805.2152
One-loop n-gluon amplitudes for n up to 20

Blackhat: Berger, Bern, LD, Febres Cordero, Forde, H. Ita, D. Kosower, D. Maître, 0803.4180, 0808.0941
One-loop n-gluon amplitudes for n up to 7;
some amplitudes needed for NLO production of Z + 3 jets



Multi-leg and multi-loop

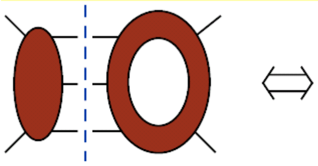
- To really push this frontier we need the aid of maximal supersymmetry:
  - N=4 for gauge theory
  - N=8 for supergravity
- Of significant interest due to
  - AdS/CFT correspondence
  - finiteness questions
- Progress in N=4 leads to progress in N=8 thanks to KLT relations between tree amplitudes Kawai, Lewellen, Tye (1986)

Multi-loop generalized unitarity

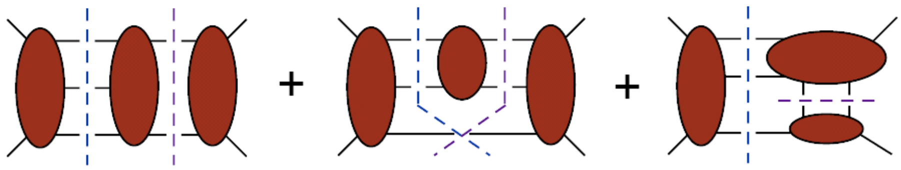
Bern, LD, Kosower (2000); Bern, Czakon, LD, Kosower, Smirnov (2006);
Bern, Carrasco, LD, Johansson, Kosower, Roiban (2007); BCJK (2007);
Cachazo, Skinner, 0801.4574; Cachazo, 0803.1988; Cachazo, Spradlin, Volovich, 0805.4832

Ordinary cuts of multi-loop amplitudes contain loop amplitudes.
But it is very convenient to work with tree amplitudes only.

For example, at 3 loops, one encounters the product of a 5-point tree and a 5-point one-loop amplitude:

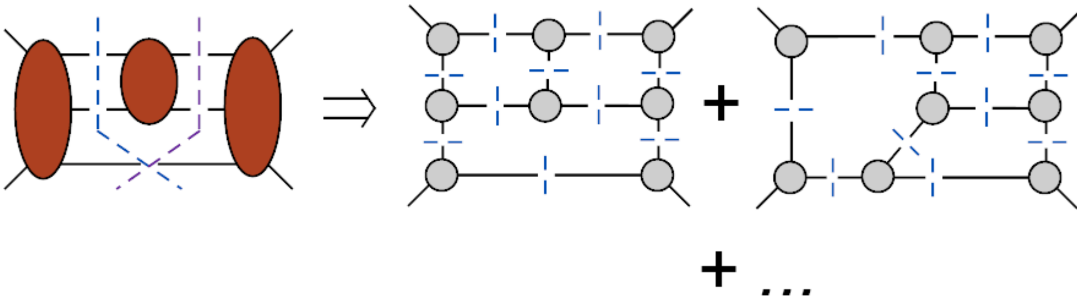


Cut 5-point loop amplitude further, into (4-point tree) x (5-point tree), in all 3 inequivalent ways:



But one can do better

Allowing for complex momenta everywhere, one can chop an amplitude entirely into 3-point trees
→ **maximal cuts or ~ leading singularities**



Advantage is that these cuts are maximally simple, yet give an excellent starting point for constructing the full answer.

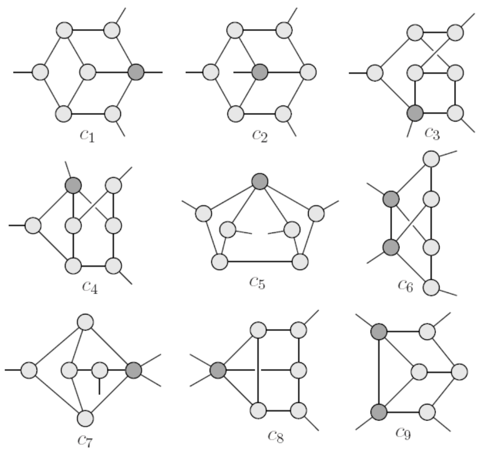
For example, in planar (leading in N_c) N=4 SYM they find all terms in the complete answer for 1, 2 and 3 loops

Finding missing terms

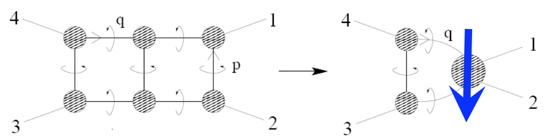
Maximal cut method:
Allowing one or two propagators to collapse from each **maximal cut**, one obtains **near-maximal cuts**

These near-maximal cuts are very useful for analyzing N=4 SYM (including nonplanar) and N=8 SUGRA at 3 loops

BCDJKR, BCJK (2007); BCDJR, to appear



Leading singularity method:
Uses consistent behavior with respect to "hidden singularities"



Cachazo, Skinner; Cachazo (2008)

Recent supersum advances for more complicated cuts

e.g. Elvang, Freedman, Kiermaier, 0808.1720

Planar N=4 SYM and AdS/CFT

recent review: Alday, Roiban, 0807.1889

- In the 't Hooft limit, $N_c \to \infty$
  $\lambda = g^2 N_c$ fixed, planar diagrams dominate
- AdS/CFT duality

  suggests that weak-coupling perturbation series in λ for large-N_c (planar) N=4 SYM should have hidden structure, because

  large λ limit ←→ weakly-coupled gravity/string theory on $AdS_5 \times S^5$

  Maldacena; Gubser, Klebanov, Polyakov; Witten

Three Hidden Structures Recently Unveiled in Planar N=4 SYM

- Exponentiation of finite terms in the amplitude (for 4 and 5 gluons)
- Dual (super)conformal invariance
- Equivalence between (MHV) amplitudes and Wilson lines

Exponentiation

Bern, LD, Smirnov (2005)

Inspired by IR structure of QCD, Mueller, Collins, Sen, Magnea, Sterman,…
based on evidence collected at 2 and 3 loops for *n=4,5* using **generalized unitarity and factorization**, **ansatz proposed**:

$$\frac{\mathcal{A}_n}{\mathcal{A}_n^{\text{tree}}} \equiv \mathcal{M}_n = \exp\Big[\sum_{l=1}^{\infty}\big[\frac{\lambda}{8\pi^2}\big]^l\Big(f^{(l)}(\epsilon)M_n^{(1)}(l\epsilon) + C^{(l)} + \mathcal{O}(\epsilon)\Big)\Big]$$

constants, indep.of kinematics

all kinematic dependence in known 1-loop amplitude (normalized by tree)

n=4 $\Rightarrow$ $\mathcal{M}_4|_{\text{finite}} = \exp\Big[\frac{1}{8}\gamma_K(\lambda)\ \ln^2\Big(\frac{s}{t}\Big) + \text{const.}\Big]$

Alday
Maldacena
0705.0303
0710.1060;
Alday talk

Confirmed at strong coupling using AdS/CFT, directly at *n=4*, indirectly at *n=5*. (But: fails for *n > 5*.)

The constants

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon\, f_1^{(l)} + \epsilon^2\, f_2^{(l)}$$

collects **3 series of constants**:

$$\underbrace{f_0^{(l)} = \frac{1}{4}\hat{\gamma}_K^{(l)} \qquad f_1^{(l)} = \frac{l}{2}\hat{\mathcal{G}}_0^{(l)}}_{\text{control IR poles}} \qquad f_2^{(l)} = (???) \qquad C^{(l)} = (???)$$

$\hat{\gamma}_K^{(l)}$, $\hat{\mathcal{G}}_0^{(l)}$ are *l*-loop coefficients of

- cusp anomalous dimension $\gamma_K(\lambda)$ (source term for differential equation for Sudakov form factor)
- "collinear" anomalous dimension $\mathcal{G}_0(\lambda) = G(-1, \lambda, \epsilon = 0)$ (integration constant for differential equation)

Beisert, Eden, Staudacher, hep-th/0610251

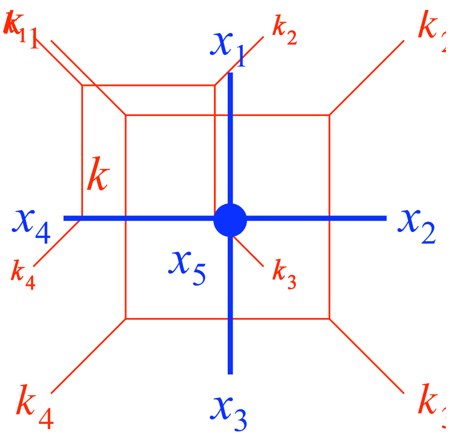
Dual Conformal Invariance

Berkovits & Sokatchev talks

A conformal symmetry acting in momentum space, on dual or sector variables x_i

First seen in N=4 SYM planar amplitudes in the loop integrals

Drummond, Henn, Smirnov, Sokatchev, hep-th/0607160



$$I = \int d^4k \frac{(k_1+k_2)^2(k_2+k_3)^2}{k^2(k-k_1)^2(k-k_1-k_2)^2(k+k_4)^2}$$

$$k_1 = x_{41}, \quad k_2 = x_{12}, \quad k_3 = x_{23}, \quad k_4 = x_{34}, \quad k = x_{45}$$

$$I = \int d^4x_5 \frac{x_{24}^2 x_{13}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

invariant under inversion:

$$x_i^\mu \to \frac{x_i^\mu}{x_i^2}$$

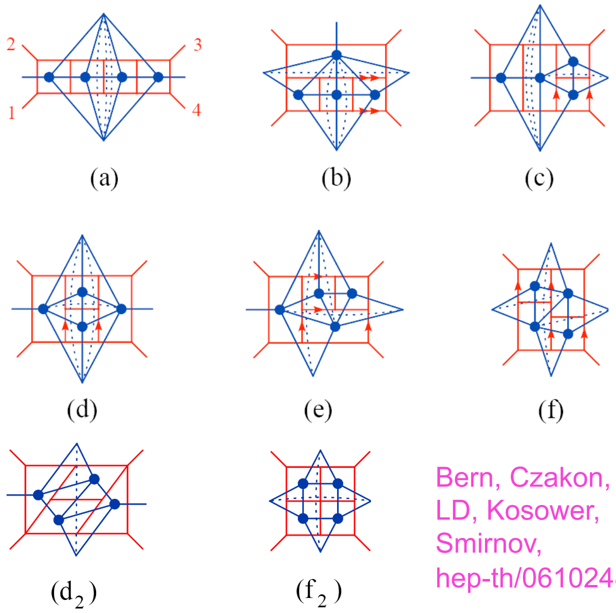
Dual conformal invariance (cont.)

• Simple graphical rules:
4 (net) lines into inner x_i
1 (net) line into outer x_i
• Dotted lines are for numerator factors

4 loop planar integrals all of this form

also true at 5 loops

Bern, Carrasco, Johansson, Kosower, 0705.1864



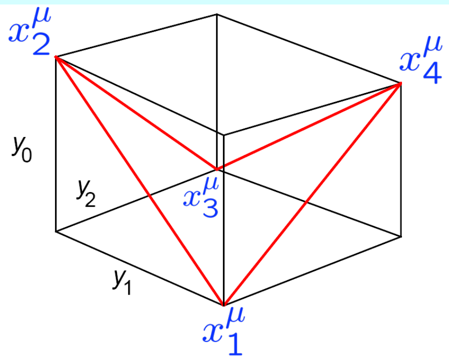
Bern, Czakon, LD, Kosower, Smirnov, hep-th/0610248

Insight from string theory

• As a property of full amplitudes, rather than integrals, dual conformal invariance follows, at strong coupling, from bosonic T duality symmetry of $AdS_5 \times S^5$.
• Also, strong-coupling calculation ~ equivalent to computation of Wilson line for n-sided polygon with vertices at x_i

Alday, Maldacena; Alday talk

Wilson line blind to helicity formalism – doesn't know MHV from non-MHV



Dual (super)conformal invariance

• Surprisingly, dual conformal invariance and Wilson line equivalence both persist to weak coupling for MHV amp's

Drummond, Korchemsky, Sokatchev, 0707.0243;
Brandhuber, Heslop, Travaglini, 0707.1153 ; } 1 loop
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223 2 loops

• Can embed dual conformal invariance into a richer dual superconformal invariance (needed to understand structure of non-MHV amplitudes)

DHKS, 0807.1095, 0808.0491; Sokatchev talk

Whole structure now explained better (at weak coupling too) by a combined bosonic and fermionic T duality symmetry leaving dilaton fixed

Berkovits, Maldacena, 0807.3196; Berkovits talk

More than 4 gluons

• Ansatz known to work for $n = 5$ (all MHV) two loops

Cachazo, Spradlin, Volovich, hep-th/0602228;
Bern, Czakon, Kosower, Roiban, Smirnov, hep-th/0604074

• Should work for $n = 5$ to all loops, assuming dual conformal invariance.

• $n = 6$ is first place it does not fix form of amplitude, due to cross ratios such as $u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}$

• There were indications of a **failure looming** for $n = 6$, based on:

• A large n, strong-coupling limit Alday, Maldacena, 0710.1060

• A Wilson line calculation

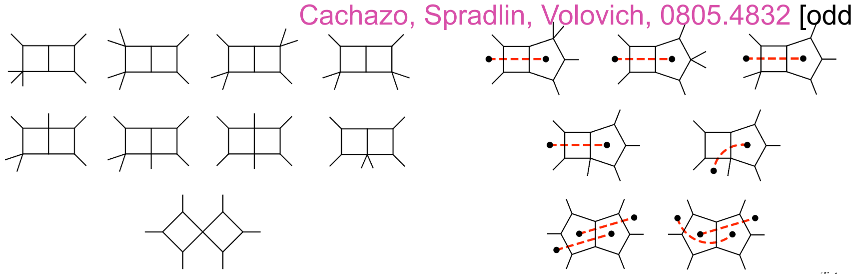
Drummond, Henn, Korchemsky, Sokatchev, 0712.4138

• A high-energy/Regge limit

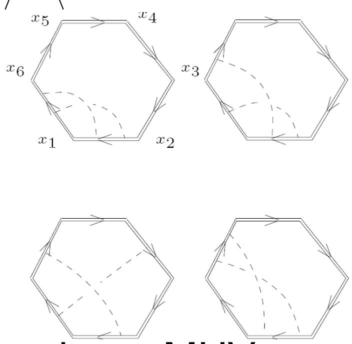
Bartels, Lipatov, Sabio Vera, 0802.2065

Tested ansatz at 2 loops, 6 gluons, MHV

Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465 [even terms]
Cachazo, Spradlin, Volovich, 0805.4832 [odd terms]



Not organic chemistry but dual conformal integrals



Ansatz definitely **breaks down** at n=6.
Yet the following properties **persist:**
1) dual conformal invariance
2) equivalence to Wilson loops →

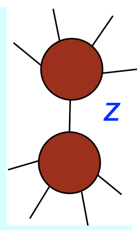
Drummond, Henn, Korchemsky, Sokatchev, 0712.4138; 0803.1466

More structure still to be unveiled, for MHV and non-MHV

Hidden cancellations in N=8 Supergravity

1. Multi-leg

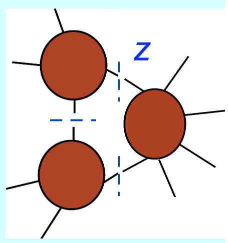
• Can construct recursive representations of tree amplitudes, using analyses of large *z* behavior



Bedford, Brandhuber, Spence, Travaligni, hep-th/0502146;
Cachazo, Svrček, hep-th/0502160;
Benincasa, Boucher-Veronneau, Cachazo, hep-th/0702032
Arkani-Hamed, Kaplan, 0801.2385

• Use unitarity to study n-graviton one loop amplitudes, relating bubble and triangle coefficients to large *z* behavior in 2- and 3-particle cuts

Bern, Carrasco, Forde, Ita, Johansson, 0707.1035



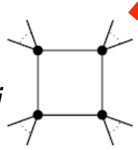
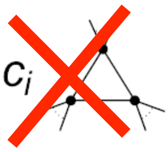
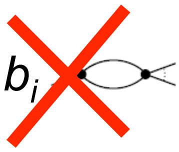
• For N=8 SUGRA → **proof of no-triangle hypothesis**
Bern, Dunbar et al. formulated & provided much evidence;
Arkani-Hamed, Cachazo, Kaplan, 0808.1446

• Proved earlier for n-graviton states using superstring-based rules
Bjerrum-Bohr, Vanhove, 0805.3682

No-triangle property

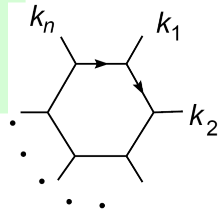
Recall the one-loop decomposition

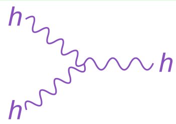
$$A^{1-\mathrm{loop}} = C + \cancel{R} + \mathcal{O}(\epsilon)$$

$$C = \sum_i d_i$$  $$+ \sum_i c_i$$  $$+ \sum_i b_i$$ 

States that $c_i = b_i = R = 0$ just as in N=4 SYM!

For a large number of gravitons, samples large momentum behavior of many high-spin vertices (but only at one loop)





$$\supset \ \ell^{\mu_1}\,\ell^{\mu_2}\,\eta^{\nu_1\rho_1}\,\eta^{\nu_2\rho_2} + \cdots$$

Hidden cancellations in N=8 Supergravity

2. Multi-loop

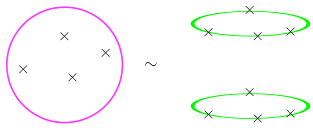
- Use Kawai-Lewellen-Tye relations, in conjunction with generalized unitarity & N=4 SYM amplitudes, to compute N=8 SUGRA amplitudes
- Examine their UV behavior
- Results to date, through 3 loops, show that N=8 SUGRA is just as well behaved in UV as N=4 SYM

Bern, Carrasco, LD, Johansson, Kosower, Roiban, hep-th/0702112;
BCDJR to appear

Kawai-Lewellen-Tye relations

KLT, 1986

Derive from relation between
open & closed string amplitudes.



Low-energy limit gives N=8 supergravity amplitudes M_n^{tree} as
quadratic combinations of N=4 SYM amplitudes A_n^{tree}

consistent with product structure of Fock space, 256 = 16 x 16

$$[\mathcal{N}=8] \;=\; [\mathcal{N}=4] \otimes [\mathcal{N}=4]$$

$$M_4^{\text{tree}}(1,2,3,4) \;=\; -i\, s_{12}\, A_4^{\text{tree}}(1,2,3,4)\, A_4^{\text{tree}}(1,2,4,3)$$

$$M_5^{\text{tree}}(1,2,3,4,5) \;=\; i\, s_{12} s_{34}\, A_5^{\text{tree}}(1,2,3,4,5)\, A_5^{\text{tree}}(2,1,4,3,5) + (2 \leftrightarrow 3)$$

$$M_6^{\text{tree}}(1,2,3,4,5,6) \;=\; \cdots$$

Multi-loop "KLT copying"

Bern, LD, Dunbar, Perelstein, Rozowsky (1998)

• Suppose we know an N=4 SYM amplitude at some loop order – both planar and nonplanar terms.
• Then we have "simple" forms for all of its generalized cuts, i.e. products of N=4 SYM trees, already summed over all internal states
• The KLT relations let us write the N=8 SUGRA cuts, which are products of N=8 SUGRA trees, summed over all internal states, very simply in terms of sums of products of two copies of the N=4 SYM cuts.

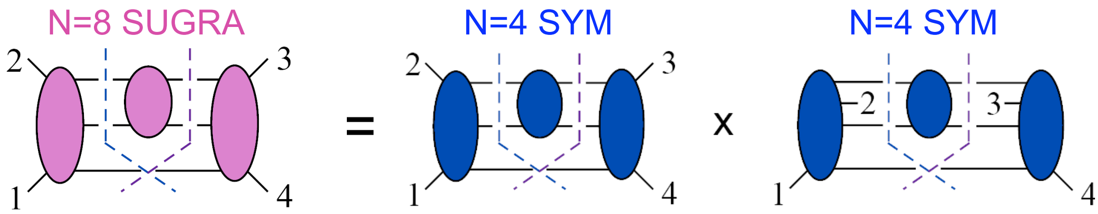
Example of KLT copying at 3 loops

Using

$$M_4^{\text{tree}}(1,2,3,4) = -i\frac{st}{u}[A_4^{\text{tree}}(1,2,3,4)]^2$$

$$M_5^{\text{tree}}(1,2,3,4,5) = -i\, s_{51}s_{23}\, A_5^{\text{tree}}(1,2,3,4,5)\, A_5^{\text{tree}}(1,4,2,3,5) + (1 \leftrightarrow 2)$$

it is easy to see that



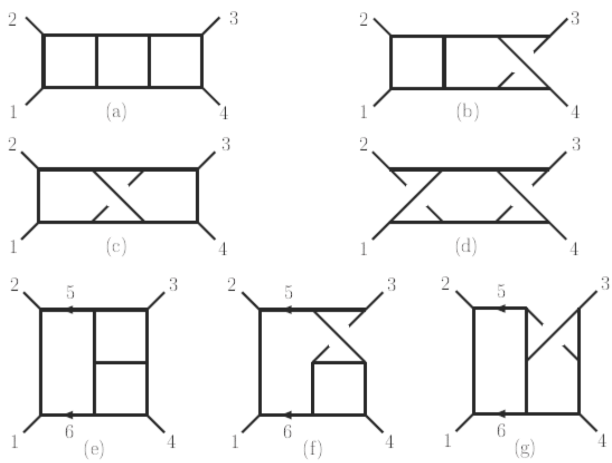
$+$ permutations $(1 \leftrightarrow 2,\ 3 \leftrightarrow 4)$

3 loop N=8 SUGRA amplitude

Nine basic integral topologies

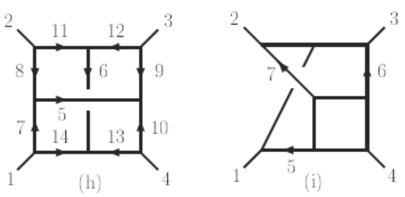
Seven (a-g) were already known (2-particle cuts → rung rule)

BDDPR (1998)



Two new ones (h,i) have no 2-particle cuts

BCDJKR (2007)



Numerator factors for the 3 loop integrals

Integral $I^{(x)}$	old numerator factors $N^{(x)}$ for $\mathcal{N}=8$ Supergravity BCDJKR (2007)
(a)–(d)	$[s_{12}^2]^2$
(e)–(g)	$[s_{12}\, s_{46}]^2$
(h)	$(s_{12}s_{89} + s_{14}s_{11,14} - s_{12}s_{14})^2 - s_{12}^2(2(s_{89} - s_{14}) + l_6^2)l_6^2 - s_{14}^2(2(s_{11,14} - s_{12}) + l_5^2)l_5^2$ $- s_{12}^2(2l_8^2l_{10}^2 + 2l_7^2l_9^2 + l_8^2l_7^2 + l_9^2l_{10}^2) - s_{14}^2(2l_{11}^2l_{13}^2 + 2l_{12}^2l_{14}^2 + l_{11}^2l_{12}^2 + l_{13}^2l_{14}^2) + 2s_{12}s_{14}l_5^2l_6^2$
(i)	$(s_{12}s_{45} - s_{14}s_{46})^2 - (s_{12}^2s_{45} + s_{14}^2s_{46} + \frac{1}{3}s_{12}s_{13}s_{14})l_7^2$

quartic in ℓ_M

$$s_{iM} = (k_i + \ell_M)^2$$

Integral $I^{(x)}$	new $N^{(x)}$ for $\mathcal{N}=8$ Supergravity BCDJR to appear
(a)–(d)	$[s_{12}^2]^2$
(e)–(g)	$s_{12}^2\, \tau_{35}\, \tau_{46}$
(h)	$(s_{12}(\tau_{26} + \tau_{36}) + s_{14}(\tau_{15} + \tau_{25}) + s_{12}s_{14})^2$ $+ (s_{12}^2(\tau_{26} + \tau_{36}) - s_{14}^2(\tau_{15} + \tau_{25}))(\tau_{17} + \tau_{28} + \tau_{39} + \tau_{4,10})$ $+ s_{12}^2(\tau_{17}\,\tau_{28} + \tau_{39}\,\tau_{4,10}) + s_{14}^2(\tau_{28}\,\tau_{39} + \tau_{17}\,\tau_{4,10}) + s_{13}^2(\tau_{17}\,\tau_{39} + \tau_{28}\,\tau_{4,10})$
(i)	$(s_{12}\,\tau_{45} - s_{14}\,\tau_{46})^2 - \tau_{27}(s_{12}^2\,\tau_{45} + s_{14}^2\,\tau_{46}) - \tau_{15}(s_{12}^2\,\tau_{47} + s_{13}^2\,\tau_{46})$ $- \tau_{36}(s_{14}^2\,\tau_{47} + s_{13}^2\,\tau_{45}) + l_5^2\, s_{12}^2\, s_{14} + l_6^2\, s_{12}\, s_{14}^2 - \frac{1}{3}l_7^2\, s_{12}\, s_{13}\, s_{14}$

(e-i) reshuffled so all terms are **quadratic** in loop momenta

$$\tau_{iM} = 2k_i \cdot \ell_M$$

Three loop N=8 SUGRA (cont.)

• Reshuffling makes manifest cancellations that lead 3-loop N=8 SUGRA to be just as well behaved in UV as N=4 SYM (also quadratic at 3 loops).

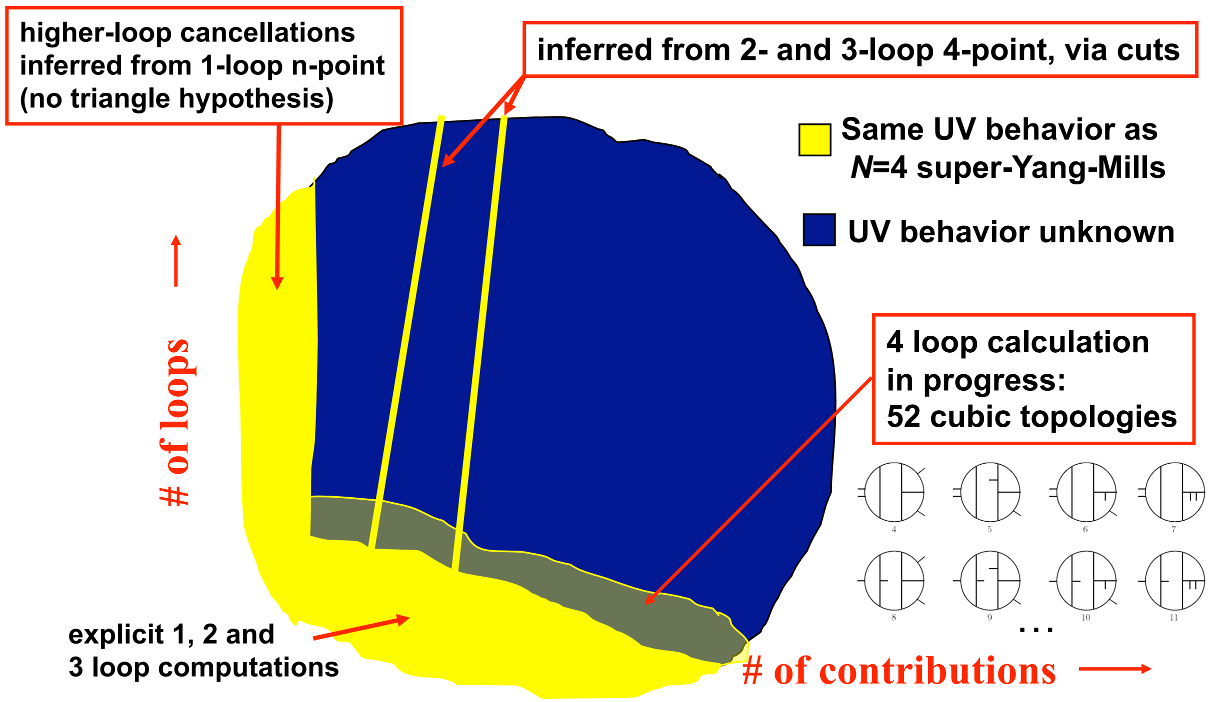
• Same feature encountered at 1 loop, for n legs: no-triangle property. B-BV; A-HCK (2008)

• Are there any further 3-loop cancellations? No.

• New form makes it easier to compute the log divergence at D=6, show that it is nonvanishing

BCDJR to appear

Beyond three loops



Could N=8 SUGRA be finite to all orders?

- If it continues to be just as well behaved in UV as N=4 SYM then obviously the answer is yes.
- Also recent very interesting conjecture that if N=8 amplitudes follow from leading singularities, they must be finite to all orders A-HCK (2008); Cachazo talk
- Explicit results through 3 loops are consistent with two more indirect arguments:
  - Superstring calculations in pure spinor formalism
    Berkovits, hep-th/0609006; Green, Russo, Vanhove, hep-th/0611273
  - M theory dualities
    Green, Russo, Vanhove, hep-th/0610299, 0807.0389; Green talk

For the final answer, stay tuned (maybe for a while)!

Conclusions

- Many computations of gauge theory scattering amplitudes exploit the helicity formalism, complex momenta, generalized unitarity and factorization.
- Multi-leg one-loop QCD amplitudes needed for LHC applications are now falling to these techniques – first analytically, and now numerically (**CutTools, Rocket, BlackHat**)
- More phenomenologically important processes now under construction.
- Same techniques can be applied to N=4 super-Yang-Mills theory and N=8 supergravity, leading to multi-loop amplitudes unveiling hidden structures:
  - exact exponentiation (n=4,5); dual (super)conformal invariance; (MHV) equivalence to Wilson lines; more to come
  - cancellations indicating possible all-orders perturbative finiteness