A perturbative approach to hydrodynamics

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based on

- Fluctuations around Bjorken Flow and the onset of turbulent phenomena, [JHEP 11, 100 (2011), with U. A. Wiedemann]
- How (non-) linear is the hydrodynamics of heavy ion collisions? [arXiv:1312.5482, with U. A. Wiedemann, A. Beraudo, L. Del Zanna, G. Inghirami, V. Rolando]
- Mode-by-mode hydrodynamics: ideas and concepts [arXiv:1401.2339]
- Statistics of initial density perturbations in heavy ion collisions and their fluid dynamic response [arXiv:1405.4393, with U. A. Wiedemann]
Introduction
What perturbations are interesting and why?

- **Initial fluid perturbations**: Event-by-event fluctuations around a background or average of fluid fields at time $\tau_0$:
  - energy density $\epsilon$
  - fluid velocity $u^\mu$
  - shear stress $\pi^{\mu\nu}$
  - more general also: baryon number density $n_B$, electric charge density, electromagnetic fields, ...

- governed by universal evolution equations
- can be used to constrain **thermodynamic and transport properties**
- contain interesting information from early times
A program to understand fluid perturbations

- Characterize initial perturbations.
- Propagated them through fluid dynamic regime.
- Determine influence on particle spectra and harmonic flow coefficients.
- Take also perturbations from non-hydro sources (jets) into account. [see talk of K. Zapp]
Characterization of initial conditions
Transverse enthalpy density

Based on Bessel-Fourier expansion and background density
[Floerchinger & Wiedemann 2013, see also Coleman-Smith, Petersen & Wolpert 2012, Floerchinger & Wiedemann 2014]

\[ w(r, \phi) = w_{BG}(r) + w_{BG}(r) \sum_{m,l} w_{l}(m) e^{im\phi} J_{m} \left( z_{l}(m) \rho(r) \right) \]

- azimuthal wavenumber $m$, radial wavenumber $l$
- $w_{l}(m)$ dimensionless
- higher $m$ and $l$ correspond to finer spatial resolution
- coefficients $w_{l}(m)$ can be related to eccentricienies
- works similar for vectors (velocity) and tensors (shear stress)
Transverse density from Glauber model
Event ensembles

- Event-by-event probability distribution

\[ p_{T_0}[w, u^\mu, \pi^{\mu\nu}, \ldots] \]

- Moments / correlation functions

\[ \langle w_{l_1}^{(m_1)} w_{l_2}^{(m_2)} \ldots w_{l_n}^{(m_n)} \rangle \]

- contain information from initial state physics / early dynamics
- universal (model independent) properties would be nice to have
- same information in cumulants (connected correlation functions)
Statistics of initial density perturbations

Independent point-sources model (IPSM)

\[
    w(\vec{x}) = \left[ \frac{1}{\tau_0} \frac{dW_{BG}}{d\eta} \right] \frac{1}{N} \sum_{j=1}^{N} \delta^{(2)}(\vec{x} - \vec{x}_j)
\]

- random positions \(\vec{x}_j\), independent and identically distributed
- probability distribution \(p(\vec{x}_j)\) reflects collision geometry
- possible to determine correlation functions analytically for central and non-central collisions [Floerchinger & Wiedemann (2014)]
- Long-wavelength modes (small \(m\) and \(l\)) that don’t resolve differences between point-like and extended sources have universal statistics.
**Scaling with number of sources**

- Connected correlation functions (cumulants) scale with $N$ like
  
  $\langle w_{l_1}^{(m_1)} \cdots w_{l_n}^{(m_n)} \rangle_c \sim \frac{1}{N^{n-1}}$

- scaling broken for non-central collisions
- $b$-dependence of term that break scaling is known

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<table>
<thead>
<tr>
<th>Moments or correlation functions</th>
<th>Cumulants or connected correlation functions</th>
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<tr>
<td>[\langle \cdots \rangle] [\ln Z[j]] [\exp] [\langle \cdots \rangle_c] [\ln Z[j]] [\exp]</td>
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**Fixed reaction plane angle $\phi_R$**

**Random reaction plane angle $\phi_R$**
Fluid dynamic response
**Perturbative expansion**

Write the hydrodynamic fields \( h = (w, u^\mu, \pi^{\mu\nu}, \pi_{\text{Bulk}}, \ldots) \)

- at initial time \( \tau_0 \) as
  \[
  h = h_0 + \epsilon h_1
  \]
  with background \( h_0 \), fluctuation part \( \epsilon h_1 \)

- at later time \( \tau > \tau_0 \) as
  \[
  h = h_0 + \epsilon h_1 + \epsilon^2 h_2 + \epsilon^3 h_3 + \ldots
  \]

Solve for time evolution in this scheme

- \( h_0 \) is solution of full, non-linear hydro equations in symmetric situation: azimuthal rotation and Bjorken boost invariant

- \( h_1 \) is solution of linearized hydro equations around \( h_0 \), can be solved mode-by-mode

- \( h_2 \) can be obtained by from interactions between modes etc.
Response to density perturbations

For a single event

\[ V^*_m = v_m e^{-i m \psi_m} \]

\[ = \sum_l S_m(m) w_l^{(m)} + \sum_{m_1, m_2, l_1, l_2} S_{m_1, m_2, l_1, l_2} w_{l_1}^{(m_1)} w_{l_2}^{(m_2)} \delta_{m, m_1 + m_2} + \ldots \]

- \( S_m(m) \) is linear dynamic response function
- \( S_{m_1, m_2, l_1, l_2} \) is quadratic dynamic response function etc.
- Symmetries imply conservation of azimuthal wavenumber
- Response functions depend on thermodynamic and transport properties, in particular viscosity.
Flow correlations from initial density correlations

Moments of flow coefficients

\[
\langle V_{m_1}^* \cdots V_{m_n}^* \rangle = S(m_1)l_1 \cdots S(m_n)l_n \langle w_{l_1}^{(m_1)} \cdots w_{l_n}^{(m_n)} \rangle + \text{non-linear terms}
\]

- combination of dynamical response coefficients and correlation functions of initial density perturbations
- linear, quadratic and higher-order terms
- For \( N \) independent sources and central collisions

\[
v_m \{n\}^n \sim \frac{1}{N^{n-1}}
\]

- holds also for extended sources
- holds also for non-linear response contributions
- can be extended to other conn. correlation functions, e.g. \( \langle V_2 V_3 V_5^* \rangle \)
- gets broken for non-central collisions
- impact parameter dependence of corrections is known
Scaling with system size

- Large (PbPb) and small systems (pPb) have different number of independent sources $N$.
- For linear dynamics one has parametrically

$$v_m\{n\} \sim S_{(m)l} \frac{N^{1/n}}{N}.$$ 

- To have $v_m\{n\}|_{\text{PbPb}} = v_m\{n\}|_{\text{pPb}}$ one needs

$$\frac{S_{(m)l}|_{\text{pPb}}}{S_{(m)l}|_{\text{PbPb}}} = \left(\frac{N_{\text{pPb}}}{N_{\text{PbPb}}}\right)^{1-\frac{1}{n}} < 1.$$ 

- Response function $S_{(m)l}$ must be smaller for pPb than for PbPb.
- $S_{(m)l}$ is independent of $n$, above equation can be true only for one $n$.
- $S_{(m)l}$ depends on system size only via initial background $w_{BG}(r)$.
  Precise dependence can be investigated more closely.
How response functions can be determined:
1. Hydrodynamic evolution
Background evolution

System of coupled $1 + 1$ dimensional non-linear partial differential equations for

- enthalpy density $w(\tau, r)$ (or temperature $T(\tau, r)$)
- fluid velocity $u^\tau(\tau, r), u^r(\tau, r)$
- two independent components of shear stress $\pi^{\mu\nu}(\tau, r)$

Can be easily solved numerically
Evolving perturbation modes

- Linearized hydro equations: set of coupled $3 + 1$ dimensional, linear, partial differential equations.
- Use Fourier expansion

$$h_j(\tau, r, \phi, \eta) = \sum_m \int \frac{dk_\eta}{2\pi} h_j^{(m)}(\tau, r, k_\eta) e^{i(m\phi + k_\eta \eta)}.$$ 

- Reduces to $1 + 1$ dimensions.
- Can be solved numerically for each initial Bessel-Fourier mode.
Mode interactions

- Non-linear terms in the evolution equations lead to mode interactions.
- Quadratic and higher order in initial perturbations.
- Can be determined from iterative solution but has not been fully worked out yet.
- Convergence can be tested with numerical solution of full hydro equations.
**Scaling tests**

- Start with single enthalpy density mode \((m = 2, l = 1)\) on top of background

\[
w(\tau_0, r, \phi) = w_{\text{BG}}(\tau_0, r) \left[1 + 2 \hat{w}_1^{(2)} J_2(k_1^{(2)} r) \cos(2\phi)\right].
\]

- Evolve this with hydro solver ECHO-QGP
  [Del Zanna *et al.*, EPJC 73, 2524 (2013), see also following talk]

- Determine Fourier components

\[
\hat{w}^{(m)}(\tau, r) = \frac{1}{w_{\text{BG}}(r)} \frac{1}{2\pi} \int d\phi \ e^{-im\phi} \ w(\tau, r, \phi)
\]
Scaling tests at first order

Compare enthalpy $\tilde{w}^{(2)}(\tau, r)$ at fixed $\tau$ for different initial weights

\[ \tilde{w}^{(2)}(\tau = \tau_0 + 10 \text{ fm/c}, r) \]

\[ \Delta_{\text{linear scaling}}[\tilde{w}^{(2)}(\tau = \tau_0 + 10 \text{ fm/c}, r)/\tilde{w}_1^{(2)}] \]
Scaling tests at second order

From symmetry considerations one expects that modes with \( m = 0 \) and \( m = 4 \) receive mainly quadratic contributions \( \sim (\tilde{\omega}_1^{(2)})^2 \).

- Hydrodynamic response to initial enthalpy density fluctuations is perturbative.
- Non-linearities can be understood order-by-order and lead to characteristic “overtones”.
- Results motivate more thorough development of fluid dynamic perturbation theory.
How response functions can be determined:

2. Kinetic freeze-out
Freeze-out surface

- Perturbative expansion can be used also at freeze-out. [Floerchinger, Wiedemann 2013]
- Freeze-out surface is azimuthally symmetric as background.
- Generalization to kinetic hadronic scattering and decay phase possible.

(solid: $\eta/s = 0.08$, dotted: $\eta/s = 0$, dashed: $\eta/s = 0.3$)
Particle distribution

for single event

\[
\ln \left( \frac{dN_{\text{single event}}}{p_T dp_T d\phi dy} \right) = \ln S_0(p_T) + \sum_{m,l} w_l^{(m)} e^{im\phi} \theta_l^{(m)}(p_T)
\]

- each mode comes with an angle, \( w_l^{(m)} = |w_l^{(m)}| e^{im\psi_l^{(m)}} \)
- each mode has different \( p_T \)-dependence, \( \theta_l^{(m)}(p_T) \)
- quadratic order correction

\[
\sum_{m_1,m_2,l_1,l_2} w_{l_1}^{(m_1)} w_{l_2}^{(m_2)} e^{i(m_1+m_2)\phi} K_{l_1,l_2}^{(m_1,m_2)}(p_T)
\]

- non-linearities from hydro evolution and freeze-out
Harmonic flow coefficients

Double differential harmonic flow coefficient (to lowest order)

\[ v_m \{2\}^2(p_T^a, p_T^b) = \sum_{l_1, l_2} \theta^{(m)}_{l_1}(p_T^a) \theta^{(m)}_{l_2}(p_T^b) \langle w^{(m)}_{l_1} w^{(m)*}_{l_2} \rangle \]

- intuitive matrix expression
- in general no factorization
- can be generalized to higher order flow cumulants
Summary and Conclusions
Conclusions

- Systematic expansion in initial fluid perturbations seems possible (good convergence properties) and very useful.
- Formalism works in praxis (see backup slides for results of “proof of principle” study).
- Initial density perturbations have some universal properties that can help to better constrain thermodynamic and transport properties.
- Fluid dynamic response allows to access correlation functions of initial perturbations.
Backup
Characterization of transverse density via eccentricities

Fluctuations in initial transverse enthalpy density $w(r, \phi)$ can be characterized in terms of eccentricities $\epsilon_{n,m}$ and angles $\psi_{n,m}$ [Ollitrault, Teaney, Yan, Luzum, and others]

$$\epsilon_{n,m} e^{im \psi_{n,m}} = \frac{\int dr \int_0^{2\pi} d\phi r^{n+1} e^{im\varphi} w(r, \varphi)}{\int dr \int_0^{2\pi} d\phi r^{n+1} w(r, \varphi)}$$

- $w(r, \phi)$ completely determined by set of all $\epsilon_{n,m}$ and $\psi_{n,m}$
- closely related method is based on cumulants [Teaney, Yan]
- no positive transverse density can be associated to small set of cumulants (beyond Gaussian order) such that higher order cumulants vanish
- generalization to velocity and shear fluctuations not known
Scaling tests at third order

From symmetry considerations one expects that modes $m = 6$ receive mainly cubic contributions $\sim (\tilde{w}_1^{(2)})^3$. 

$w_{BG}(\tau,r)\tilde{w}^{(6)}(\tau,r)$ [GeV/fm$^3$], $\tau=\tau_0+10$ fm/c

$w_{BG}(\tau,r)\tilde{w}^{(6)}(\tau,r)/(\tilde{w}_1^{(2)})^3$, $\tau=\tau_0+10$ fm/c
Scaling tests embedded in realistic event

Embed mode \((m = 2, l = 1)\) into realistic fluctuating event and compare to embedding into pure background.
Scaling tests with several initial modes

Start with linear combination of \((m = 2, l = 2)\) and \((m = 3, l = 1)\) modes and test scaling for \(m = 1\) and \(m = 5\) response.
**Generalized Glauber model**

- Fluctuations due to nucleon positions: used so far
  
  $$\epsilon(\tau, \mathbf{x}, y) = \sum_{i=1}^{N_{\text{part}}} \epsilon_w(\tau, \mathbf{x} - \mathbf{x}_i, y), \quad u^\mu = (1, 0, 0, 0)$$

- can be generalized to include also velocity fluctuations
  
  $$T^{\mu\nu}(\tau, \mathbf{x}, y) = \sum_{i=1}^{N_{\text{part}}} T^{\mu\nu}_w(\tau, \mathbf{x} - \mathbf{x}_i, y)$$

- More generally describe primordial fluid fields by
  - expectation values $$\langle \epsilon(\tau_0, \mathbf{x}, y) \rangle, \langle u^\mu(\tau_0, \mathbf{x}, y) \rangle, \langle n_B(\tau_0, \mathbf{x}, y) \rangle$$
  - correlation functions $$\langle \epsilon(\tau_0, \mathbf{x}, y) \epsilon(\tau_0, \mathbf{x}', y') \rangle$$, etc.

- Origin of this fluctuations is initial state physics and early-time, non-equilibrium dynamics.
also the velocity field will fluctuate at the initialization time $\tau_0$
- take here transverse velocity for every participant to be Gaussian distributed with width $0.1c$
- vorticity $|\partial_1 u^2 - \partial_2 u^1|$ and divergence $|\partial_1 u^1 + \partial_2 u^2|$
“Proof of principle” study: One-particle spectrum

Initial conditions from Glauber Monte Carlo Model

\[ S(p_T) = \frac{dN}{(2\pi p_T dp_T d\eta d\phi)} \]

Points: 5% most central collisions, ALICE [PRL 109, 252301 (2012)]
Curves: Our calculation, no hadron rescattering and decays after freeze-out.
Harmonic flow coefficients for central collisions

Triangular flow for charged particles

Points: 2% most central collisions, ALICE [PRL 107, 032301 (2011)]
Curves: Different maximal resolution $l_{\text{max}}$
Harmonic flow coefficients for central collisions

Elliptic flow for charged particles

Points: 2% most central collisions, ALICE [PRL 107, 032301 (2011)]
Solid curves: Different maximal resolution $l_{\text{max}}$
Dashed curve: Mode $(m = 2, l = 1)$ suppressed by factor 0.7
Harmonic flow coefficients for central collisions

Flow coefficient $v_4$ for charged particles

Points: 2% most central collisions, ALICE [PRL 107, 032301 (2011)]
Curves: Different maximal resolution $l_{\text{max}}$
Harmonic flow coefficients for central collisions

Flow coefficient $v_5$ for charged particles

Points: 2% most central collisions, ALICE [PRL 107, 032301 (2011)]
Curves: Different maximal resolution $l_{\text{max}}$
Harmonic flow coefficients, central, particle identified

![Graphs showing $v_2(p_T)$, $v_3(p_T)$, $v_4(p_T)$, and $v_5(p_T)$ for various particles like $p + p$, $K^+ + K^-$, $\pi^+ + \pi^-$, as functions of $p_T$. Each graph plots $p_T$ in GeV on the x-axis and harmonic flow coefficients on the y-axis.]