Maximal Supersymmetry and Duality

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Interplay of maximal supersymmetry (N=8 in four dimensions) and duality symmetries of M/String theory leads to powerful constraints.

How powerful?

Outline:

1) Maximal supersymmetry and low energy expansion
   Higher derivative interactions – important at high curvature.
   e.g. IIB in d=10. Differential equations on moduli space;
   Exact non-perturbative coefficients.

2) Four-graviton scattering amplitude
   Duality with multi-loop Feynman diagrams of
   eleven-dimensional supergravity compactified on two-torus.

3) Connections with maximal supergravity
   UV divergence properties.
1) **Maximal supersymmetry and low energy expansion**

- **Closed Superstring/M -Theory** reduces at low energy to maximal supergravity - IIA or IIB in d=10 and N=8 in d=4 - plus higher derivative terms.

- **Moduli-dependent coefficients** encode exact behaviour of higher-dimension terms.
  Perturbative + non-perturbative dependence on couplings

  e.g. 4-graviton amplitude - effective action

  \[
  \frac{1}{\alpha'} \int d^{10}x \sqrt{G} \left( e^{-2\phi} R + \alpha'^3 \mathcal{F}(\phi, \ldots) R^4 + \cdots \right)
  \]

  \(\alpha'\) series important for high curvatures

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**Type IIB supergravity**

**Fields:** coset scalars \(\Omega = \Omega_1 + i\Omega_2\) \(\Omega_2 \equiv e^{-\phi} = g_B^{-1}\)

\[
\partial_\mu \Omega_1/\Omega_2, \quad \Omega_2^{-\frac{1}{2}}(F_{\mu\nu\rho} + i\Omega_2 H_{\mu\nu\rho})
\]

\(P_\mu, \lambda, G_{\mu\nu\rho}, \psi_\mu, F_5, g_{\mu\nu}\)

Dilaton, dilatino, 3-form, gravitino, 5-form, metric

\(u_\Phi: -2, -3/2, -1, -1/2, 0, 0\)

\(U(1)\) charges in \(SL(2,R)/U(1)\)

Coset becomes \(SL(2,Z)\backslash SL(2,R)/U(1)\) in string theory

\[\Rightarrow\]

Pattern of \(u\) non-conserving higher-order interactions.
Higher-derivative terms in IIB:

Consider composite operator \( P_{2n+2}^{(u)} \): U(1) charge \( u \), dimension \( \Delta = 2n + 2 \)

- \( \mathcal{R}^4 \) \( u = 0 \), \( \Delta = 8 \);
- \( \lambda^{16} \) \( u = -24 \), \( \Delta = 8 \);
- \((GG)^p \mathcal{R}^4\) \( u = 0 \), \( \Delta = 2p + 8 \).

**SL(2,Z) - invariant action:**

\[
S^{(n)} = \alpha^{-n-4} \sum_{u,i} \int d^{10}x \mathcal{F}_n^{(u)i} (\Omega, \bar{\Omega}) P_{2n+2}^{(-u)i}
\]

Index \( i \) labels degenerate terms

\( \mathcal{F}_n^{(u)i} \) has holomorphic and antiholomorphic weights \( \mp u/2 \).

\[
\mathcal{F}_n^{(u)i} (\Omega, \bar{\Omega}) \rightarrow \left( \frac{c\Omega + d}{c\Omega + d} \right)^{u/2} \mathcal{F}_n^{(u)i} (\Omega, \bar{\Omega}) \quad \Omega \rightarrow \frac{a\Omega + b}{c\Omega + d}
\]

How is \( \mathcal{F}_n^{(u)i} \) constrained by supersymmetry??

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Consequences of supersymmetry

**Invariance of action**

\[
\sum_{m=0}^{\infty} \delta^{(m)} \sum_{n=0}^{\infty} S^{(n)} = 0
\]

i.e.,

\[
(\delta^{(0)} + \alpha\delta^{(3)} + \ldots)(S^{(0)} + \alpha^3 S^{(3)} + \ldots) = 0
\]

**On-shell algebra**

\[
[\delta,\delta] \Phi = [\delta^{(0)} + \alpha^3 \delta^{(3)} + \ldots, \delta^{(0)} + \alpha^3 \delta^{(3)} + \ldots] \Phi
\]

\[
= a \cdot P \Phi + \Phi \quad \text{eqn. of motion} + \delta_{gauge} \Phi
\]

Strongly constrains the form of \( \mathcal{F}_n^{(u)}, \delta^{(m)} \)

Difficult to implement in detail in absence of off-shell superspace formalism. **Modified torsion constraints.**

Consider general form of component supersymmetry.
Classical IIB supersymmetry transformations

\[ \delta^{(0)} \Omega = 2 \lambda \epsilon \Omega_2 \]

\[ \delta^{(0)} \Phi^{(u)} = \hat{\delta}^{(0)} \Phi^{(u)} + \tilde{\delta}^{(0)}_u \Phi^{(u)} \]

where \( \Phi^{(u)} \) is any field with U(1) charge \( u \) and

\[ \tilde{\delta}^{(0)}_u \Phi^{(u)} = u (\lambda \epsilon - \lambda^* \epsilon^*) \Phi^{(u)} \]

Classical supersymmetry:

\[ \delta^{(0)} S^{(n)} = \alpha'^{n-4} \int d^{10} x \sum_u \left( \mathcal{F}^{(u)}_n i \hat{\delta}^{(0)} \left( \mathcal{P}_{2n+2}^{(-u)} i \right) \right) \\
\]

\[ -2i D \mathcal{F}^{(u)}_n \lambda \epsilon \mathcal{P}_{2n+2}^{(-u)} i + 2i \bar{D} \mathcal{F}^{(u)}_n \lambda^* \epsilon^* \mathcal{P}_{2n+2}^{(-u)} i \]

where \( D = i \Omega_2 \frac{\partial}{\partial \Omega_2} - \frac{u}{4} \) is modular covariant derivative on charge \( u \).

\[ D f^{(u)} = f^{(u+1)} \]

Add \( \sum_m \delta^{(m)} S^{(n-m)} \) terms, and require closure of superalgebra, \( \left[ \delta^{(m)} , \delta^{(n)} \right] \Phi \approx 0 \), leads to expression of general form (suppressing superscripts and coefficients)

\[ D \mathcal{F}_n = \mathcal{F}_n + \mathcal{F}_{m_1} \mathcal{F}_{n-m_1} + \mathcal{F}_{m_1} \mathcal{F}_{m_2} \mathcal{F}_{n-m_1-m_2} + \cdots + \mathcal{F}_{m_1} \mathcal{F}_{m_1+m_2} \cdots \mathcal{F}_{n-m_1-\cdots-m_n-1} + \cdots \]

Detailed coefficients need a more complete analysis.

Apply \( \bar{D} \) to above equation:

\[ \Rightarrow \text{Inhomogeneous Laplace (Poisson) equation} \]

\[ \bar{D} \bar{D} \mathcal{F}_n = \bar{D} \mathcal{F}_n + \bar{D}(\ldots \ldots \ldots) \]

\[ = \mathcal{F}_n + \mathcal{F}_{m_1} \mathcal{F}_{n-m} + \ldots \]
Simple cases can be analyzed in detail:

(a) Simple nondegenerate examples : \( (\text{index } i \text{ on } F_n^{(u)} \text{ i is redundant}) \)

\[
D F_n^{(u)} = c_u F_n^{(u+2)} \quad \bar{D} F_n^{(u+2)} = \bar{c}_{u+2} F_n^{(u)}
\]

i.e. Laplace eigenvalue equation

\[
\bar{D} D F_n^{(u)} = c_u \bar{c}_{u+2} F_n^{(u)}
\]

\[u = 0, \quad c_0 \bar{c}_2 = s(s - 1) \quad \text{where} \quad n = 2s = \frac{1}{2} \Delta - 1
\]

\[
\nabla_\Omega^2 = \Omega_2^2 \partial_\Omega \partial_\Omega \rightarrow \nabla_\Omega^2 F_n^{(0)} = s(s - 1) F_n^{(0)}
\]

Solution is nonholomorphic Eisenstein series

\[
F_n^{(0)} = E_s = \sum_{(\hat{m}, \hat{n}) \neq (0,0)} \frac{\Omega_2^s}{|\hat{m} + \hat{n}\Omega|^2s}
\]

Natural \( SL(2,\mathbb{Z}) \) generalization of Riemann Zeta Values

\[
\sim 2\zeta(2s)\Omega_2^2 + (\ldots)\zeta(2s - 1)\Omega_2^{1-s} + \sum_{k \neq 0} \mu(k, s) \left( e^{2\pi ik\Omega} + \text{c.c.} \right) (1 + O(\Omega_2^{-1}))
\]

- TREE-level terms
- GENUS-(s - \frac{1}{2}) term
- D-INSTANTON terms with pert. corrections

non-renormalization at higher loops

examples :

\[
E_{\frac{3}{2}} \mathcal{R}^4, \quad E_{\frac{3}{2}} D^4 \mathcal{R}^4
\]

ii) \( U(1) \) -violating processes at order \( n=3 \):

\[
F_3^{(u)} = D^u F_3^{(0)} = D^u E_{\frac{3}{2}}
\]

examples :

\[
F_3^{(8)} G^8, \quad F_3^{(24)} \lambda^{16}
\]
iii) Higher order: \( \mathcal{F}^{(0)}_6 \mathcal{D}^6 \mathcal{R}^4 \) \( (u = 0, \ n = 6) \)

\[
(\nabla^2_{\Omega} - 12) \mathcal{F}^{(0)}_6 = -6 E^3_2 E^3_2
\]

Not (yet) derived purely from supersymmetry but motivated by four-graviton scattering amplitude.

[Other examples in very recent paper by Basu + Sethi.]

(b) Degenerate cases:

In general Laplace eigenvalue equation generalizes to inhomogeneous simultaneous equations:

\[
(\delta_{ij} \bar{D} D - \lambda^{(u)}_{n;ij}) \mathcal{F}^{(u)}_n j = \sum_{j,k,m,v} f_{ijk}^{mn} \mathcal{F}^{(v)}_m j \mathcal{F}^{(u-v)}_n k + \ldots
\]

**Interesting SL(2,Z) generalization of Multiple Zeta Values.**

**(Zagier)**

Illustrated by four-graviton amplitude.

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2) **Four-graviton scattering amplitude**

Tiny subsector of complete theory type II theory derivatives of curvature (zero fluxes, fixed dilaton).

\[ \mathcal{R}^4, \ \partial^4 \mathcal{R}^4, \ldots, \partial^{2k} \mathcal{R}^4, \ldots \]

linearized Weyl curvatures contracted with familiar sixteen-index tensor

Low-energy expansion of string perturbation theory:

**TREE-LEVEL (Virasoro-Shapiro Model)** - all-orders expansion.

**ONE-LOOP** (Genus-one world-sheet) - recent results in d=9, 10.

**TWO-LOOP** (Genus-two world-sheet) - little explicitly known.

Boundary “data” for non-perturbative structure.
**Genus-one amplitude:**
\[ I = \int \frac{d^2 \tau}{\tau^2} F(\tau, \bar{\tau}; s, t, u) \]

Integral of modular function

\[ A_{4}^{h=1} = \mathcal{R}^4 I(s, t, u) \]

Expansion in powers of
\[ \sigma_2 = s^2 + t^2 + u^2 \quad \text{and} \quad \sigma_3 = s^3 + t^3 + u^3 = 3stu \]

Analytic part -
subtract threshold cuts

\[ I^{an} = \frac{\pi \alpha'}{3} + \frac{\pi \alpha'}{3} + \sum \frac{\zeta(3)}{3} \sigma_3 + \frac{\pi \alpha'}{3} + \frac{\pi \alpha'}{3} + \sum \frac{\zeta(3)}{3} \sigma_3 + \frac{\alpha'^6}{97} \frac{97}{1080} \zeta(5) \sigma_2 \sigma_3 \]

\[ -\alpha'^{12} \left( \frac{1}{30} \zeta(3)^2 \sigma_3 + \frac{61}{1080} \zeta(3)^2 \sigma_3 \right) + \ldots \]

\[ \cdot \text{no } S^2 \text{ or } S^4 \text{ terms} \]

MBG, Russo, Vanhove. arXiv:0801.0322

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**Compactify on circle radius** \( r \quad (d=9) \)

\[ A_{4}^{h=1}(r; s, t) = \frac{\pi}{3} \left[ r + r^{-1} + \sigma_2 \left( \frac{\zeta(3)}{15} r^3 + \frac{\zeta(3)}{15} r^{-3} \right) \right. \]

\[ + \sigma_3 \left( \frac{\zeta(5)}{3} r^5 + \frac{\zeta(3)}{3} r + \frac{\zeta(3)}{3} r^{-1} + \frac{\zeta(5)}{63} r^{-5} \right) \]

\[ + \sigma_2^2 \left( \frac{\zeta(7)}{315} r^7 + \frac{2 \zeta(3)}{3} r \log(r^2 \lambda_4) + \frac{\zeta(5)}{36} r^{-3} + \frac{\zeta(3)^2}{315} r^{-5} + \frac{\zeta(7)}{1050} r^{-7} \right) \]

\[ + \sigma_2 \sigma_3 \left( \frac{7 \zeta(9)}{2970} r^9 + \frac{\zeta(3)}{21} r^3 + \frac{97 \zeta(5)}{1080} r + \frac{29 \zeta(5)}{135} r^{-1} + O(r^{-3}) \right) \]

\[ + \sigma_2^2 \left( \frac{3 \zeta(11)}{8008} r^{11} + \frac{2 \zeta(3) \zeta(5)}{525} r^{-5} + \frac{11 \zeta(5)}{210} r \log(r^2 \lambda_6) + \frac{\zeta(3)^2}{30} r + \frac{\zeta(3)}{30} r^{-1} + O(r^{-3}) \right) \]

\[ + \sigma_3^2 \left( \frac{109 \zeta(11)}{225225} r^{11} + \frac{8 \zeta(3) \zeta(5)}{1575} r^5 + \frac{\zeta(5)}{15} r \log(r^2 \lambda_6) + \frac{61 \zeta(3)^2}{1080} r + \frac{61 \zeta(3)^2}{6144} r^{-1} + O(r^{-3}) \right) \]

\[ + O(r^{-3}) \]

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**Intriguing pattern of coefficients**
- rational numbers \( X \) products of zeta values.

**Relevant to M-theory compactified on** \( T^2 \)
What is non-perturbative completion??

\[ \text{SL}(2,\mathbb{Z}) - \text{invariant effective IIB action} \quad \text{(string frame)} \]

\[
\alpha'^4 S = \int d^{10}x \sqrt{g} \left( e^{-2\phi} R + \alpha'^3 e^{-\phi/2} E_4 R^4 + \alpha'^5 e^{\phi/2} E_2 D^4 R^4 + \ldots \right)
\]

[What is the complete list of \( O(1/\alpha') \) interactions??]

- absence of superspace formalism makes things difficult
- exact dependence on \( F_5 : \quad R^4 \rightarrow \frac{1}{\alpha'}(R + DF_5 + F_5^2)^4 \)
  gives info concerning AdS/CFT plasma viscosity (Buchel, Myers, Paulos, Sinha)
- Stretched horizon of stringy black holes

Higher derivative interactions??

Clues from M-theory/String Theory duality -

Connections with eleven-dimensional supergravity

Recall: CLASSICALLY:

Eleven-dimensional M-theory on \( T^2 \) is dual to type II on a circle of radius \( r_A = r_B^{-1} \)

Torus volume: \[ \mathcal{V} = \exp \left( \frac{1}{3} \phi^B \right) r_B^{-\frac{3}{2}} \]

Complex structure: \[ \Omega = \Omega_1 + i\Omega_2 = \text{Complex IIB coupling} \]
\[ \Omega_1 = C^{(0)} = C^{(1)}_g, \quad \Omega_2 = \exp (-\phi^B) = r_A \exp (-\phi^A). \]

Type IIB in d=10: \[ \mathcal{V} \rightarrow 0 \quad \longrightarrow \quad r_B \rightarrow \infty \]

Type IIA in d=10: \[ R_{10} \rightarrow \infty \quad \longrightarrow \quad r_A \rightarrow \infty \]
What about quantum effects ??

Feynman diagrams $L$ loops - UV divergent $\sim (d-2)L^{d-2}$ (in 11 dimensions on two-torus)

Regulate, e.g., momentum scale $\Lambda$.

Naive degree of divergence $\sim \Lambda^{d-2}$. 

Actual degree of divergence much less due to overall factor of $R^4$ (eight powers of momentum).

Further powers of $S$, $T$, $U$ as $L$ increases - lower degree of divergence (see last part of talk).

Subtract divergences with counterterms - unknown coefficients encoding short distance features of $M$-theory. Some of these (how many?) are determined by requiring consistency with string perturbation theory.

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$L=1$ One loop in 11 dimensions on $T^2$

Sum of all Feynman diagrams:

\[
R \sim k k h \rightarrow R^4
\]

Box diagram of \textbf{SCALAR} field theory

Sum over windings of loop around cycles of $T^2$

Winding numbers $\tilde{m}$, $\tilde{n}$

\[
\Lambda^3 V \quad \text{divergence in zero winding number sector}
\]

\[
\text{suppressed in limit } V \rightarrow 0 \quad \hat{m} = \hat{n} = 0
\]

\[
V^{-\frac{1}{2}} \quad \text{from non-zero windings}
\]
L=1

One loop in 11 dimensions on $T^2$

Sum of all Feynman diagrams:

\[ \mathcal{R} \sim k k h \]

Box diagram of **SCALAR** field theory

(i) LOW ENERGY $S, T, U \to 0$

\[
A = \sum_{(\hat{m}, \hat{n}) \neq (0,0)} \frac{\Omega_2^2}{|\hat{m} + \hat{n}\Omega|^3} v^{-\frac{1}{2}} \mathcal{R}^4 = E_{\frac{3}{2}} v^{-\frac{1}{2}} \mathcal{R}^4
\]

11-dim. Mandelstam variables - capital letters

TEN-DIMENSIONAL IIB limit:

\[
\rightarrow \quad \frac{1}{\alpha'} e^{-\frac{1}{2} \phi^L} E_{\frac{3}{2}}(\Omega, \tilde{\Omega}) \mathcal{R}^4
\]

$E_{\frac{3}{2}}(\Omega, \tilde{\Omega})$ contains TREE - LEVEL and GENUS-ONE string perturbative terms together with non-perturbative D-instantons

(ii) HIGHER ORDERS in $S, T, U$:

Infinite series of terms in IIA limit: $(n^A \to \infty)$

\[
c_h e^{2(h-1)\phi^A} s_h \mathcal{R}^4
\]

finite coefficients - no contributions from higher loops!
Higher-loop 11-dim. sugra on $T^2$

$L=2 \cdot$ TWO LOOPS - factor out overall $S^2 \hat{R}^4$
resulting in scalar field theory diagrams
(Bern, Dixon, Dunbar, Perelstein, Rozowsky 1998)

$S^2 \hat{R}^4$ \hspace{2cm} $S^2 \hat{R}^4$

+ $T$ and $U$ diagrams

Sum over windings of both loops around cycles of $T^2$
Winding numbers $\hat{m}_1, \hat{m}_2, \hat{n}_1, \hat{n}_2$  
(MBG, Vanhove 1999)

Use one-loop counterterm for sub-divergences

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Evaluation of two-loop integrals

![Diagram](image)

Redefinition of three Schwinger parameters

$$L_1, L_2, L_3 \rightarrow V, \tau_1, \tau_2$$

(and four vertex postions)

where

$$\tau_1 = \frac{L_2}{L_1 + L_2}, \quad \tau_2 = \frac{\sqrt{\Delta}}{L_1 + L_2}, \quad V = \sqrt{\Delta}$$

$$\Delta = L_1 L_2 + L_2 L_3 + L_3 L_1$$

c.f. Complex structure and volume of `torus'

$$\tau = \tau_1 + i\tau_2$$

(c.f. genus-one world-sheet)

Integration Domain (a)

= 3 copies of SL(2,Z)
fund. domain in (b)
Inhomogeneous Laplace equations

\[ A = \mathcal{I}(S, T, U) \mathcal{R}^4 = \sum_{(p,q)} \sigma^p_2 \sigma^q_3 I_{(p,q)} \mathcal{R}^4 \]

General terms: 
\[ I_{(p,q)} = \sum_i h^i_{(p,q)} \text{ where} \]

\[ (\nabla^2_\Omega - i(i + 1)) h^i_{(p,q)} = \sum_{r,s} c^{rs}_{(p,q)} E_r E_s \]

Modular invariant coefficient
at lower orders of \( \alpha' \)

Some examples:

\begin{enumerate}
  \item Limit \( S, T, U \to 0 \) gives \( \sigma_2 \mathcal{R}^4 \sim D^4 \mathcal{R}^4 \)

Ten-dimensional IIB

\[ \to \alpha' e^{\frac{1}{2} \phi^B} E_{\frac{5}{2}} (\Omega, \bar{\Omega}) \sigma_2 \mathcal{R}^4 \]

(recall \( (\nabla^2_\Omega - \frac{15}{4}) E_{\frac{5}{2}} = 0 \) so source term is zero)

\( E_{\frac{5}{2}} (\Omega, \bar{\Omega}) \) contains TREE-LEVEL and GENUS-TWO terms together with non-perturbative D-instantons coinciding with tree-level and genus-two string results
ii) Next order in $S, T, U$ gives $\sigma_3 \mathcal{R}^4 \sim D^6 \mathcal{R}^4$

Ten-dimensional IIB

$$S_{D^6 \mathcal{R}^4} = \alpha' e^{\phi_B} \mathcal{E}_{(0,1)}(\Omega) D^6 \mathcal{R}^4$$

$\mathcal{V} \to 0$

$$\nabla^2_{\Omega} \mathcal{E}_{(0,1)} - 12 \mathcal{E}_{(0,1)} = -6 E_3^3 E_3^3$$

New effect: $E_3$ - coefficient of $\mathcal{R}^4$

Mixing of $\delta^{(3)} S^{(3)}$ with $\delta^{(0)} S^{(6)}$ leads to source term in Poisson eqn. for coefficient of $S^3 \mathcal{R}^4$ term.

$\mathcal{E}_{(0,1)}$ contains genus: 0, 1, 2, 3: Agrees with string pert. theory as far as can be checked.

i.e. genus 0, 1, 3

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iii) Expand Two-Loop Supergravity to higher orders in $S, T, U$:

(MBG, Russo, Vanhove arXiv:0807.0389)

Leads to $d=9$ modular invariant interactions of form:

$$\frac{1}{r_B^m} \alpha'^{r_{p+3q}} \mathcal{E}_{(p,q)}^{(m+1)}(\Omega, \bar{\Omega}) \sigma_2^p \sigma_3^q \mathcal{R}^4 \quad m = 4p + 6q - 7$$

$$\frac{1}{r_B^2} \alpha' \mathcal{E}_{(0,0)}^{(2)}(\Omega, \bar{\Omega}) \sigma_2^2 \mathcal{R}^4$$

$$\frac{1}{r_B^3} \alpha^5 \mathcal{E}_{(1,1)}^{(4)}(\Omega, \bar{\Omega}) \sigma_2 \sigma_3 \mathcal{R}^4$$

$$\frac{1}{r_B^3} \alpha^6 \mathcal{E}_{(3,0)}^{(6)}(\Omega, \bar{\Omega}) \sigma_2^3 \mathcal{R}^4$$

$$\frac{1}{r_B^5} \alpha^6 \mathcal{E}_{(0,2)}^{(6)}(\Omega, \bar{\Omega}) \sigma_3^2 \mathcal{R}^4$$
New feature: Each modular function is the sum of solutions of Poisson equations,

\[ \mathcal{E}_{(1,1)}^{(4)} = \sum_{r=0}^{5} \mathcal{E}_{(1,1)}^{(4)j} \]

where

\[ (\Delta \Omega - j(j + 1)) \mathcal{E}_{(1,1)}^{(4)j} = -2v_j E^{2} E^{2} - 24w_j \zeta(2) E^{1} E^{1} \]

\( v_j, w_j \) are constants

\( E^{2} \sim \Omega_2 \log \Omega_2 + \ldots \)

- Solve for perturbative coefficients:
  Many agreements with string pert. theory and unitarity.
- Higher supergravity loops (\( L > 2 \)) will reproduce further string theory terms.

3) **Connections with maximal supergravity**

Higher-loop supergravity?? All L

**THREE LOOPS** - extra power of S
- anticipated from successes of one and two loops
- explicit construction (Bern, Carrasco, Johansson, Dixon, Kosower, Roiban)

**HIGHER LOOPS** - are there further powers of S???

Little known of details beyond three loops - but, subject to important assumptions. Duality with string theory points to possibly important constraints.
Simple dimensional argument:

\( L \) - LOOP Maximal Supergravity in \( d \) dimensions

Naive divergence (count vertices and propagators)

\[ A_L \sim \Lambda^{(d-2)L+2} \]

External momentum factors reduce divergence of sum of all Feynman diagrams

\[ A_L \sim S^{\beta_L} \mathcal{R}^4 \Lambda^{(d-2)L-6-2\beta_L} \]

**QUESTION** – what is the value of \( \beta_L \)?

Direct multi-loop calculations 1982 - 2006

\( L = 1 \quad \beta_L = 0, \quad A_1 \sim \mathcal{R}^4 \Lambda^{d-8} \)

\( L = 2 \quad \beta_L = 2, \quad A_2 \sim S^2 \mathcal{R}^4 \Lambda^{2d-14} \)

i.e. \( \mathcal{R}^4 \) not renormalized beyond 1 loop – hence NO 3-LOOP \( \mathcal{R}^4 \) counterterm in \( d=4 \).

\( L = 3 \quad \beta_L = 3, \quad A_2 \sim S^3 \mathcal{R}^4 \Lambda^{3d-18} \)

i.e. \( S^2 \mathcal{R}^4 \) not renormalized beyond 2 loops – NO 5-LOOP \( S^2 \mathcal{R}^4 \) counterterm in \( d=4 \).

\( L = 4 \quad \beta_L = 4, \quad A_2 \sim S^4 \mathcal{R}^4 \Lambda^{4d-22} \)

i.e. \( S^3 \mathcal{R}^4 \) not renormalized beyond 2 loops – NO 6-LOOP \( S^3 \mathcal{R}^4 \) counterterm in \( d=4 \).
Fermionic zero mode argument (Berkovits, 2006)

\[
\beta_L = L \quad L \leq 5, \quad \beta_L \geq 6 \quad L \geq 6
\]

Based on pure spinor string theory – builds in full supersymmetry

No UV divergence up to \textbf{9 LOOPS} in \(d=4\)

\[(\text{MBG, Russo, Vanhove 2006a})\]

– manifestly \(S^6 \mathcal{R}^4\) duality-invariant counterterm.

**IF** \(\beta_L = L\) for all \(L\) \(\Rightarrow A_L \sim S^L \mathcal{R}^4 \Lambda^{(d-4)L-6}\)

Motivated by duality of eleven-dim. supergravity and string theory. \(\text{(MBG, J. Russo, P. Vanhove 2006b)}\)

Ultraviolet finite when: \(d < 4 + \frac{6}{L}\)

i.e., finite for all \(L\) when \(d=4\) – (as in maximal Yang-Mills).

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Consider \(L\)-loop \(11\)-dim. SUGRA on circle (radius \(R_{11}\))

Compactified expression (arbitrary \(\beta_L\)) contains powers of \((S R_{11}^2)^\nu\) and \((R_{11} \Lambda)^{-w}\) \((w > 0\) for subdivergence)

Transforming to IIA parameters \(R_{11}^3 \rightarrow g_A^2, \ S \rightarrow s \ R_{11}\)

\[A_L \sim s^{\beta_L + \nu} \ g_A^{2(\nu + \frac{1}{3}(\beta_L - w))} \mathcal{R}^4 \Lambda^{9L - 6 - 2\beta_L - w}\]

i) \(L=1\): \(\beta_1 = 0, \ w = 3\)

\[s^h g_A^{2(h-1)} \mathcal{R}^4 \ (h = \nu > 1)\]

ii) \(L > 1: \beta_L \geq 2\) contributes to lower powers of \(g_A\)

\[s^h g_A^{2(h-1)} g_A^{-\frac{2}{3}(2\beta_L + w - 3)} \mathcal{R}^4 \Lambda^{9L - 6 - 2\beta_L - w}\]

Hence leading behaviour at genus \(h\): \(s^h \mathcal{R}^4 \ (\beta_h = h)\)
Comment on relation to superstring:

CANNOT decouple maximal supergravity quantum field theory from closed string (in $d > 3$ dimensions). (MBG, Ooguri Schwarz)

Suggests that maximal supergravity probably does not make sense in isolation from string theory for $d>3$. [i.e., String theory is crucial UV completion.]

VIZ: Open string theory reduces to $N=4$ maximally supersymmetric Yang–Mills theory. CAN be decoupled from string theory in $d=4$ (as in AdS/CFT limit of D3-branes). N=4 Yang–Mills is UV finite and is a sensible decoupled local quantum field theory.
From \( L=1 \) to \( L=2 \) \hspace{1cm} 16 \text{ years} \hspace{0.5cm} 1982-1998

\( L=2 \) to \( L=3 \) \hspace{1cm} 8 \text{ years} \hspace{0.5cm} 1998-2006

\( L=3 \) to \( L=4 \) \hspace{1cm} 4 \text{ years} \hspace{0.5cm} 2006-2010 \ ?

\[ \text{Converges} \]

\[ L = \infty \hspace{1cm} 2014 \]