# *A perturbative approach to hydrodynamics*

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based on

- Fluctuations around Bjorken Flow and the onset of turbulent phenomena, [JHEP 11, 100 (2011), with U. A. Wiedemann]
- Mode-by-mode fluid dynamics for relativistic heavy ion collisions [Phys. Lett. B, 728, 407 (2014), with U. A. Wiedemann]
- Characterization of initial fluctuations for the hydrodynamical description of heavy ion collisions, [Phys. Rev. C 88, 044906 (2013), with U. A. Wiedemann]
- Kinetic freeze-out, particle spectra and harmonic flow coefficients from mode-by-mode hydrodynamics, [Phys. Rev. C 89 (2014) 034914, with U. A. Wiedemann]
- How (non-) linear is the hydrodynamics of heavy ion collisions? [arXiv:1312.5482, with U. A. Wiedemann, A. Beraudo, L. Del Zanna, G. Inghirami, V. Rolando]
- Mode-by-mode hydrodynamics: ideas and concepts [arXiv:1401.2339]
- Statistics of initial density perturbations in heavy ion collisions and their fluid dynamic response [arXiv:1405.4393, with U. A. Wiedemann]

# *Introduction*

# *What perturbations are interesting and why?*

- **Initial fluid perturbations**: Event-by-event fluctuations around a background or average of fluid fields at time $\tau_0$:
  - energy density $\epsilon$
  - fluid velocity $u^\mu$
  - shear stress $\pi^{\mu\nu}$
  - more general also: baryon number density $n_B$, electric charge density, electromagnetic fields, ...
- governed by universal evolution equations
- can be used to constrain thermodynamic and transport properties
- contain interesting information from early times

# *A program to understand fluid perturbations*

1. Characterize initial perturbations.
2. Propagated them through fluid dynamic regime.
3. Determine influence on particle spectra and harmonic flow coefficients.
4. Take also perturbations from non-hydro sources (jets) into account. [see talk of K. Zapp]

# *Characterization of initial conditions*

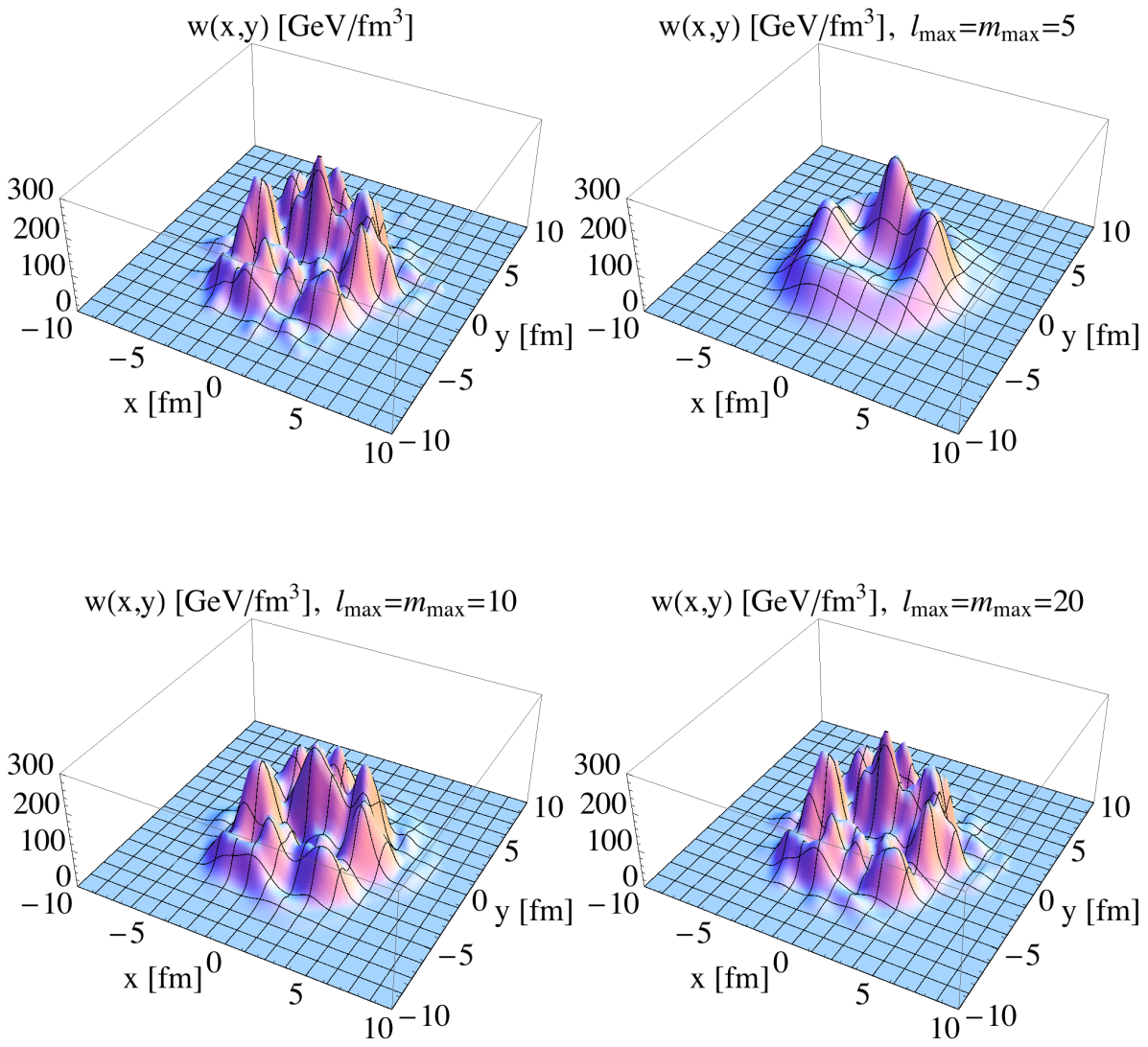
# *Transverse enthalpy density*

Based on Bessel-Fourier expansion and background density [Floerchinger & Wiedemann 2013, see also Coleman-Smith, Petersen & Wolpert 2012, Floerchinger & Wiedemann 2014]

$$w(r,\phi) = w_{\mathsf{BG}}(r) + w_{\mathsf{BG}}(r) \sum_{m,l} w_l^{(m)}\, e^{im\phi}\, J_m\left(z_l^{(m)} \rho(r)\right)$$

- azimuthal wavenumber $m$, radial wavenumber $l$
- $w_l^{(m)}$ dimensionless
- higher $m$ and $l$ correspond to finer spatial resolution
- coefficients $w_l^{(m)}$ can be related to eccentricienies
- works similar for vectors (velocity) and tensors (shear stress)

# *Transverse density from Glauber model*



# *Event ensembles*

- Event-by-event probability distribution

$$p_{\tau_0}[w, u^\mu, \pi^{\mu\nu}, \ldots]$$

- Moments / correlation functions

$$\left\langle w_{l_1}^{(m_1)} w_{l_2}^{(m_2)} \ldots w_{l_n}^{(m_n)} \right\rangle$$

  - contain information from initial state physics / early dynamics
  - universal (model independent) properties would be nice to have
  - same information in cumulants (connected correlation functions)

# *Statistics of initial density perturbations*

Independent point-sources model (IPSM)

$$w(\vec{x}) = \left[\frac{1}{\tau_0}\frac{dW_{\mathsf{BG}}}{d\eta}\right]\frac{1}{N}\sum_{j=1}^{N}\delta^{(2)}(\vec{x}-\vec{x}_j)$$

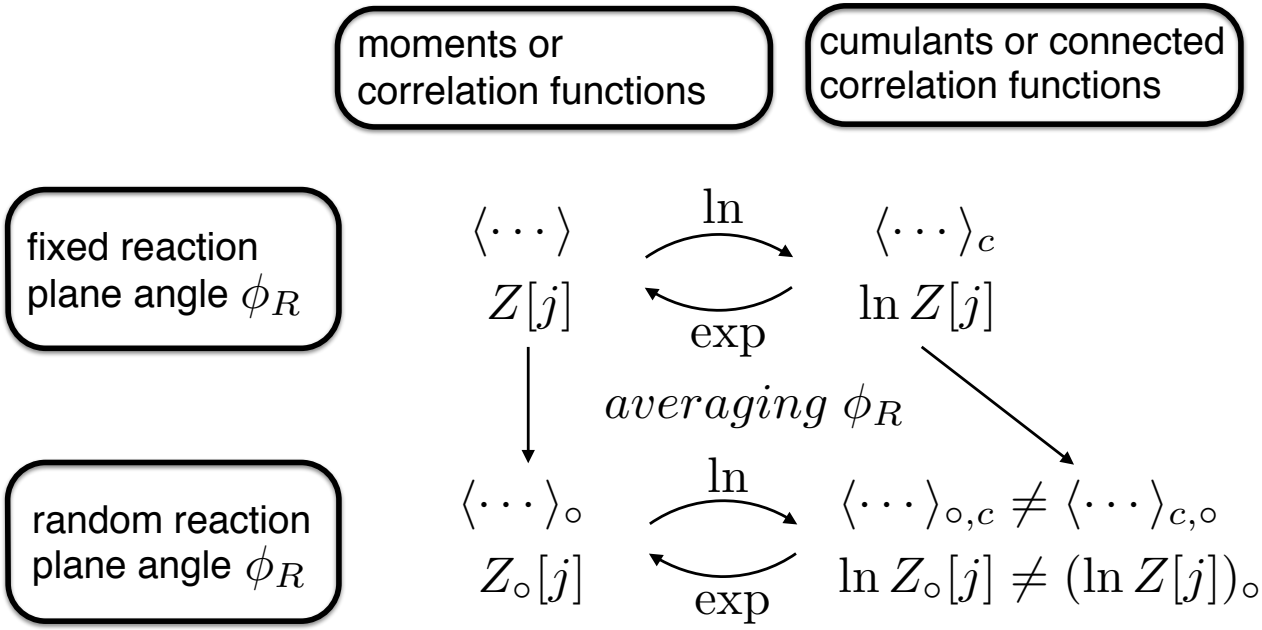
- random positions $\vec{x}_j$, independent and identically distributed
- probability distribution $p(\vec{x}_j)$ reflects collision geometry
- possible to determine correlation functions analytically for *central* and *non-central* collisions [Floerchinger & Wiedemann (2014)]
- Long-wavelength modes (small $m$ and $l$) that don't resolve differences between point-like and extended sources have *universal statistics*.

# *Scaling with number of sources*

- Connected correlation functions (cumulants) scale with $N$ like [see also Ollitrault & Yan (2014), Bzdak & Skokov (2014)]

$$\langle w_{l_1}^{(m_1)} \cdots w_{l_n}^{(m_n)} \rangle_c \sim \frac{1}{N^{n-1}}$$

- scaling broken for non-central collisions
- $b$-dependence of term that break scaling is known



# *Fluid dynamic response*

# *Perturbative expansion*

Write the hydrodynamic fields $h = (w, u^\mu, \pi^{\mu\nu}, \pi_{\text{Bulk}}, \ldots)$

- at initial time $\tau_0$ as

$$h = h_0 + \epsilon\, h_1$$

  with background $h_0$, fluctuation part $\epsilon\, h_1$

- at later time $\tau > \tau_0$ as

$$h = h_0 + \epsilon\, h_1 + \epsilon^2 h_2 + \epsilon^3 h_3 + \ldots$$

Solve for time evolution in this scheme

- $h_0$ is solution of full, non-linear hydro equations in symmetric situation: azimuthal rotation and Bjorken boost invariant
- $h_1$ is solution of linearized hydro equations around $h_0$, can be solved mode-by-mode
- $h_2$ can be obtained by from interactions between modes etc.

# *Response to density perturbations*

For a single event

$$V_m^* = v_m e^{-i\,m\,\psi_m}$$
$$= \sum_l S_{(m)l}\, w_l^{(m)} + \sum_{\substack{m_1,m_2,\\ l_1,l_2}} S_{(m_1,m_2)l_1,l_2}\, w_{l_1}^{(m_1)}\, w_{l_2}^{(m_2)}\, \delta_{m,m_1+m_2} + \dots$$

- $S_{(m)l}$ is linear dynamic response function
- $S_{(m_1,m_2)l_1,l_2}$ is quadratic dynamic response function etc.
- Symmetries imply conservation of azimuthal wavenumber
- Response functions depend on thermodynamic and transport properties, in particular viscosity.

# *Flow correlations from initial density correlations*

Moments of flow coefficients

$$\left\langle V_{m_1}^* \cdots V_{m_n}^* \right\rangle = S_{(m_1)l_1} \cdots S_{(m_n)l_n} \left\langle w_{l_1}^{(m_1)} \cdots w_{l_n}^{(m_n)} \right\rangle$$

$$+ \text{ non-linear terms}$$

- combination of dynamical response coefficients and correlation functions of initial density perturbations
- linear, quadratic and higher-order terms
- For $N$ independent sources and central collisions

$$v_m\{n\}^n \sim \frac{1}{N^{n-1}}$$

  - holds also for extended sources
  - holds also for non-linear response contributions
  - can be extended to other conn. correlation functions, e. g. $\langle V_2 V_3 V_5^* \rangle$
  - gets broken for non-central collisions
  - impact parameter dependence of corrections is known

# *Scaling with system size*

- Large (PbPb) and small systems (pPb) have different number of independent sources $N$.
- For linear dynamics one has parametrically

$$v_m\{n\} \sim S_{(m)l} \frac{N^{1/n}}{N}.$$

- To have $v_m\{n\}|_{\mathsf{PbPb}} = v_m\{n\}|_{\mathsf{pPb}}$ one needs

$$\frac{S_{(m)l}|_{\mathsf{pPb}}}{S_{(m)l}|_{\mathsf{PbPb}}} = \left(\frac{N_{\mathsf{pPb}}}{N_{\mathsf{PbPb}}}\right)^{1-\frac{1}{n}} < 1.$$

  - Response function $S_{(m)l}$ must be smaller for pPb than for PbPb.
  - $S_{(m)l}$ is independent of $n$, above equation can be true only for one $n$.
  - $S_{(m)l}$ depends on system size only via initial background $w_{\mathsf{BG}}(r)$. Precise dependence can be investigated more closely.

*How response functions can be determined:*

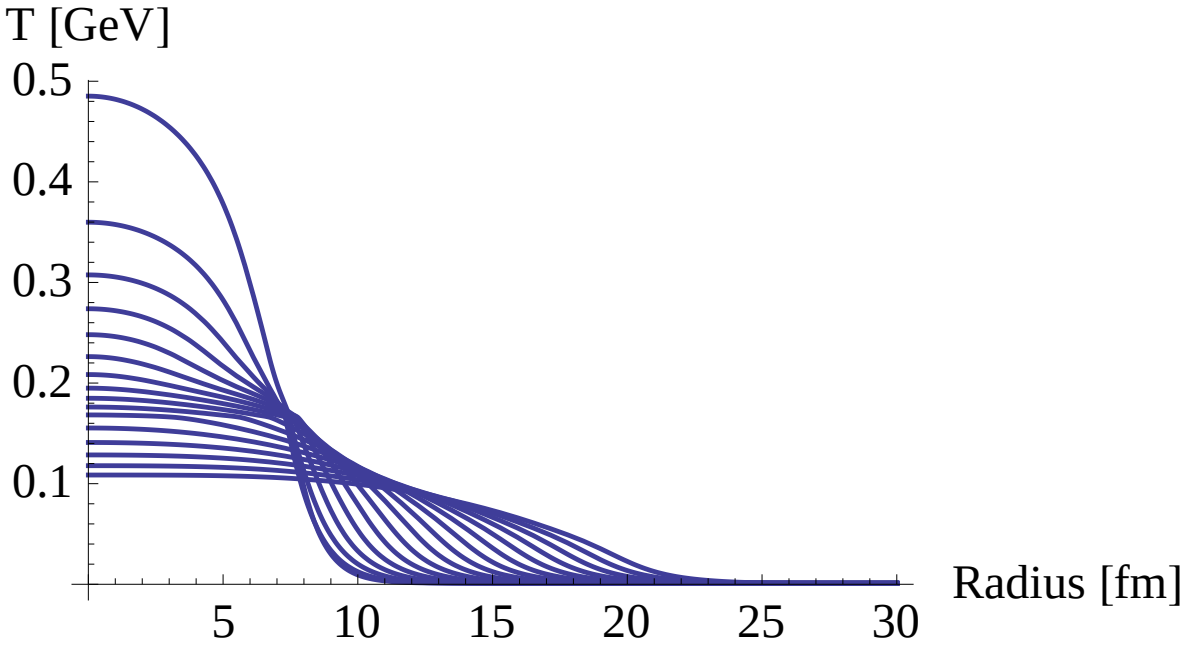
*1. Hydrodynamic evolution*

# *Background evolution*

System of coupled $1+1$ dimensional non-linear partial differential equations for

- enthalpy density $w(\tau, r)$ (or temperature $T(\tau, r)$)
- fluid velocity $u^\tau(\tau, r), u^r(\tau, r)$
- two independent components of shear stress $\pi^{\mu\nu}(\tau, r)$

Can be easily solved numerically

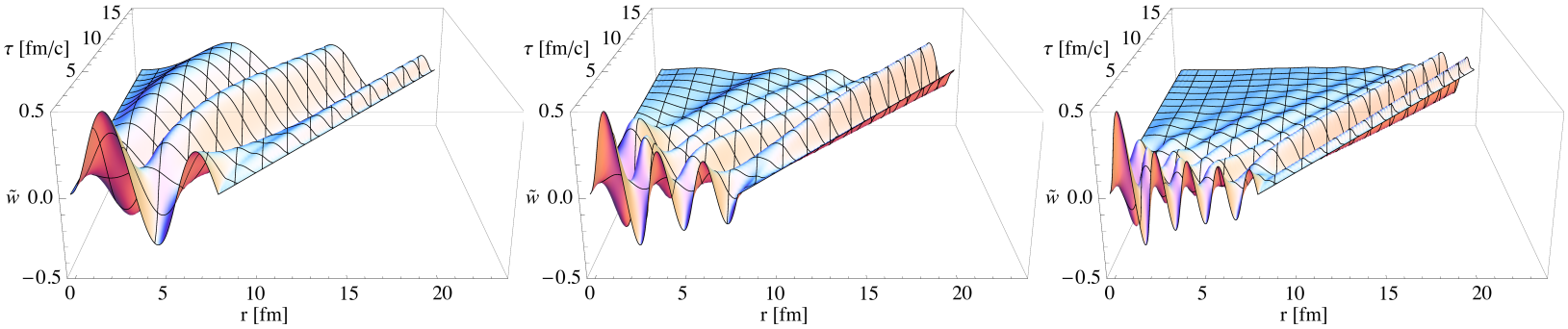


# *Evolving perturbation modes*

- Linearized hydro equations: set of coupled $3+1$ dimensional, linear, partial differential equations.
- Use Fourier expansion

$$h_j(\tau, r, \phi, \eta) = \sum_m \int \frac{dk_\eta}{2\pi}\, h_j^{(m)}(\tau, r, k_\eta)\, e^{i(m\phi + k_\eta \eta)}.$$

- Reduces to $1+1$ dimensions.
- Can be solved numerically for each initial Bessel-Fourier mode.



# *Mode interactions*

- Non-linear terms in the evolution equations lead to mode interactions.
- Quadratic and higher order in initial perturbations.
- Can be determined from iterative solution but has not been fully worked out yet.
- Convergence can be tested with numerical solution of full hydro equations.

# *Scaling tests*

- Start with single enthalpy density mode $(m = 2, l = 1)$ on top of background

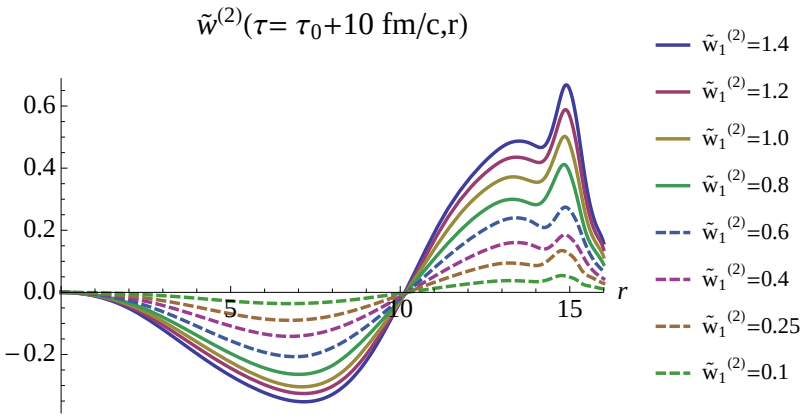
$$w(\tau_0, r, \phi) = w_{\mathsf{BG}}(\tau_0, r) \left[1 + 2\,\tilde{w}_1^{(2)} J_2(k_1^{(2)} r)\,\cos(2\phi)\right].$$

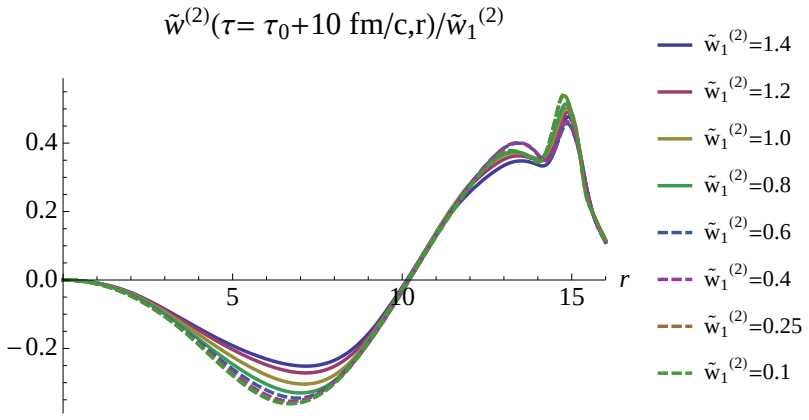
- Evolve this with hydro solver ECHO-QGP
  [Del Zanna *et al.*, EPJC 73, 2524 (2013), see also following talk]
- Determine Fourier components

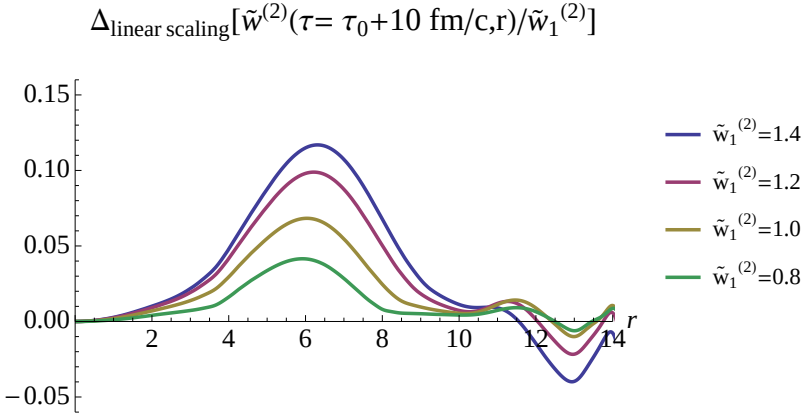
$$\tilde{w}^{(m)}(\tau, r) = \frac{1}{w_{\mathsf{BG}}(r)} \frac{1}{2\pi} \int d\phi\, e^{-im\phi}\, w(\tau, r, \phi)$$

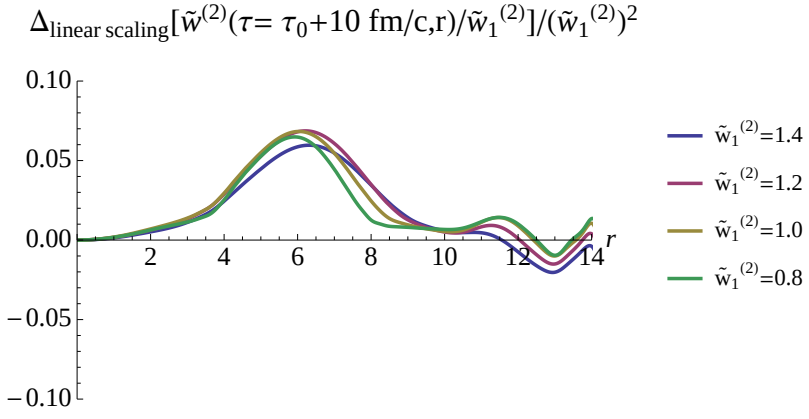
# *Scaling tests at first order*

Compare enthalpy $\tilde{w}^{(2)}(\tau, r)$ at fixed $\tau$ for different initial weights



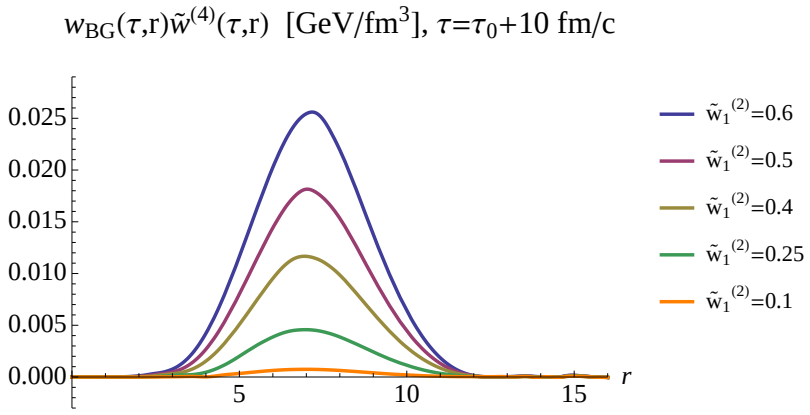



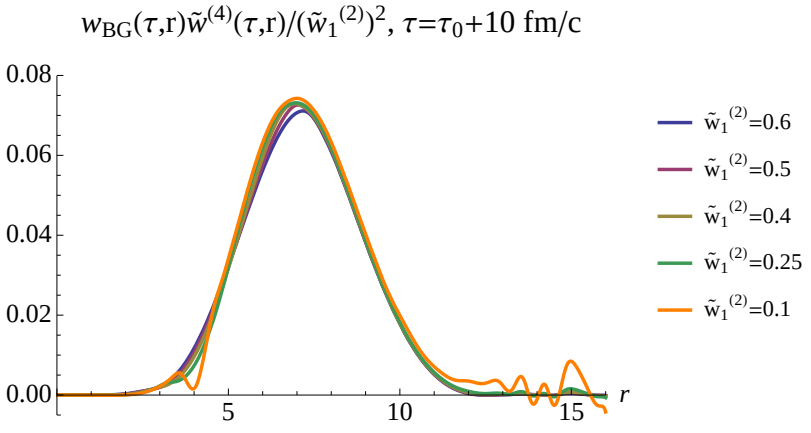



# *Scaling tests at second order*

From symmetry considerations one expects that modes with $m = 0$ and $m = 4$ receive mainly quadratic contributions $\sim \left(\tilde{w}_1^{(2)}\right)^2$.



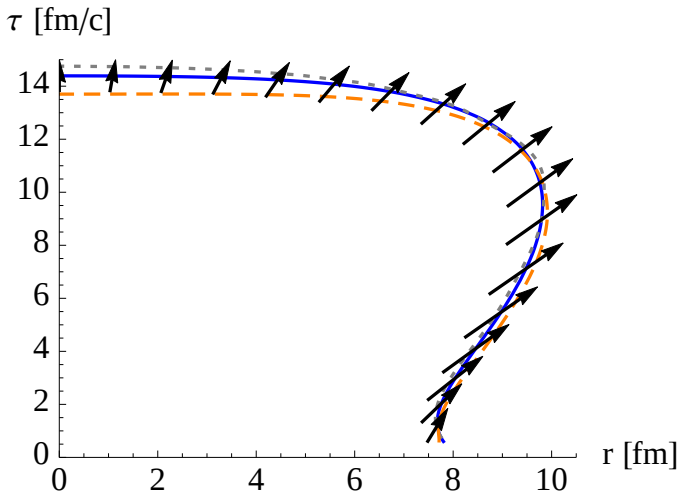

- Hydrodynamic response to initial enthalpy density fluctuations is perturbative.
- Non-linearities can be understood order-by-order and lead to characteristic "overtones".
- Results motivate more thorough development of fluid dynamic perturbation theory.

*How response functions can be determined:*

*2. Kinetic freeze-out*

# *Freeze-out surface*

- Perturbative expansion can be used also at freeze-out. [Floerchinger, Wiedemann 2013]
- Freeze-out surface is azimuthally symmetric as background.
- Generalization to kinetic hadronic scattering and decay phase possible.




(solid: $\eta/s = 0.08$, dotted: $\eta/s = 0$, dashed: $\eta/s = 0.3$)

# *Particle distribution*

for single event

$$\ln\left(\frac{dN^{\text{single event}}}{p_T dp_T d\phi dy}\right) = \underbrace{\ln S_0(p_T)}_{\text{from background}} + \underbrace{\sum_{m,l} w_l^{(m)} e^{im\phi} \theta_l^{(m)}(p_T)}_{\text{from fluctuations}}$$

- each mode comes with an angle, $w_l^{(m)} = |w_l^{(m)}|\, e^{im\psi_l^{(m)}}$
- each mode has different $p_T$-dependence, $\theta_l^{(m)}(p_T)$
- quadratic order correction

$$\sum_{m_1,m_2,l_1,l_2} w_{l_1}^{(m_1)} w_{l_2}^{(m_2)}\; e^{i(m_1+m_2)\phi}\; \kappa_{l_1,l_2}^{(m_1,m_2)}(p_T)$$

- non-linearities from hydro evolution and freeze-out

# *Harmonic flow coefficients*

Double differential harmonic flow coefficient (to lowest order)

$$v_m\{2\}^2(p_T^a, p_T^b) = \sum_{l_1,l_2} \theta_{l_1}^{(m)}(p_T^a)\ \theta_{l_2}^{(m)}(p_T^b)\ \langle w_{l_1}^{(m)} w_{l_2}^{(m)*} \rangle$$

- intuitive matrix expression
- in general no factorization
- can be generalized to higher order flow cumulants

# *Summary and Conclusions*

# *Conclusions*

- Systematic expansion in initial fluid perturbations seems possible (good convergence properties) and very useful.
- Formalism works in praxis (see backup slides for results of "proof of principle" study).
- Initial density perturbations have some universal properties that can help to better constrain thermodynamic and transport properties.
- Fluid dynamic response allows to access correlation functions of initial perturbations.

*Backup*

# *Characterization of transverse density via eccentricities*

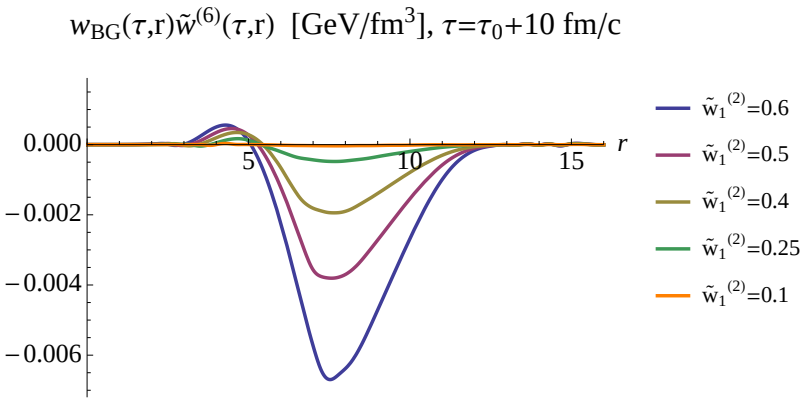
Fluctuations in initial transverse enthalpy density $w(r,\phi)$ can be characterized in terms of eccentricities $\epsilon_{n,m}$ and angles $\psi_{n,m}$ [Ollitrault, Teaney, Yan, Luzum, and others]

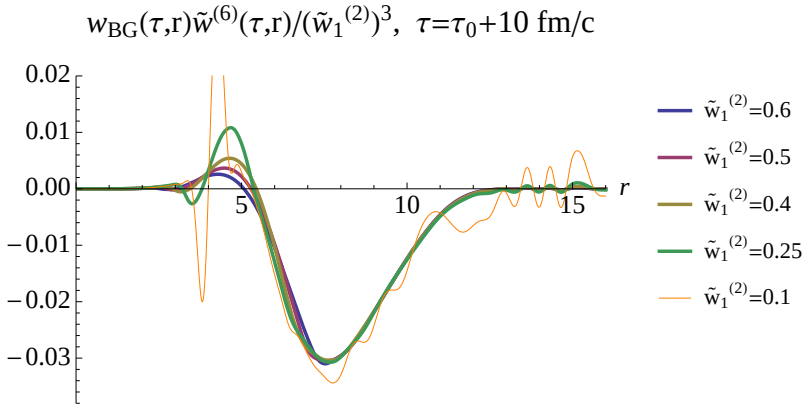
$$\epsilon_{n,m}\, e^{im\,\psi_{n,m}} = \frac{\int dr \int_0^{2\pi} d\varphi\, r^{n+1}\, e^{im\varphi}\, w(r,\varphi)}{\int dr \int_0^{2\pi} d\varphi\, r^{n+1}\, w(r,\varphi)}$$

- $w(r,\phi)$ completely determined by set of all $\epsilon_{n,m}$ and $\psi_{n,m}$
- closely related method is based on cumulants [Teaney, Yan]
- no positive transverse density can be associated to small set of cumulants (beyond Gaussian order) such that higher order cumulants vanish
- generalization to velocity and shear fluctuations not known

# *Scaling tests at third order*

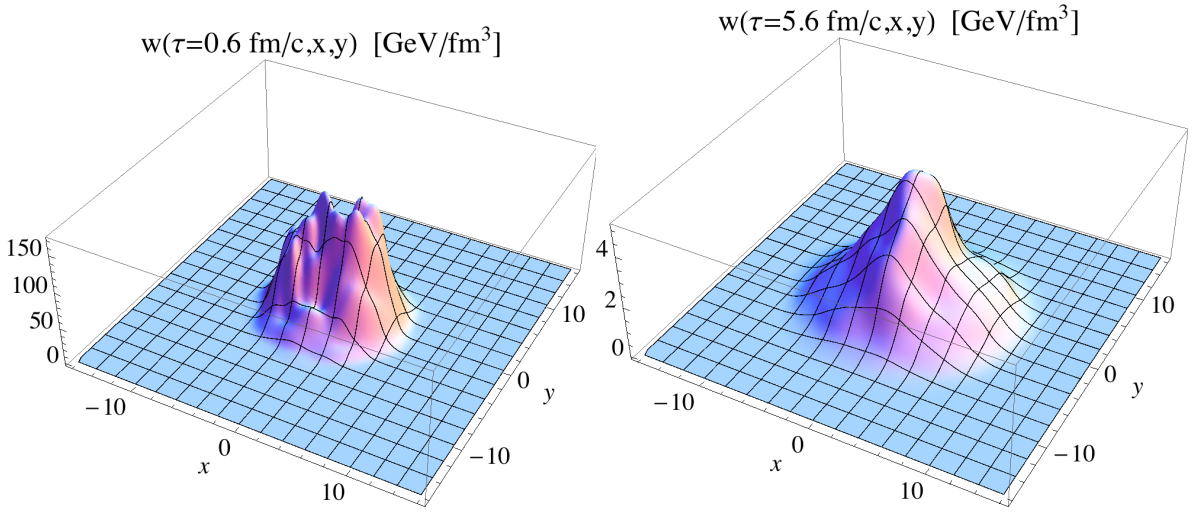
From symmetry considerations one expects that modes $m = 6$ receive mainly cubic contributions $\sim \left(\tilde{w}_1^{(2)}\right)^3$.

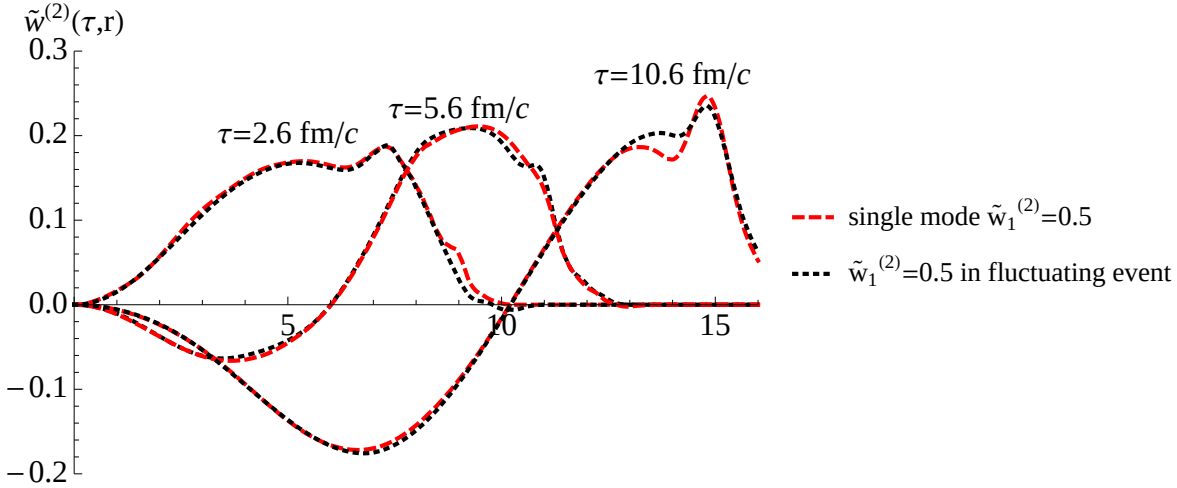


# *Scaling tests embedded in realistic event*

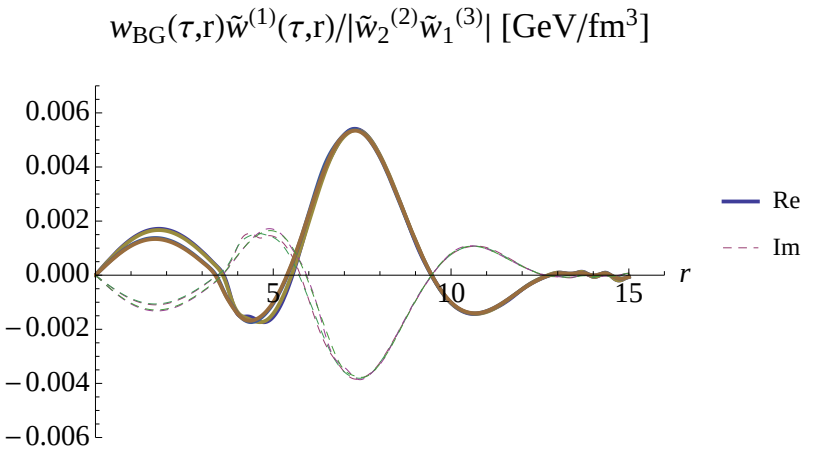
Embed mode $(m = 2, l = 1)$ into realistic fluctuating event and compare to embedding into pure background.

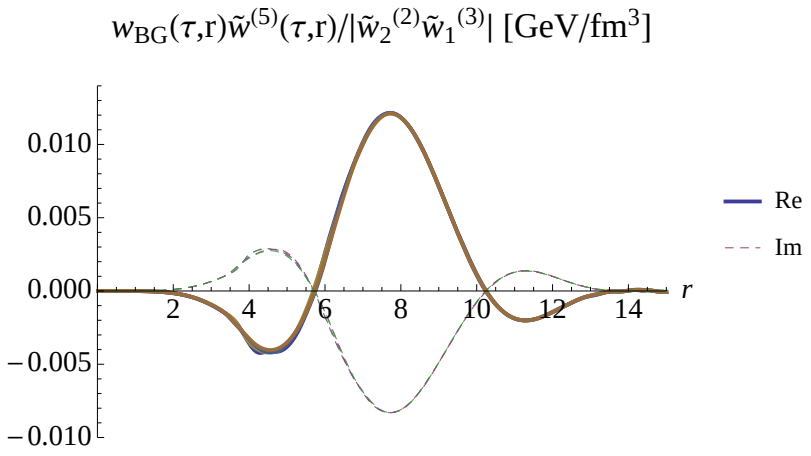


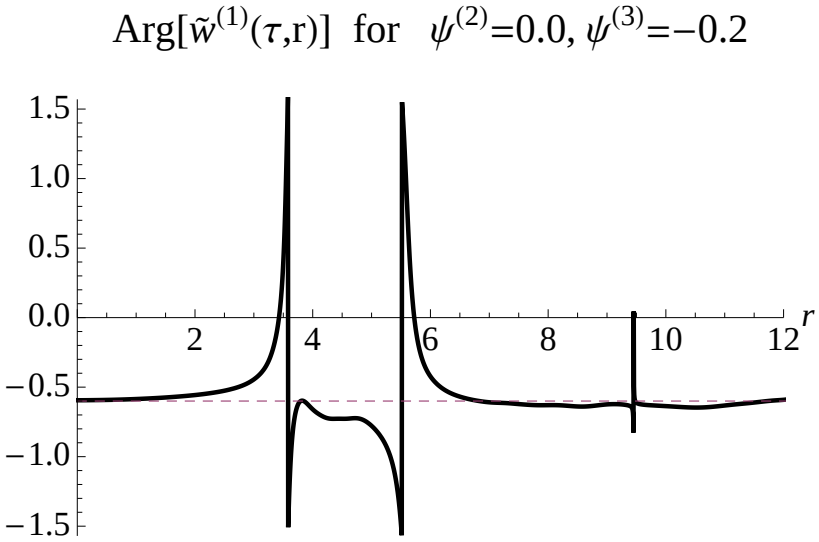



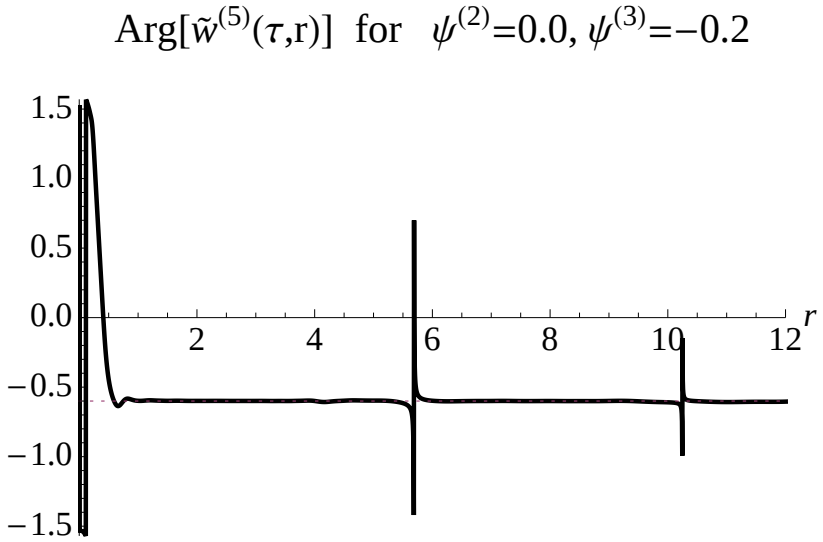
# *Scaling tests with several initial modes*

Start with linear combination of $(m = 2, l = 2)$ and $(m = 3, l = 1)$ modes and test scaling for $m = 1$ and $m = 5$ response.



# *Generalized Glauber model*

- Fluctuations due to nucleon positions: used so far

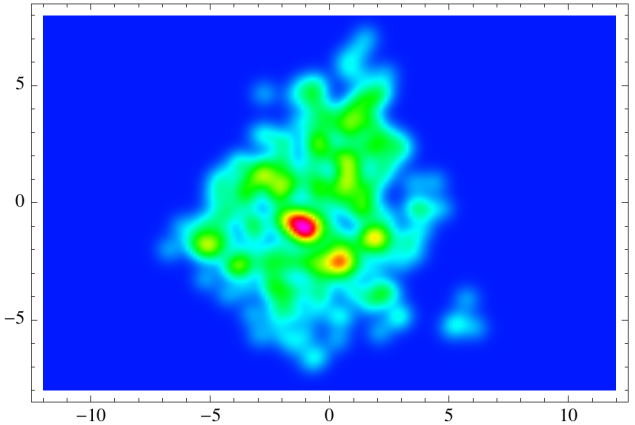
$$\epsilon(\tau, \mathbf{x}, y) = \sum_{i=1}^{N_{\text{part}}} \epsilon_w(\tau, \mathbf{x} - \mathbf{x}_i, y), \qquad u^\mu = (1, 0, 0, 0)$$

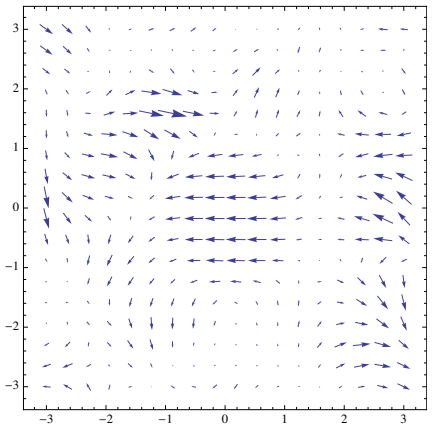
- can be generalized to include also velocity fluctuations

$$T^{\mu\nu}(\tau, \mathbf{x}, y) = \sum_{i=1}^{N_{\text{part}}} T_w^{\mu\nu}(\tau, \mathbf{x} - \mathbf{x}_i, y)$$

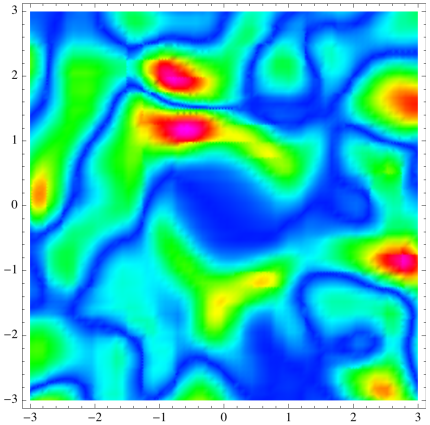
- More generally describe primordial fluid fields by
  - expectation values $\langle \epsilon(\tau_0, \mathbf{x}, y) \rangle, \langle u^\mu(\tau_0, \mathbf{x}, y) \rangle, \langle n_B(\tau_0, \mathbf{x}, y) \rangle$
  - correlation functions $\langle \epsilon(\tau_0, \mathbf{x}, y)\, \epsilon(\tau_0, \mathbf{x}', y') \rangle$, etc.
- Origin of this fluctuations is initial state physics and early-time, non-equilibrium dynamics.

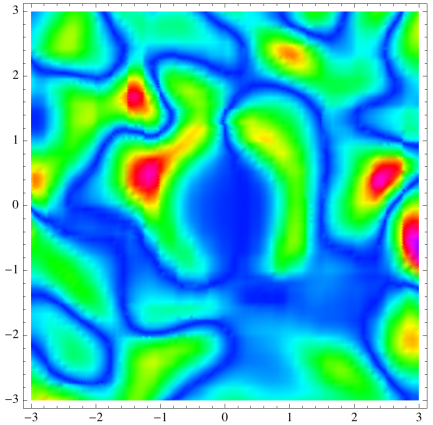
# *Velocity fluctuations*




- also the velocity field will fluctuate at the initialization time $\tau_0$
- take here transverse velocity for every participant to be Gaussian distributed with width $0.1c$
- vorticity $|\partial_1 u^2 - \partial_2 u^1|$ and divergence $|\partial_1 u^1 + \partial_2 u^2|$

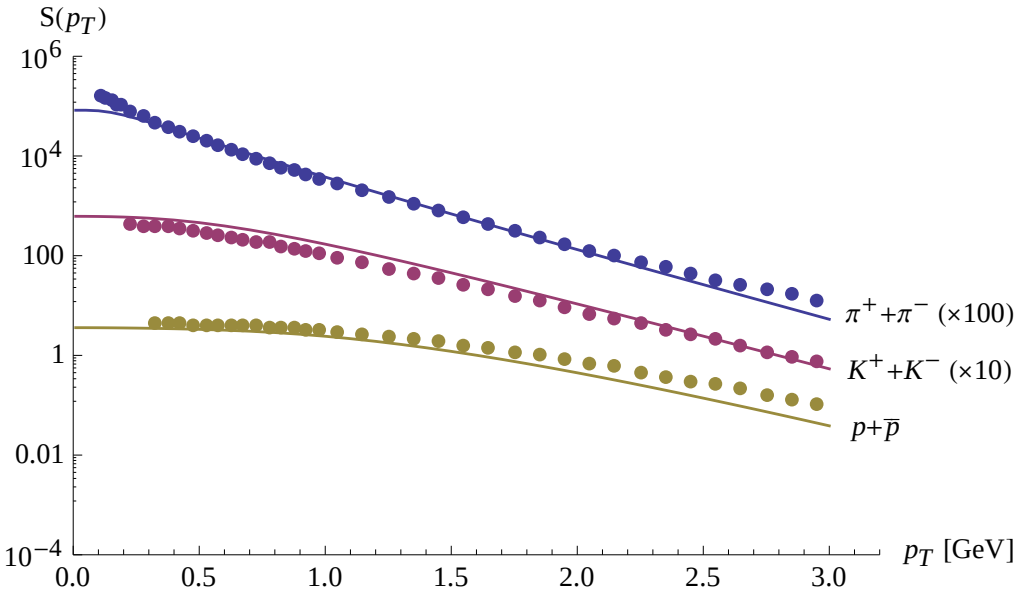


# *"Proof of principle" study: One-particle spectrum*

Initial conditions from Glauber Monte Carlo Model

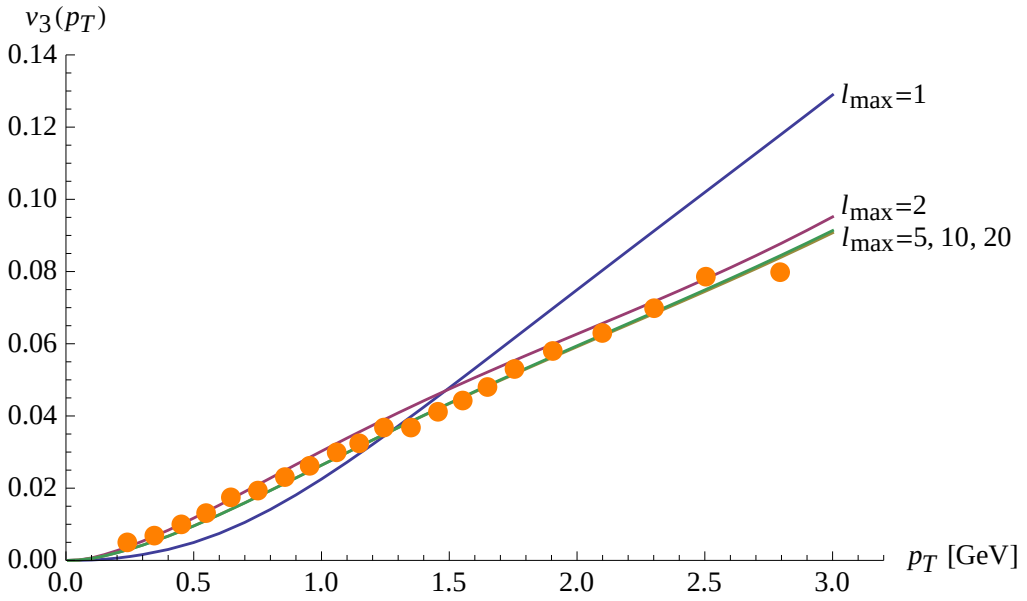
$$S(p_T) = dN/(2\pi p_T dp_T d\eta d\phi)$$




Points: 5% most central collisions, ALICE [PRL 109, 252301 (2012)]
Curves: Our calculation, no hadron rescattering and decays after freeze-out.

# *Harmonic flow coefficients for central collisions*

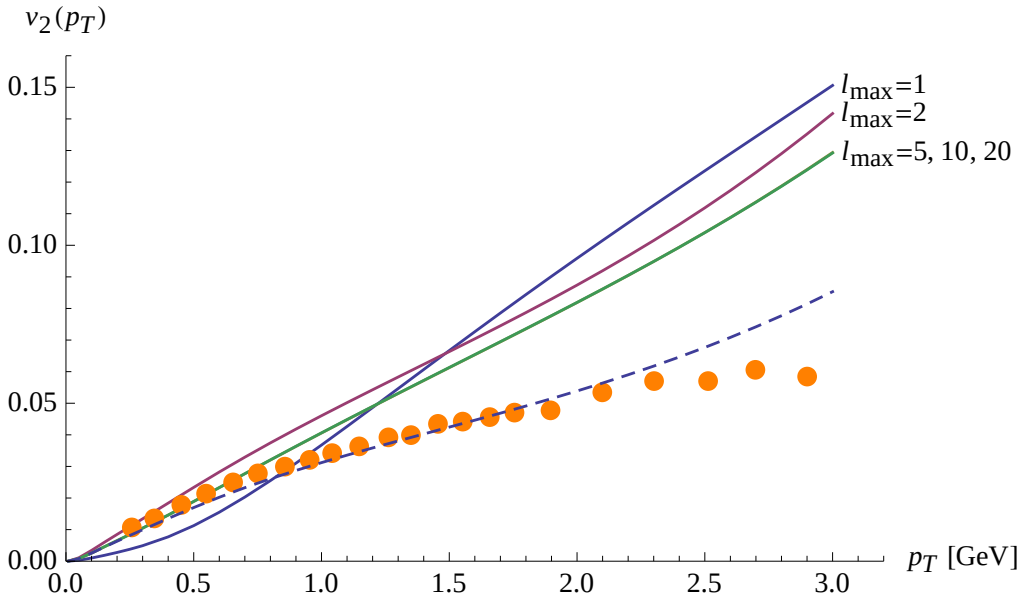
Triangular flow for charged particles




Points: 2% most central collisions, ALICE [PRL 107, 032301 (2011)]
Curves: Different maximal resolution $l_{\max}$

# *Harmonic flow coefficients for central collisions*

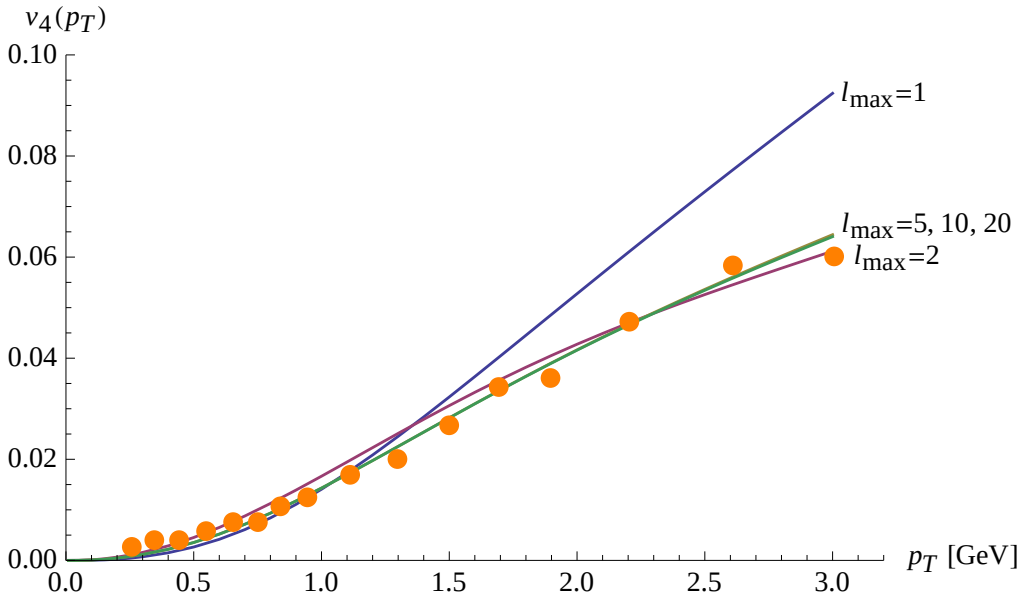
Elliptic flow for charged particles




Points: 2% most central collisions, ALICE [PRL 107, 032301 (2011)]
Solid curves: Different maximal resolution $l_{\text{max}}$
Dashed curve: Mode $(m = 2, l = 1)$ suppressed by factor 0.7

# *Harmonic flow coefficients for central collisions*

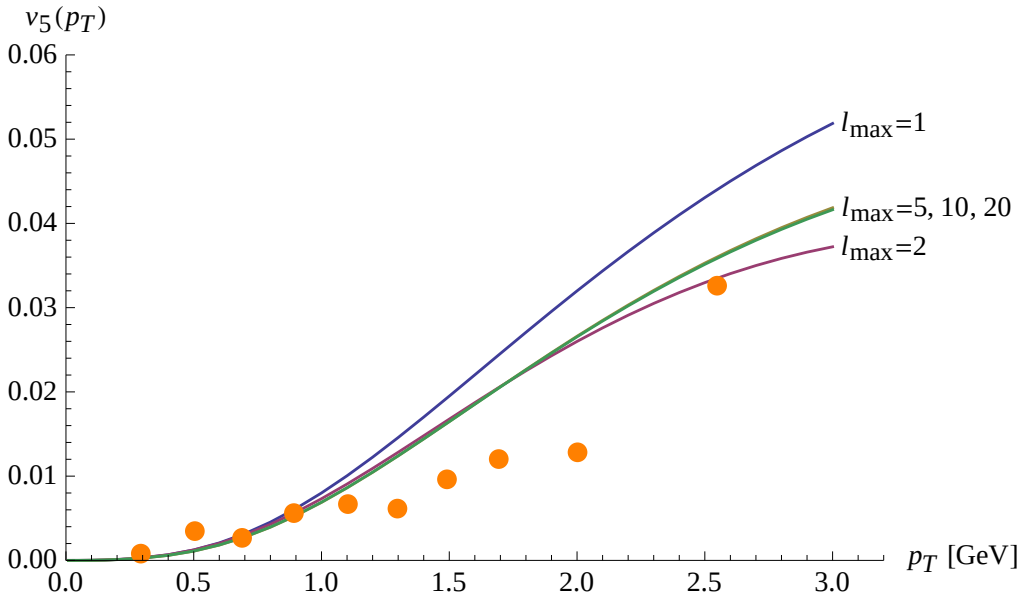
Flow coefficient $v_4$ for charged particles




Points: 2% most central collisions, ALICE [PRL 107, 032301 (2011)]
Curves: Different maximal resolution $l_{\max}$

# *Harmonic flow coefficients for central collisions*

Flow coefficient $v_5$ for charged particles




Points: 2% most central collisions, ALICE [PRL 107, 032301 (2011)]
Curves: Different maximal resolution $l_{\max}$

# *Harmonic flow coefficients, central, particle identified*