Maximal Supersymmetry and Duality

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Interplay of maximal supersymmetry (N=8 in four dimensions) and duality symmetries of M/String theory leads to powerful constraints.
How powerful?

Outline:

1) Maximal supersymmetry and low energy expansion
   Higher derivative interactions – important at high curvature.
   e.g. IIB in d=10 . Differential equations on moduli space;
   Exact non-perturbative coefficients.

2) Four-graviton scattering amplitude
   Duality with multi-loop Feynman diagrams of
   eleven-dimensional supergravity compactified on two-torus.

3) Connections with maximal supergravity
   UV divergence properties.

1) Maximal supersymmetry and low energy expansion

- Closed Superstring/M -Theory reduces at low energy to maximal supergravity - IIA or IIB in d=10 and N=8 in d=4 - plus higher derivative terms.

- Moduli-dependent coefficients encode exact behaviour of higher-dimension terms.
  Perturbative + non-perturbative dependence on couplings

e.g. 4-graviton amplitude – effective action

$$\frac{1}{\alpha'^4}\int d^{10}x\,\sqrt{G}\,\left(e^{-2\phi}\,R+\alpha'^3\,\mathcal{F}(\phi,\cdots)\,R^4+\cdots\right)$$

Moduli-dependent coefficients

String pert. expansion $\mathcal{F}(\phi,\dots)=\sum_{h=0}^{\infty}e^{2(h-1)\phi}\,f_h+\text{nonpert.}$

α' series important for high curvatures

Type IIB supergravity

Fields: coset scalars $\Omega=\Omega_1+i\Omega_2$ $\Omega_2\equiv e^{-\phi}=g_B^{-1}$

	$\partial_\mu\Omega/\Omega_2$		$\Omega_2^{-\frac{1}{2}}(F_{\mu\nu\rho}+i\Omega_2 H_{\mu\nu\rho})$			
	P_μ ,	λ ,	$G_{\mu\nu\rho}$,	ψ_μ ,	F_5 ,	$g_{\mu\nu}$
	Dilaton,	dilatino ,	3-form,	gravitino,	5-form,	metric
u_Φ :	-2,	-3/2,	-1,	-1/2,	0,	0

U(1) charges in SL(2,R)/U(1)

Coset becomes SL(2,Z)\SL(2,R)/U(1) in string theory

$\Longrightarrow$

Pattern of u non-conserving higher-order interactions.

Higher-derivative terms in IIB:

Consider composite operator $\mathcal{P}^{(u)}_{2n+2}$: U(1) charge u, dimension $\Delta = 2n+2$

e.g. $\mathcal{R}^4 \quad u=0\,, \quad \Delta = 8;$

$\lambda^{16} \quad u=-24\,, \quad \Delta = 8;$

$(G\bar{G})^p\,\mathcal{R}^4 \quad u=0\,, \quad \Delta = 2p+8;$

SL(2,Z) - invariant action:

$$S^{(n)} = \alpha'^{\,n-4} \sum_{u\,i} \int d^{10}x\, \mathcal{F}^{(u)\,i}_n(\Omega,\bar{\Omega})\, \mathcal{P}^{(-u)\,i}_{2n+2}$$

Index i labels degenerate terms

$\mathcal{F}^{(u)\,i}_n$ has holomorphic and antiholomorphic weights $\mp u/2$.

$$\mathcal{F}^{(u)\,i}_n(\Omega,\bar{\Omega}) \to \left(\frac{c\bar{\Omega}+d}{c\Omega+d}\right)^{u/2} \mathcal{F}^{(u)\,i}_n(\Omega,\bar{\Omega}) \qquad \Omega \to \frac{a\Omega+b}{c\Omega+d}$$

How is $\mathcal{F}^{(u)\,i}_n$ constrained by supersymmetry??

Consequences of supersymmetry

Invariance of action
$$\sum_{m=0}^{\infty} \delta^{(m)} \sum_{n=0}^{\infty} S^{(n)} = 0$$

i.e,
$$(\delta^{(0)} + \alpha'^3\delta^{(3)} + \ldots)(S^{(0)} + \alpha'^3 S^{(3)} + \ldots) = 0$$

On-shell algebra
$$[\delta\,,\delta]\Phi = [\delta^{(0)} + \alpha'^3\delta^{(3)} + \ldots\,, \delta^{(0)} + \alpha'^3\delta^{(3)} + \ldots]\Phi$$
$$= a\cdot P\,\Phi + \Phi \text{ eqn. of motion} + \delta_{gauge}\Phi$$

Strongly constrains the form of $\mathcal{F}^{(u)}_n$, $\delta^{(m)}$

Difficult to implement in detail in absence of off-shell superspace formalism. Modified torsion constraints.

Consider general form of component supersymmetry.

Classical IIB supersymmetry transformations

$$\delta^{(0)}\Omega = 2\lambda\epsilon\,\Omega_2$$

Compensating U(1) transform.

$$\delta^{(0)}\Phi^{(u)} = \hat{\delta}^{(0)}\Phi^{(u)} + \tilde{\delta}^{(0)}_u\Phi^{(u)}$$

where $\Phi^{(u)}$ is any field with U(1) charge u and

$$\tilde{\delta}^{(0)}_u\Phi^{(u)} = u\left(\lambda\epsilon - \lambda^*\epsilon^*\right)\Phi^{(u)}$$

Classical supersymmetry :

$$\delta^{(0)}S^{(n)} = \alpha'^{\,n-4}\int d^{10}x\sum_u\Big(\mathcal{F}^{(u)\,i}_n\,\hat{\delta}^{(0)}\left(\mathcal{P}^{(-u)\,i}_{2n+2}\right)$$
$$-2i\mathcal{D}\,\mathcal{F}^{(u)\,i}_n\,\lambda\epsilon\,\mathcal{P}^{(-u)\,i}_{2n+2} + 2i\bar{\mathcal{D}}\,\mathcal{F}^{(u)\,i}_n\,\lambda^*\epsilon^*\,\mathcal{P}^{(-u)\,i}_{2n+2}\Big)$$

where $\mathcal{D} = i\Omega_2\frac{\partial}{\partial\Omega} - \frac{u}{4}$ is modular covariant derivative on charge u.
$\mathcal{D}f^{(u)} = f^{(u+1)}$

Add $\sum_m \delta^{(m)}S^{(n-m)}$ terms, and require closure of superalgebra, $[\delta^{(m)},\delta^{(n)}]\,\Phi \approx 0$, leads to expression of general form (suppressing superscripts and coefficients)

$$\boxed{\begin{aligned}\mathcal{D}\mathcal{F}_n = \mathcal{F}_n + \mathcal{F}_{m_1}\mathcal{F}_{n-m_1} + \mathcal{F}_{m_1}\mathcal{F}_{m_2}\mathcal{F}_{n-m_1-m_2}\\ +\cdots+\mathcal{F}_{m_1}\mathcal{F}_{m_1+m_2}\ldots\mathcal{F}_{n-m_1-\cdots-m_{n-1}}+\ldots\end{aligned}}$$

Detailed coefficients need a more complete analysis.

Apply $\bar{\mathcal{D}}$ to above equation :

$\Longrightarrow$ Inhomogeneous Laplace (Poisson) equation

$$\boxed{\begin{aligned}\bar{\mathcal{D}}\,\mathcal{D}\mathcal{F}_n &= \bar{\mathcal{D}}\mathcal{F}_n + \bar{\mathcal{D}}(\ldots\ldots\ldots)\\ &= \mathcal{F}_n + \mathcal{F}_{m_1}\mathcal{F}_{n-m} + \ldots\end{aligned}}$$

Simple cases can be analyzed in detail:

(a) Simple nondegenerate examples : (index i on $\mathcal{F}_n^{(u)\,i}$ is redundant)

$$\mathcal{D}\,\mathcal{F}_n^{(u)} = c_u\,\mathcal{F}_n^{(u+2)} \qquad \bar{\mathcal{D}}\,\mathcal{F}_n^{(u+2)} = \bar{c}_{u+2}\,\mathcal{F}_n^{(u)}$$

i.e. Laplace eigenvalue equation

$$\bar{\mathcal{D}}\,\mathcal{D}\,\mathcal{F}_n^{(u)} = c_u\bar{c}_{u+2}\,\mathcal{F}_n^{(u)}$$

e.g. i) U(1) preserving

$$u = 0\,, \qquad c_0\bar{c}_2 = s(s-1) \qquad \text{where} \qquad n = 2s = \tfrac{1}{2}\Delta - 1$$

$$\nabla^2_\Omega = \Omega_2^2\partial_\Omega\partial_{\bar\Omega} \longrightarrow \boxed{\nabla^2_\Omega\,\mathcal{F}_n^{(0)} = s(s-1)\,\mathcal{F}_n^{(0)}}$$

Solution is nonholomorphic Eisenstein series

$$\mathcal{F}_n^{(0)} = E_s = \sum_{(\hat m,\hat n)\neq(0,0)} \frac{\Omega_2^s}{|\hat m + \hat n\Omega|^{2s}}$$

Natural SL(2,Z) generalization of Riemann Zeta Values

$$\sim 2\zeta(2s)\Omega_2^s + (\ldots)\zeta(2s-1)\Omega_2^{1-s} + \sum_{k\neq 0}\mu(k,s)\,\left(e^{2\pi i k\Omega} + c.c.\right)\left(1 + O(\Omega_2^{-1}\right)$$

TREE-level terms GENUS-$(s-\frac{1}{2})$ term D-INSTANTON terms with pert. corrections

non-renormalization at higher loops

examples : $E_{\frac{3}{2}}\,\mathcal{R}^4\,, \qquad E_{\frac{5}{2}}\,D^4\mathcal{R}^4$

ii) U(1) -violating processes at order n=3:

$$\mathcal{F}_3^{(u)} = \mathcal{D}^u\,\mathcal{F}_3^{(0)} = \mathcal{D}^u\,E_{\frac{3}{2}}$$

examples : $\mathcal{F}_3^{(8)}\,G^8\,, \qquad \mathcal{F}_3^{(24)}\,\lambda^{16}$

iii) Higher order: $\mathcal{F}_6^{(0)}\,\mathcal{D}^6\mathcal{R}^4$ $(u=0\,,\ n=6)$

$$(\nabla^2_\Omega - 12)\,\mathcal{F}_6^{(0)} = -6E_{\frac{3}{2}}\,E_{\frac{3}{2}}$$

Not (yet) derived purely from supersymmetry but motivated by four-graviton scattering amplitude.

[Other examples in very recent paper by Basu + Sethi.]

(b) Degenerate cases :

In general Laplace eigenvalue equation generalizes to inhomogeneous simultaneous equations :

$$\boxed{(\delta_{ij}\,\bar{D}\,D - \lambda^{(u)}_{n;ij})\mathcal{F}_n^{(u)\,j} = \sum_{j,k,m,v} f_{ijk}^{mn}\,\mathcal{F}_m^{(v)\,j}\,\mathcal{F}_{n-m}^{(u-v)\,k} + \ldots}$$

Lower order source coefficients

Interesting SL(2,Z) generalization of Multiple Zeta Values. (Zagier)

Illustrated by four-graviton amplitude.

2) Four - graviton scattering amplitude

Tiny subsector of complete theory type II theory derivatives of curvature (zero fluxes, fixed dilaton).

$$\mathcal{R}^4,\ \partial^4\mathcal{R}^4, \ldots, \partial^{2k}\mathcal{R}^4, \ldots$$

linearized Weyl curvatures contracted with familiar sixteen-index tensor

Low-energy expansion of string perturbation theory:

TREE-LEVEL (Virasoro-Shapiro Model) – all-orders expansion.

ONE-LOOP (Genus-one world-sheet) – recent results in d=9, 10.

TWO-LOOP (Genus-two world-sheet) – little explicitly known.

Boundary "data" for non-perturbative structure.

Genus-one amplitude:

$$I = \int_F \frac{d^2\tau}{\tau_2^2} F(\tau,\bar\tau; s,t,u)$$

Integral of modular function

$$A_4^{h=1} = \mathcal{R}^4\, I(s,t,u)$$

Expansion in powers of $\sigma_2 = s^2+t^2+u^2$
and $\sigma_3 = s^3+t^3+u^3 = 3stu$

Analytic part -
subtract threshold cuts

$\mathcal{R}^4$ $\quad D^6\mathcal{R}^4$ $\quad D^{10}\mathcal{R}^4$

$$I^{an} = \frac{\pi\alpha'}{3} + 0\,\sigma_2 + \frac{\pi\alpha'^6}{3}\zeta(3)\,\sigma_3 + 0\,\sigma_2^2 + \alpha'^6\frac{97}{1080}\zeta(5)\,\sigma_2\sigma_3$$
$$-\alpha'^{12}\left(\frac{1}{30}\zeta(3)^2\,\sigma_2^3 + \frac{61}{1080}\zeta(3)^2\,\sigma_3^2\right) + \ldots$$

$D^{12}\mathcal{R}^4$

- no s^2 or s^4 terms

MBG, Russo,Vanhove arXiv:0801.0322

Compactify on circle radius r (d=9)

$$\mathrm{A}_4^{h=1}(r;s,t) = \frac{\pi}{3}\Big[r + r^{-1} + \sigma_2\left(\frac{\zeta(3)}{15}\,r^3 + \frac{\zeta(3)}{15}\,r^{-3}\right)$$
$$+\sigma_3\left(\frac{\zeta(5)}{63}\,r^5 + \frac{\zeta(3)}{3}\,r + \frac{\zeta(3)}{3}\,r^{-1} + \frac{\zeta(5)}{63}\,r^{-5}\right)$$
$$+\sigma_2^2\left(\frac{\zeta(7)}{315}\,r^7 + \frac{2\zeta(3)}{15}\,r\,\log(r^2\,\lambda_4) + \frac{\zeta(5)}{36}\,r^{-3} + \frac{\zeta(3)^2}{315}\,r^{-5} + \frac{\zeta(7)}{1050}\,r^{-7}\right)$$
$$+\sigma_2\sigma_3\left(\frac{7\zeta(9)}{2970}\,r^9 + \frac{\zeta(3)^2}{21}\,r^3 + \frac{97\zeta(5)}{1080}\,r + \frac{29\zeta(5)}{135}\,r^{-1} + O(r^{-3})\right)$$
$$+\sigma_2^3\left(\frac{3\zeta(11)}{8008}\,r^{11} + \frac{2\zeta(3)\zeta(5)}{525}\,r^5 + \frac{11\zeta(5)}{210}\,r\,\log(r^2\,\lambda_6) + \frac{\zeta(3)^2}{30}\,r + \frac{\zeta(3)^2}{30}\,r^{-1} + O(r^{-3})\right)$$
$$+\sigma_3^2\left(\frac{109\zeta(11)}{225225}\,r^{11} + \frac{8\zeta(3)\zeta(5)}{1575}\,r^5 + \frac{\zeta(5)}{15}\,r\,\log(r^2\,\lambda_6) + \frac{61\zeta(3)^2}{1080}\,r + \frac{61\zeta(3)^2}{6144}\,r^{-1}\right.$$
$$\left.+O(r^{-3})\right) + O(e^{-r})\Big]$$

Intriguing pattern of coefficients
- rational numbers X products of zeta values.

Relevant to M-theory compactified on T^2

What is non-perturbative completion ??

SL(2,Z) - invariant effective IIB action (string frame)

Einstein-Hilbert

Higher-derivative interactions

$$\alpha'^4 S = \int d^{10}x\sqrt{g}\left(e^{-2\phi}R + \alpha'^3 e^{-\phi/2}E_{\frac{3}{2}}\mathcal{R}^4 + \alpha'^5 e^{\phi/2}E_{\frac{5}{2}}D^4\mathcal{R}^4 + \dots\right)$$

[What is the complete list of $O(1/\alpha')$ interactions ??

- absence of superspace formalism makes things difficult
- exact dependence on F_5 : $\mathcal{R}^4 \to \frac{1}{\alpha'}(\mathcal{R} + \mathcal{D}F_5 + F_5^2)^4$
  gives info concerning AdS/CFT plasma viscosity
  (Buchel, Myers, Paulos, Sinha)
- Stretched horizon of stringy black holes]

Higher derivative interactions ??

Clues from M-theory/String Theory duality -

Connections with eleven-dimensional supergravity

Recall: CLASSICALLY:

Eleven-dimensional M-theory on T^2 is dual to type II on a circle of radius $r_A = r_B^{-1}$

Torus volume: $\mathcal{V} = \exp\left(\frac{1}{3}\phi^B\right) r_B^{-\frac{4}{3}}$

Complex structure: $\Omega = \Omega_1 + i\Omega_2 =$ Complex IIB coupling

$$\Omega_1 = C^{(0)} = C_9^{(1)}, \qquad \Omega_2 = \exp\left(-\phi^B\right) = r_A \exp\left(-\phi^A\right).$$

Type IIB in d=10: $\boxed{\mathcal{V} \to 0 \longrightarrow r_B \to \infty}$

Type IIA in d=10: $\boxed{R_{10} \to \infty \longrightarrow r_A \to \infty}$

What about quantum effects ??

Feynman diagrams L loops - UV divergent (in 11 dimensions on two-torus)

Regulate, e.g., momentum scale Λ.

Naive degree of divergence $\Lambda^{(d-2)L+2}$.

Actual degree of divergence much less due to overall factor of $\mathcal{R}^4$ (eight powers of momentum).

Further powers of S, T, U as L increases - lower degree of divergence (see last part of talk).

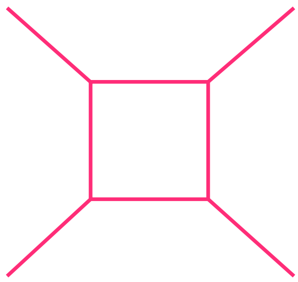
Subtract divergences with counterterms - unknown coefficients encoding short distance features of M-theory. Some of these (how many ?) are determined by requiring consistency with string perturbation theory.

L=1 One loop in 11 dimensions on T²

Sum of all Feynman diagrams:

$\mathcal{R} \sim k\, k\, h$

$\mathcal{R}^4$



Box diagram of **SCALAR** field theory

Sum over windings of loop around cycles of T²

Winding numbers $\hat{m},\ \hat{n}$

$\Lambda^3\, \mathcal{V}$ divergence in zero winding number sector $\hat{m} = \hat{n} = 0$

suppressed in limit $\mathcal{V} \to 0$

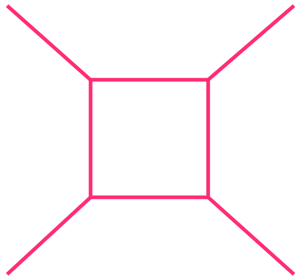
$\mathcal{V}^{-\frac{1}{2}}$ from non-zero windings

L=1 One loop in 11 dimensions on T^2

Sum of all Feynman diagrams:

$\mathcal{R} \sim k\, k\, h$

$\mathcal{R}^4$



Box diagram of **SCALAR** field theory

(i) LOW ENERGY $S, T, U \to 0$

11-dim. Mandelstam variables - capital letters

$$A = \sum_{(\hat{m},\hat{n}) \neq (0,0)} \frac{\Omega_2^{\frac{3}{2}}}{|\hat{m} + \hat{n}\Omega|^3} \mathcal{V}^{-\frac{1}{2}} \mathcal{R}^4 \quad = E_{\frac{3}{2}} \mathcal{V}^{-\frac{1}{2}} \mathcal{R}^4$$

TEN-DIMENSIONAL IIB limit:

$$\underset{\mathcal{V}\to 0}{\rightarrow} \frac{1}{\alpha'} e^{-\frac{1}{2}\phi^B} E_{\frac{3}{2}}(\Omega, \bar{\Omega})\, \mathcal{R}^4$$

$E_{\frac{3}{2}}(\Omega, \bar{\Omega})$ contains TREE - LEVEL and GENUS-ONE string perturbative terms together with non-perturbative D-instantons

(ii) HIGHER ORDERS in S, T, U :

Infinite series of terms in IIA limit: $(r^A \to \infty)$

$$c_h\, e^{2(h-1)\phi^A} s^h\, \mathcal{R}^4$$

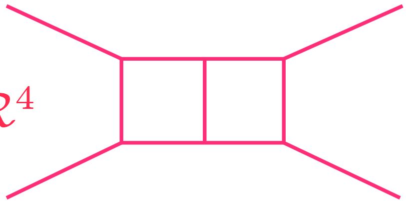
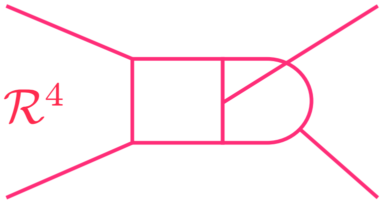
genus-h

finite coefficients – no contributions from higher loops !

Higher-loop 11-dim. sugra on T²

L=2 • TWO LOOPS – factor out overall $S^2\,\hat{\mathcal{R}}^4$
resulting in **scalar** field theory diagrams

(Bern, Dixon, Dunbar, Perelstein, Rozowsky 1998)

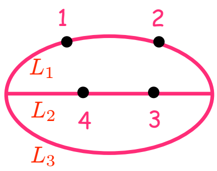
$S^2\mathcal{R}^4$  $S^2\mathcal{R}^4$ 

\+ T and U diagrams

Sum over windings of both loops around cycles of T²
Winding numbers $\hat{m}_1, \hat{m}_2, \hat{n}_1, \hat{n}_2$ (MBG, Vanhove 1999)

Use one-loop counterterm for sub-divergences

Evaluation of two-loop integrals



Redefinition of three Schwinger parameters

$L_1,\, L_2,\, L_3 \;\rightarrow\; V,\, \tau_1,\, \tau_2$ (and four vertex postions)

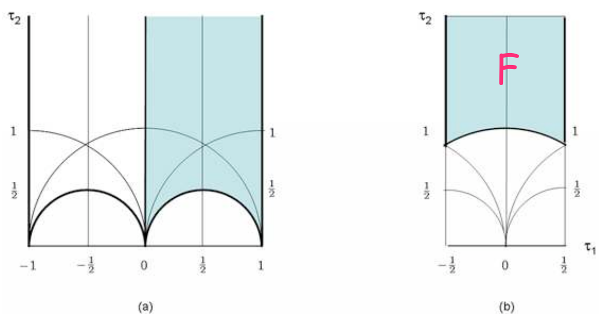
where $\underbrace{\tau_1 = \frac{L_2}{L_1+L_2}, \qquad \tau_2 = \frac{\sqrt{\Delta}}{L_1+L_2}},$ $\qquad V = \sqrt{\Delta}$

$\Delta = L_1L_2 + L_2L_3 + L_3L_1$

c.f. Complex structure and volume of \`torus' !

$\tau = \tau_1 + i\tau_2$ (c.f. genus-one world-sheet)

Integration Domain (a)
=
3 copies of SL(2,Z)
fund. domain in (b)



Inhomogeneous Laplace equations

$$A = \mathcal{I}(S,T,U)\,\mathcal{R}^4 = \sum_{(p,q)} \sigma_2^p \sigma_3^q \, I_{(p,q)}\, \mathcal{R}^4$$

General terms: $I_{(p,q)} = \sum_i h^i_{(p,q)}$ where

i.e., SUM of modular functions

$$\boxed{(\nabla^2_\Omega - i(i+1))h^i_{(p,q)} = \sum_{r,s} c^{rs}_{(p,q)}\, E_r\, E_s}$$

Modular invariant coefficient
at lower orders of α'

Some examples:

i) Limit $S,T,U \to 0$ gives $\sigma_2\, \mathcal{R}^4 \sim D^4 \mathcal{R}^4$

Ten-dimensional IIB

$$\underset{\mathcal{V}\to 0}{\to}\ \alpha'\, e^{\frac{1}{2}\phi^B}\ E_{\frac{5}{2}}(\Omega,\bar{\Omega})\, \sigma_2 \mathcal{R}^4$$

(recall $(\nabla^2_\Omega - \frac{15}{4})\,E_{\frac{5}{2}} = 0$ so source term is zero)

$E_{\frac{5}{2}}(\Omega,\bar{\Omega})$ contains TREE - LEVEL and GENUS-TWO terms together with non-perturbative D-instantons

coinciding with tree-level and genus-two string results

ii) Next order in S, T, U gives $\sigma_3\,\mathcal{R}^4 \sim D^6\mathcal{R}^4$

Ten-dimensional IIB

$$S_{D^6 R^4} = \alpha'\, e^{\phi^B}\,\mathcal{E}_{(0,1)}(\Omega)\, D^6 R^4$$

$$\mathcal{V} \to 0$$

$$\boxed{\nabla^2_\Omega \mathcal{E}_{(0,1)} - 12\mathcal{E}_{(0,1)} = -6E_{\frac{3}{2}}\,E_{\frac{3}{2}}}$$

$E_{\frac{3}{2}}$ - coefficient of $\mathcal{R}^4$

New effect:

Mixing of $\delta^{(3)}\,S^{(3)}$ with $\delta^{(0)}\,S^{(6)}$ leads to source term in Poisson eqn. for coefficient of $S^3\,\mathcal{R}^4$ term.

$\mathcal{E}_{(0,1)}$ contains genus: 0, 1, 2, 3: Agrees with string pert. theory as far as can be checked.

i.e. genus 0, 1, 3

iii) Expand Two-Loop Supergravity to higher orders in S,T,U :

(MBG, Russo,Vanhove **arXiv:0807.0389**)

Leads to d=9 modular invariant interactions of form :

$$\frac{1}{r_B^m}\,\alpha'^{2p+3q}\,\mathcal{E}^{(m+1)}_{(p,q)}(\Omega,\bar\Omega)\,\sigma_2^p\,\sigma_3^q\,\mathcal{R}^4 \qquad m = 4p+6q-7$$

$$\frac{1}{r_B}\,\alpha'^4\,\mathcal{E}^{(2)}_{(2,0)}(\Omega,\bar\Omega)\,\sigma_2^2\,\mathcal{R}^4 \qquad \frac{1}{r_B^3}\,\alpha'^5\,\mathcal{E}^{(4)}_{(1,1)}(\Omega,\bar\Omega)\,\sigma_2\sigma_3\,\mathcal{R}^4$$

$$\frac{1}{r_B^5}\,\alpha'^6\,\mathcal{E}^{(6)}_{(3,0)}(\Omega,\bar\Omega)\,\sigma_2^3\,\mathcal{R}^4 \qquad \frac{1}{r_B^5}\,\alpha'^6\,\mathcal{E}^{(6)}_{(0,2)}(\Omega,\bar\Omega)\,\sigma_3^2\,\mathcal{R}^4$$

New feature: Each modular function is the sum of solutions of Poisson equations,

e.g, $\mathcal{E}^{(4)}_{(1,1)} = \sum_{r=0}^{5} \mathcal{E}^{(4)\,j}_{(1,1)}$

where

$$(\Delta_\Omega - j(j+1))\,\mathcal{E}^{(4)\,j}_{(1,1)} = -2v_j\, E_{\frac{3}{2}}\, E_{\frac{3}{2}} - 24w_j\zeta(2)\, E_{\frac{1}{2}}\, E_{\frac{1}{2}}$$

v_j , w_j are constants $E_{\frac{1}{2}} \sim \Omega_2 \log \Omega_2 + \ldots$

- Solve for perturbative coefficients:
  Many agreements with string pert. theory and unitarity.
- Higher supergravity loops (L>2) will reproduce further string theory terms.

3) Connections with maximal supergravity

Higher-loop supergravity?? All L

THREE LOOPS - extra power of S

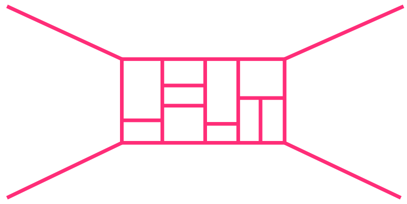
- anticipated from successes of one and two loops
- explicit construction (Bern, Carrasco, Johansson, Dixon, Kosower, Roiban)

HIGHER LOOPS - are there further powers of S???

Little known of details beyond three loops – but, **subject to important assumptions,** duality with string theory points to possibly important constraints.

Simple dimensional argument :

L - LOOP Maximal Supergravity
in d dimensions



Naive divergence
(count vertices and propagators)

$$A_L \sim \Lambda^{(d-2)L+2}$$

External momentum factors
reduce divergence of sum
of all Feynman diagrams

$$A_L \sim S^{\beta_L}\, \mathcal{R}^4\, \Lambda^{(d-2)L-6-2\beta_L}$$

QUESTION – what is the value of β_L ?

Direct multi-loop calculations 1982 - 2006

$L = 1 \quad \beta_L = 0\,, \quad A_1 \sim \mathcal{R}^4\, \Lambda^{d-8}$

$L = 2 \quad \beta_L = 2\,, \quad A_2 \sim S^2 \mathcal{R}^4\, \Lambda^{2d-14}$

i.e $\mathcal{R}^4$ not renormalized beyond 1 loop –
hence NO 3-LOOP $\mathcal{R}^4$ counterterm in d=4.

$L = 3 \quad \beta_L = 3\,, \quad A_2 \sim S^3 \mathcal{R}^4\, \Lambda^{3d-18}$

i.e $S^2\, \mathcal{R}^4$ not renormalized beyond 2 loops –
NO 5-LOOP $S^2\, \mathcal{R}^4$ counterterm in d=4

$L = 4 \quad \beta_L = 4\,, \quad A_2 \sim S^4 \mathcal{R}^4\, \Lambda^{4d-22}$

i.e $S^3\, \mathcal{R}^4$ not renormalized beyond 2 loops –
NO 6-LOOP $S^3\, \mathcal{R}^4$ counterterm in d=4

Fermionic zero mode argument (Berkovits,2006)

$$\beta_L = L \qquad L \le 5\,, \qquad\qquad \beta_L \ge 6 \qquad L \ge 6$$

Based on pure spinor string theory – builds in full supersymmetry

No UV divergence up to $\boxed{\text{9 LOOPS}}$ in d=4

(MBG, Russo, Vanhove 2006a)

– manifestly $S^6\,\mathcal{R}^4$ duality-invariant counterterm.

IF $\beta_L = L$ for all L $\Rightarrow A_L \sim S^L\,\mathcal{R}^4\,\Lambda^{(d-4)L-6}$

Motivated by duality of eleven-dim. supergravity and string theory.

(MBG, J. Russo, P. Vanhove 2006b)

Ultraviolet finite when: $d < 4 + \frac{6}{L}$

i.e., finite for all L when d=4 – (as in maximal Yang-Mills).

Consider L-loop 11- dim. SUGRA on circle (radius R_{11})

Compactified expression (arbitrary β_L) contains powers of $(S\,R_{11}^2)^\nu$ and $(R_{11}\Lambda)^{-w}$ (w > 0 for subdivergence)

Transforming to IIA parameters $R_{11}^3 \to g_A^2\,,\ S \to s\,R_{11}$

$$A_L \sim s^{\beta_L+\nu}\,g_A^{2(\nu+\frac{1}{3}(\beta_L-w))}\,\mathcal{R}^4\,\Lambda^{9L-6-2\beta_L-w}$$

i) L=1 : $\beta_1 = 0\,,\ w = 3$ $\qquad s^h\,g_A^{2(h-1)}\,\mathcal{R}^4 \qquad (h = \nu > 1)$

ii) L > 1 : $\beta_L \ge 2$ contributes to lower powers of g_A

$$s^h\,g_A^{2(h-1)}\,g_A^{-\frac{2}{3}(2\beta_L+w-3)}\,\mathcal{R}^4\,\Lambda^{9L-6-2\beta_L-w}$$

Hence leading behaviour at genus h : $s^h\,\mathcal{R}^4 \qquad (\beta_h = h)$

Comment on relation to superstring:

CANNOT decouple maximal supergravity quantum field theory from closed string (in d > 3 dimensions). (MBG, Ooguri Schwarz)

Suggests that maximal supergravity probably does not make sense in isolation from string theory for d>3.
[i.e., String theory is crucial UV completion.]

VIZ: Open string theory reduces to N=4 maximally supersymmetric Yang—Mills theory. CAN be decoupled from string theory in d=4 (as in AdS/CFT limit of D3-branes).
N=4 Yang—Mills is UV finite and is a sensible decoupled local quantum field theory.

From L=1 to L=2	16 years 1982-1998
L=2 to L=3	8 years 1998-2006
L=3 to L=4	4 years 2006-2010 ?
…………..	

Converges

$L = \infty$ 2014