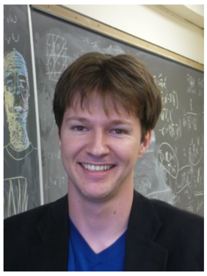
Developments in BPS Wall-Crossing

Strings 2008, Cern, August 22, 2008

Work done with
Davide Gaiotto and Andy Neitzke
arXiv:0807.4723

And, to appear…



Outline

1. Review BPS Wall-Crossing
2. The Kontsevich-Soibelman formula
3. N=2,D=4 Field Theory on $\mathbb{R}^3 \times S^1$
4. Twistor Space
5. One-particle corrections
6. Multi-particle corrections: Riemann-Hilbert
7. Differential Equations
8. Summary & Concluding Remarks

Introduction

This talk is about the BPS spectrum of theories with d=4,N=2

Recently there has been some progress in understanding how the BPS spectrum depends on the vacuum.

These are called Wall-Crossing Formulae (WCF)

Last year: WCF derived with Frederik Denef

Subsequently, Kontsevich & Soibelman proposed a remarkable wall-crossing formula for generalized Donaldson-Thomas invariants of CY 3-folds

This talk will give a physical explanation & derivation of the KS formula

Review of BPS Wall-Crossing-I

Consider a theory on $\mathbb{R}^4$ with $\mathcal{N} = 2$ SUSY

For $u \in \mathcal{M}_v$, the moduli space of vacua,

let $\mathcal{H}_u$ be the one-particle Hilbert space.

*Low energy theory: an unbroken rank r **abelian** gauge theory*

Γ: Symplectic lattice of electric & magnetic charges γ

$$\mathcal{H}_u = \oplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma,u}$$

BPS -II

$\mathcal{N}=2$ central charge operator $\hat{Z}=Z_\gamma(u)$ on $\mathcal{H}_{\gamma,u}$

$\mathcal{H}^{BPS}_{u,\gamma}$: Subspace of $\mathcal{H}_{\gamma,u}$ with $E=|Z_\gamma(u)|$

$$\Omega(\gamma;u):=-\frac{1}{2}\mathrm{Tr}_{\mathcal{H}^{BPS}_{\gamma,u}}(2J_3)^2(-1)^{2J_3}$$

Some BPS states are boundstates of other BPS states.

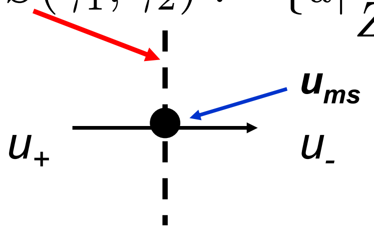
$\psi\in\mathcal{H}^{BPS}_{\gamma,u}$, a boundstate of BPS states: $\gamma=\gamma_1+\gamma_2$,

ψ will decay as u crosses a *wall of marginal stability* where $Z_{\gamma_1}(u)$ and $Z_{\gamma_2}(u)$ are parallel.

(Cecotti,Fendley,Intriligator,Vafa; Seiberg&Witten)

Semi-Primitive Wall-Crossing

Marginal Stability Wall: $MS(\gamma_1,\gamma_2):=\{u|\frac{Z_{\gamma_1}(u)}{Z_{\gamma_2}(u)}\in\mathbb{R}_+\}$



Denef & Moore gave formulae for $\Delta\Omega$

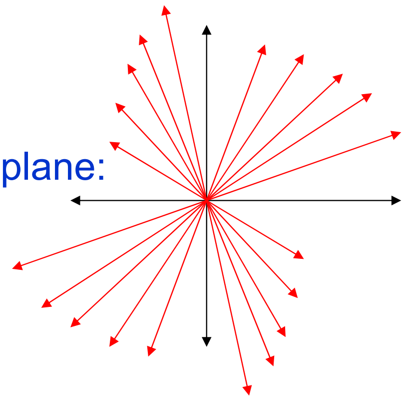
for decays of the form $\gamma\to\gamma_1+N\gamma_2\qquad N\geq 1$

Based on Denef's multi-centered solutions of sugra, and quiver quantum mechanics.

Do not easily generalize to $\gamma\to N_1\gamma_1+N_2\gamma_2\qquad N_i\geq 1$

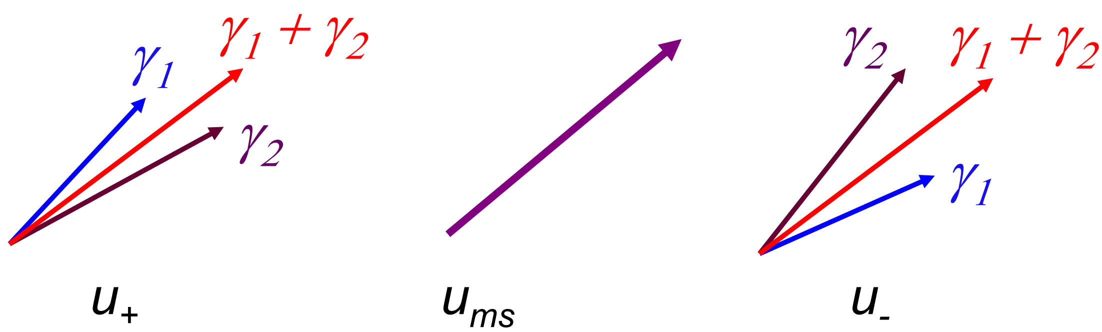
BPS Rays

For each $\gamma \in \Gamma$ associate a ray in the ζ plane:



$$\ell_\gamma := Z_\gamma(u)\mathbb{R}_-$$

As u crosses an MS wall some BPS rays will coalesce:



Symplectic transformations

$$T := \Gamma^* \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^{2r}$$

$\mathcal{K}_\gamma$: Symplectic transformations of T

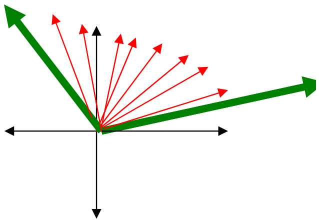
KS assign a group element S_γ to each BPS ray ℓ_γ

$$S_\gamma := \overrightarrow{\prod_{\gamma' \| \gamma}} \mathcal{K}_{\gamma'}^{\Omega(\gamma';u)}$$

KS WCF

For generic u, and convex cone $\mathcal{V}$ in the ζ-plane

$$\mathcal{A}(\mathcal{V}) := \prod^{\rightarrow}_{\ell_\gamma \in \mathcal{V}} S_\gamma = \prod^{\rightarrow}_{-Z_\gamma(u) \in \mathcal{V}} \mathcal{K}_\gamma^{\Omega(\gamma;u)}$$



Main statement: The product is ***INDEPENDENT OF u***

This is a wall-crossing formula !!

KS Transformations

$T = \Gamma^* \otimes \mathbb{C}^* \Rightarrow$ Fourier modes: $X_\gamma : T \to \mathbb{C}^*$

$$\epsilon^{ij} = \langle \gamma^i, \gamma^j \rangle \Rightarrow \varpi^T := \frac{1}{2}\epsilon_{ij} \frac{dX^i}{X^i} \frac{dX^j}{X^j}$$

$$\mathcal{K}_\gamma : X_{\gamma'} \to X_{\gamma'} e^{\langle \gamma, \gamma' \rangle \log(1 - X_\gamma)}$$

Example for r=1: $T = \mathbb{C}^* \times \mathbb{C}^*$ $\qquad \varpi^T = \frac{dx}{x}\frac{dy}{y}$

$$\mathcal{K}_{a,b} : (x,y) \to (x(1-x^a y^b)^b, y(1-x^a y^b)^{-a})$$

Seiberg-Witten Theory

Consider a $d=4, \mathcal{N}=2$ field theory with a semisimple gauge group of rank r.

$\mathcal{M}_v \cong \mathbb{C}^r$ $\quad (u_2 = \langle \mathrm{Tr}\Phi^2 \rangle,\ u_3 = \langle \mathrm{Tr}\Phi^3 \rangle, \cdots)$

$\Gamma \to \mathcal{M}_v$ ***is now a local system***

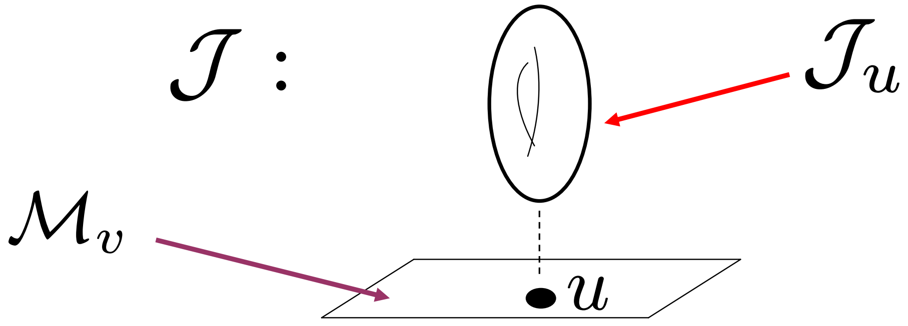
Locally, we may choose a duality frame $\Gamma \cong \Gamma_{el} \oplus \Gamma_{mg}$

Special coordinates $a^I = Z_{\alpha^I}(u)$

$$Z_\gamma(u) = a \cdot \gamma_{el} + a_D \cdot \gamma_{mg}$$

Low Energy Theory on R4

Torus fibration: $\mathcal{J}_u := \Gamma_u^* \otimes (\mathbb{R}/2\pi\mathbb{Z})$



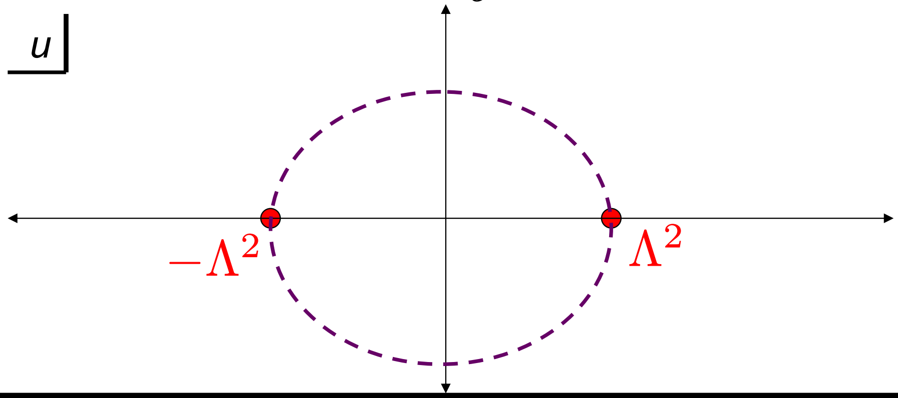
Generic fibers $\mathcal{J}_u$: *Abelian varieties*

Choosing a duality frame:

$$\mathcal{L} = -\frac{1}{4\pi}\mathrm{Im}\tau_{IJ}(da^I * d\bar{a}^J + F^I * F^J) + \frac{1}{4\pi}\mathrm{Re}\tau_{IJ}F^I F^J + \cdots$$

Example of G=SU(2)

$$\Sigma_u : \qquad y + \frac{\Lambda^4}{y} = x^2 - 2u$$



$$\mathcal{K}_{2,-1}\mathcal{K}_{0,1} = \mathcal{K}_{0,1}\mathcal{K}_{2,1}\mathcal{K}_{4,1}\cdots\mathcal{K}_{2,0}^{-2}\cdots\mathcal{K}_{4,-1}\mathcal{K}_{2,-1}$$

It's true!!!

Low Energy theory on $\mathbb{R}^3 \times S^1$

(Seiberg & Witten)

3D sigma model with target space $\mathcal{J}$

$$\varphi_e^I = \oint_{S^1} A_4^I dx^4$$

Periodic coordinates

$$\varphi_{m,I} = \oint_{S^1} (A_{D,4})_I dx^4$$

for $\mathcal{J}_u$

Susy ⇒ $\mathcal{J}$ *is hyperkähler*

Semiflat Metric

R = radius of S^1 $R \to \infty$

KK reduce and dualize the 3D gauge field:

$$g^{\rm sf} = R {\rm Im}\tau |da|^2 + (R{\rm Im}\tau)^{-1} |dz|^2$$

$$dz_I = d\varphi_{m,I} - \tau_{IJ} d\varphi_e^J$$

The Main Idea

- g^{sf} is quantum-corrected by BPS states (instanton = worldline of BPS particle on S^1)
- So, quantum corrections depend on the BPS spectrum
- The spectrum jumps, but the true metric g must be smooth across MS walls.
- This implies a WCF!

Twistor Space

$$\mathcal{Z} := \mathcal{J} \times \mathbb{C}P^1 \quad \stackrel{p}{\rightarrow} \quad \mathbb{C}P^1$$

$\mathcal{J}_\zeta$ has complex structure $\zeta \in \mathbb{C}P^1$

A HK metric g is equivalent to a fiberwise holomorphic symplectic form

$$\varpi \in \Omega^2_{\mathcal{Z}/\mathbb{C}P^1} \otimes \mathcal{O}(2)$$

$$\varpi_\zeta = \zeta^{-1}\omega_+ + \omega_3 + \zeta\omega_-, \qquad \zeta \in \mathbb{C}^*$$

Holomorphic Fourier Modes

$\mathcal{J}_\zeta \rightarrow \mathcal{M}_v$ is a torus fibration

$\exists$ a basis of holomorphic functions $\mathcal{X}_\gamma(\zeta), \quad \gamma \in \Gamma$

Restriction to torus $(\mathcal{J}_\zeta)_u \Rightarrow \mathcal{X}_\gamma(\zeta) = \exp\left[i\theta_\gamma + \cdots\right]$

$$\mathcal{J}_u = \Gamma_u^* \otimes (\mathbb{R}/2\pi\mathbb{Z}) \qquad T_u = \Gamma_u^* \otimes \mathbb{C}^*$$

$$\mathcal{X}(\zeta) : \mathcal{J}_\zeta \rightarrow T \qquad \mathcal{X}_\gamma(\zeta) = \mathcal{X}(\zeta)^*(X_\gamma)$$

$$\varpi_\zeta = \mathcal{X}(\zeta)^*(\varpi^T)$$

Semiflat holomorphic Fourier modes

$$\mathcal{X}_\gamma^{\rm sf} = \exp\left[\pi R \zeta^{-1} Z_\gamma + i\theta_\gamma + \pi R \zeta \bar{Z}_\gamma\right]$$

(Neitzke, Pioline, & Vandoren)

Strategy: Compute quantum corrections to $\mathcal{X}_\gamma^{\rm sf}$

$$\mathcal{X}_\gamma = \mathcal{X}_\gamma^{\rm sf} \mathcal{X}_\gamma^{\rm inst}$$

Recover the metric from $\varpi_\zeta = \mathcal{X}(\zeta)^*(\varpi^T)$

One Particle Corrections

- Work near a point u* where one HM, $\mathcal{H}$ becomes massless
- Dominant QC's from instantons of these BPS particles
- Choose a duality frame where $\mathcal{H}$ has electric charge q>0
- Do an effective Lagrangian computation

Periodic Taub-NUT

$r = 1$: Gibbons-Hawking metric on $U(1)$ fibration over $\mathbb{R}^2 \times S^1$

$$g^{\rm PTN} = V^{-1}(d\varphi_m + A)^2 + V\left[|da|^2 + (d\varphi_e)^2\right]$$

$$V = V^{\rm sf} + V^{\rm inst}$$

$$V^{\rm sf} = -q^2 R \left(\log\frac{a}{\Lambda} + \log\frac{\bar a}{\bar\Lambda}\right)$$

$$V^{\rm inst} = q^2 R \sum_n e^{inq\varphi_e} K_0(R|nqa|)$$

(S&W, Ooguri & Vafa, Seiberg & Shenker)

Twistor coordinates for PTN

From the metric $g^{\rm PTN}$ we compute ϖ^{PTN}

$$\varpi^{PTN} = \mathcal{X}^*(\varpi^T) = \frac{d\mathcal{X}_e}{\mathcal{X}_e}\frac{d\mathcal{X}_m}{\mathcal{X}_m}$$

$$\mathcal{X}_e(\zeta) = \exp[i\varphi_e + \cdots]$$

$$\mathcal{X}_m(\zeta) = \exp[i\varphi_m + \cdots]$$

→ *Differential equation for twistor coordinates*

Explicit PTN twistor coordinates

$$\mathcal{X}_e(\zeta) = \mathcal{X}_e^{\rm sf} = \exp[R\zeta^{-1}a + i\varphi_e + R\zeta\bar{a}]$$

$$\mathcal{X}_m(\zeta) = \mathcal{X}_m^{\rm sf}\mathcal{X}_m^{\rm inst}$$

$$\mathcal{X}_m^{\rm sf} = \exp[R\zeta^{-1}a_D + i\varphi_m + R\zeta\bar{a}_D]$$

$$\mathcal{X}_m^{\rm inst}(\zeta) \sim \exp\left\{ q\int_{\ell_{\gamma_e}} [d\zeta'] \frac{\log[1-\mathcal{X}_e(\zeta')^q]}{\zeta'-\zeta} \right\}$$

Key features of the coordinates

1. As a function of ζ, $\mathcal{X}_m$ is *discontinuous* across the electric BPS ray ℓ_{γ_e}.

The discontinuity is given by a KS transformation!

$$(\mathcal{X}_e, \mathcal{X}_m)^{ccw} = \mathcal{K}_{\gamma_e}(\mathcal{X}_e, \mathcal{X}_m)^{cw}$$

2. $\mathcal{X}_m(\zeta) \quad \sim_{\zeta\to 0,\infty} \quad \mathcal{X}_m^{\rm sf}(\zeta)$

As befits instanton corrections.

Multi-Particle Contributions

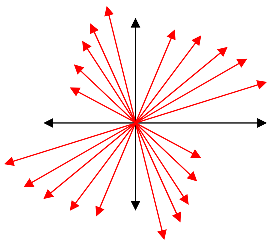
- To take into account the instanton corrections from *ALL* the BPS particles we cannot use an effective field theory computation.
- Mutually nonlocal fields in L_{eff} are illegal!
- We propose to circumvent this problem by reformulating the instanton corrections as a Riemann-Hilbert problem in the ζ plane.

Riemann-Hilbert problem

$\mathcal{X}(\zeta) : \mathcal{J}_\zeta \to T$ *Piecewise holomorphic family*

1. Across each BPS ray ℓ_γ

$$\mathcal{X}(\zeta)^{ccw} = S_\gamma \mathcal{X}(\zeta)^{cw}$$



2. $\mathcal{X}(\zeta) \to \mathcal{X}^{\rm sf}(\zeta)$

Exponentially fast for $\zeta \to 0, \infty$

Solution to the RH problem

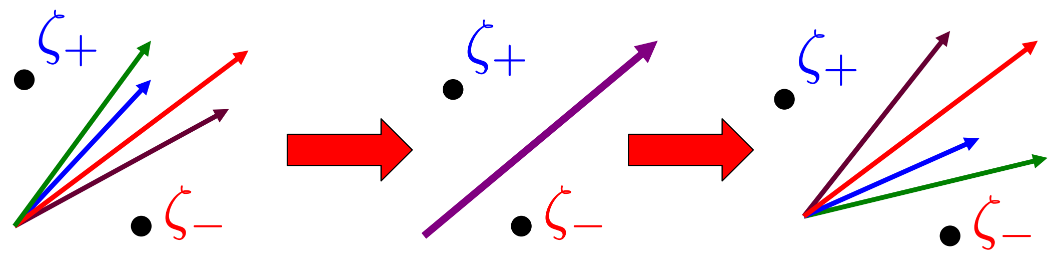
$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}^{\rm sf}_\gamma(\zeta)\cdot$$
$$\cdot \exp\left\{\sum_{\gamma'}\Omega(\gamma';u)\langle\gamma,\gamma'\rangle\int_{\ell_{\gamma'}}[d\zeta']\frac{\log(1-\mathcal{X}_{\gamma'}(\zeta'))}{\zeta'-\zeta}\right\}$$

⇒ *Explicit instanton expansion as a sum over trees*

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}^{\rm sf}_\gamma(\zeta)\mathcal{X}^{\rm inst}_\gamma(\zeta)$$

KS WCF = Continuity of the metric

As u crosses a wall, BPS rays pile up



KS WCF ⇒ *Discontinuity from* ζ_+ *to* ζ_- *is unchanged*

⇒ ϖ *and hence g is continuous across a wall !*

Differential Equations

The Riemann-Hilbert problem is equivalent to a flat system of differential equations:

$$\zeta \partial_\zeta \mathcal{X} = (\zeta^{-1} A_\zeta^- + A_\zeta^0 + \zeta A_\zeta^+)\mathcal{X}$$ *$U(1)_R$ symmetry*

$$R \partial_R \mathcal{X} = (\zeta^{-1} A_R^- + A_R^0 + \zeta A_R^+)\mathcal{X}$$ *scale symmetry*

$$\partial_u \mathcal{X} = (\zeta^{-1} A_u^- + A_u^0)\mathcal{X}$$ *holomorphy*

⇒ *Stokes factors are independent of u,R*

⇒ *Compute at large R: Stokes factors = KS factors S_γ*

Summary

- We constructed the HK metric for circle compactification of SW theories.
- Quantum corrections to the dim. red. metric g^{sf} encode the BPS spectrum.
- Continuity of the metric across walls of MS is equivalent to the KS WCF.
- Use the twistor transform to include quantum corrections of mutually nonlocal particles

Other Things We Have Studied

- The $\mathcal{X}_\gamma$ are Wilson-'tHooft-Maldacena loop operators, and generate the chiral ring of a 3D TFT
- Analogies to tt* geometry of Cecotti & Vafa
- Relations to Hitchin systems and D4/NS5 branes following Cherkis & Kapustin.

Open Problems

- Singularities at superconformal points
- Relations to integrable systems?
- Meaning of KS ``motivic WCF formula''?
- Relation to the work of Joyce
  & Bridgeland/Toledano Laredo
- Generalization to SUGRA
- QC's to hypermultiplet moduli spaces