Ghosts in Keldysh-Schwinger formalism

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1 Introduction

In field theories obeying a gauge symmetry the number of fields exceeds the number of physical degrees of freedom. To get rid of unphysical degrees of freedom in a manifestly Lorentz covariant way, one introduces the fictitious fields known as Faddeev-Popov ghosts which play a crucial role in nonAbelian field theories where unphysical degrees of freedom interact with physical ones. Here we show how to introduce ghosts into the Keldysh-Schwinger formalism and find forms of free ghosts Green's functions. We consider a system of quarks and gluons which is, in general, out of equilibrium but the system is assumed to be translationally invariant. It is thus homogeneous (in coordinate space) but the momentum distribution is arbitrary. In particular, the system can be strongly anisotropic. The translational invariance greatly simplifies our analysis, as each two-point function depends on its two arguments only through their difference.

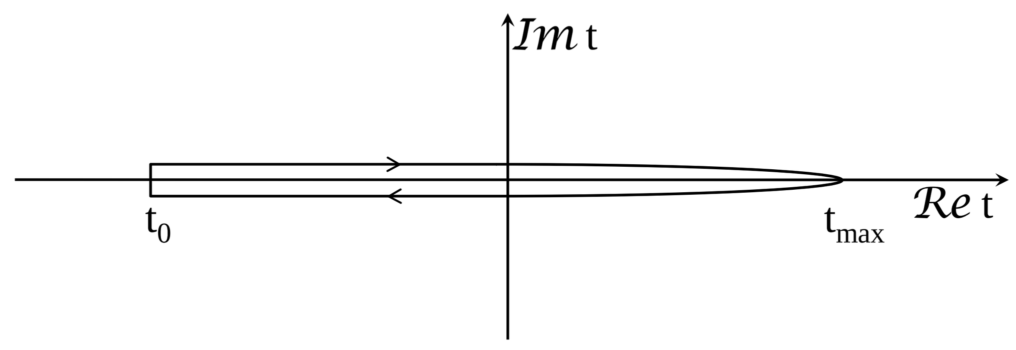
The complete analysis of the problem is presented in [1].

2 Keldysh-Schwinger formalism

The main object of the Keldysh-Schwinger approach is the contour-ordered Green's function

$$i\mathcal{D}^{ab}_{\mu\nu}(x,y) \stackrel{\rm def}{=} \frac{{\rm Tr}\big[\rho(t_0)\,\tilde{T} A^a_\mu(x) A^b_\nu(y)\big]}{{\rm Tr}[\rho(t_0)]},$$

where the time arguments are located on the Keldysh contour



The contour Green's function carries information about microscopic interactions in the system under consideration and its statistical properties. It involves four Green's functions with real time arguments $\mathcal{D}^c$, $\mathcal{D}^a$, $\mathcal{D}^>$, $\mathcal{D}^<$.

- $\mathcal{D}^c$ describes a particle disturbance propagating forward in time, and an antiparticle disturbance propagating backward in time,
- $\mathcal{D}^a$ describes a particle disturbance propagating backward in time, and an antiparticle disturbance propagating forward in time,
- $\mathcal{D}^{\gtrless}$ play a role of the phase-space densities of (quasi-)particles, so they can be treated as quantum analogs of the classical distribution functions.

The free Green's functions D can be found solving the equation of motion and in the Feynman gauge the functions read

$$\begin{aligned}
\big(D^{ab}_{\mu\nu}\big)^>(p) &= \frac{i\pi}{E_p} g_{\mu\nu}\delta^{ab}\Big[\delta(p_0-E_p)\big(n_g(\mathbf{p})+1\big)+\delta(p_0+E_p)n_g(-\mathbf{p})\Big],\\
\big(D^{ab}_{\mu\nu}\big)^<(p) &= \frac{i\pi}{E_p} g_{\mu\nu}\delta^{ab}\Big[\delta(p_0-E_p)n_g(\mathbf{p})+\delta(p_0+E_p)\big(n_g(-\mathbf{p})+1\big)\Big],\\
\big(D^{ab}_{\mu\nu}\big)^{\delta}(p) &= \mp g_{\mu\nu}\delta^{ab}\Big[\frac{1}{p^2\pm i0^+}\mp\frac{i\pi}{E_p}\Big(\delta(p_0-E_p)n_g(\mathbf{p})+\delta(p_0+E_p)n_g(-\mathbf{p})\Big)\Big],
\end{aligned}$$

where $n_g(\mathbf{p})$ is a distribution function of gluons which are assumed to be unpolarized with respect to spin and color degrees of freedom.

The free Green's functions of a fermion field can be derived in a similar way by solving the appropriate equations of motion. One could also find the Green's functions of ghost fields solving the equations of motion but it is fairly unclear what is the distribution function of ghosts.

3 Generating functional & Slavnov identities

The problem of ghosts in statistical systems is to be solved through the path integral formulation in which a generating functional is the starting point. The generating functional of Keldysh-Schwinger formalism is given as

$$\begin{aligned}
W[J,\chi,\chi^*] \;=\; & N\int DA'(\mathbf{x})\,DA''(\mathbf{x})\,Dc'(\mathbf{x})\,Dc''(\mathbf{x})\,Dc^{*\prime}(\mathbf{x})\,Dc^{*\prime\prime}(\mathbf{x})\\
&\times\;\rho\big[A'(\mathbf{x}),c'(\mathbf{x}),c^{*\prime}(\mathbf{x})\big|A''(\mathbf{x}),c''(\mathbf{x}),c^{*\prime\prime}(\mathbf{x})\big]\;W_0[J,\chi,\chi^*],
\end{aligned}$$

where $A'(\mathbf{x})$, $A''(\mathbf{x})$, $c'(\mathbf{x})$, $c''(\mathbf{x})$, $c^{*\prime}(\mathbf{x})$, $c^{*\prime\prime}(\mathbf{x})$ are boundary fields and ρ is a density matrix which describes the system of fields at $t=-\infty$. $W_0[J,\chi,\chi^*]$ is the analog of the generating functional of the vacuum QCD and is of the form

$$\begin{aligned}
W_0[J,\chi,\chi^*] \;=\; & N_0\int_{\substack{A(-\infty+i0^+,\mathbf{x})=A'(\mathbf{x})\\ A(-\infty-i0^+,\mathbf{x})=A''(\mathbf{x})}}\mathcal{D}A(x)\int_{\substack{c(-\infty+i0^+,\mathbf{x})=c'(\mathbf{x})\\ c(-\infty-i0^+,\mathbf{x})=c''(\mathbf{x})}}\mathcal{D}c(x)\\
&\times\int_{\substack{c^*(-\infty+i0^+,\mathbf{x})=c^{*\prime}(\mathbf{x})\\ c^*(-\infty-i0^+,\mathbf{x})=c^{*\prime\prime}(\mathbf{x})}}\mathcal{D}c^*(x)\exp\Big[i\int_C d^4x\,\mathcal{L}_{\rm eff}(x)\Big],
\end{aligned}$$

where the time integral is along the Keldysh contour and the effective Lagrangian is

$$\begin{aligned}
\mathcal{L}_{\rm eff} \;=\; & -\frac{1}{4}F^{\mu\nu}_a F^a_{\mu\nu}+\bar{\psi}(i\gamma_\mu D^\mu-m)\psi-\frac{1}{2\alpha}\big(\partial^\mu A^a_\mu\big)^2\\
&-c^*_a(\partial^\mu\partial_\mu\delta^{ab}-g\partial^\mu f^{abc}A^c_\mu)c_b+J^\mu_a A^a_\mu+\chi^*_a c_a+\chi_a c^*_a.
\end{aligned}$$

Contrary to the vacuum field theory, the generating functional of the Keldysh-Schwinger formalism cannot be expressed in a closed explicit form even for a free theory because of the unspecified density operator. Nevertheless, the generating functional provides various relations among the Green's functions, in particular, the Slavnov-Taylor identities.

The general Slavnov-Taylor identity results from the invariance of the generating functional with respect to the infinitesimal gauge transformations $A^a_\mu \to A^a_\mu + f^{abc}\omega^b A^c_\mu - \frac{1}{g}\partial_\mu\omega^a$ where $|\omega| \ll 1$. We assume that the gauge transformation does not work at $t=-\infty$, that is $\omega(t=-\infty,\mathbf{x})=0$, and consequently the density matrix ρ remains unchanged. Requiring invariance of the generating functional with respect to the gauge transformation we get the general Slavnov-Taylor identity

$$\Big\{i\partial^\mu_{(z)}\frac{\delta}{\delta J^\mu_d(z)}-\int_C d^4x\,J^\mu_a(x)\Big(\partial^{(x)}_\mu\delta^{ab}+igf^{abc}\frac{\delta}{\delta J^\mu_c(x)}\Big)M^{-1}_{bd}\Big[\frac{1}{i}\frac{\delta}{\delta J}\Big|x,z\Big]\Big\}W[J,\chi^*,\chi]=0,$$

which holds in the Feynman gauge; M^{-1} is essentially the ghost Green's function. Differentiating the general relation with respect to $J^\nu_e(y)$ and putting $\chi=\chi^*=J=0$, we obtain

$$\partial^\mu_{(z)}\mathcal{D}^{ab}_{\mu\nu}(z,y)=\partial^{(y)}_\nu\Delta_{ab}(y,z),$$

which relates to each other the contour Green's functions of interacting gluons and free ghosts. Locating the time arguments y_0 and z_0 on the upper or lower branch of the Keldysh contour, we get the relations for the Green's functions of real arguments. Since the system under study is translationally invariant, the Fourier transformed identity is

$$-p^\mu\mathcal{D}^{ab}_{\mu\nu}(p)=p_\nu\Delta_{ab}(-p),$$

which relates the longitudinal part of the gluon Green's function to the free ghost function. The relation also expresses the well-known fact that the longitudinal part of the gluon Green's function is not modified by interaction and consequently the polarization tensor is purely transversal.

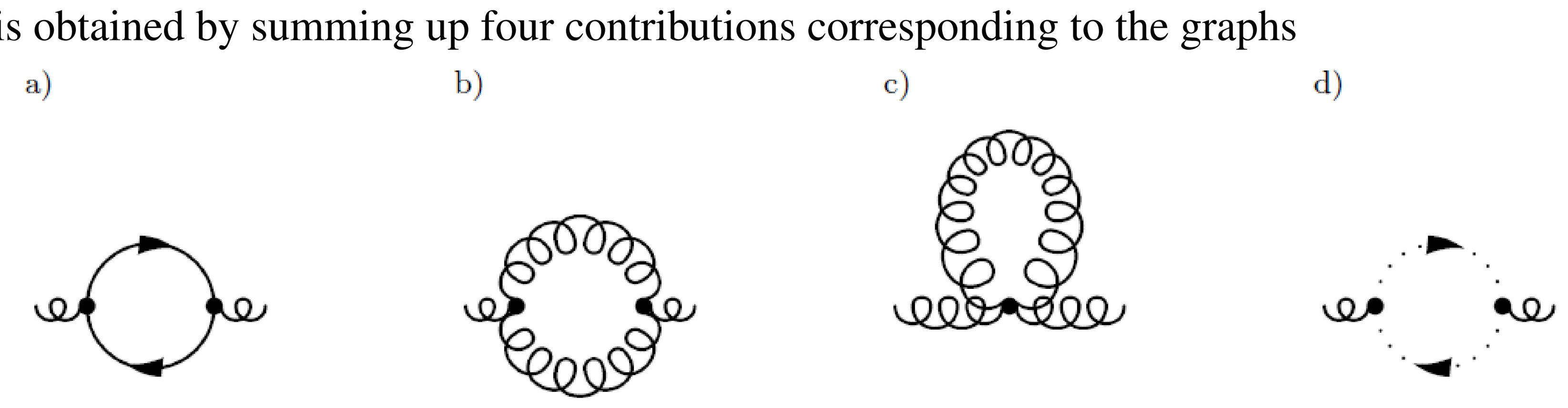
With the explicit expressions of the gluon functions ($D^>$, $D^<$, D^δ), the specific Slavnov-Taylor relation provides the Green's functions of free ghosts

$$\begin{aligned}
\Delta^>_{ab}(p) &= -\delta^{ab}\frac{i\pi}{E_p}\Big[\delta(E_p-p_0)\big(n_g(\mathbf{p})+1\big)+\delta(E_p+p_0)n_g(-\mathbf{p})\Big],\\
\Delta^<_{ab}(p) &= -\delta^{ab}\frac{i\pi}{E_p}\Big[\delta(E_p-p_0)n_g(\mathbf{p})+\delta(E_p+p_0)\big(n_g(-\mathbf{p})+1\big)\Big],\\
\Delta^{\delta}_{ab}(p) &= \pm\delta^{ab}\Big[\frac{1}{p^2\pm i0^+}\mp\frac{i\pi}{E_p}\Big(\delta(p_0-E_p)n_g(\mathbf{p})+\delta(p_0+E_p)n_g(-\mathbf{p})\Big)\Big].
\end{aligned}$$

As seen, the gluon distribution function $n_g(\mathbf{p})$, which describes the *physical* gluons, enters the Green's functions of *unphysical* ghosts.

4 Gluon polarization tensor

As an application of the Green's functions of the free ghosts we compute the retarded polarization tensor of a quark-gluon plasma. Our computation is performed within the hard loop approach applicable to anisotropic systems. The retarded polarization tensor is an important characteristic of a plasma system, as it carries information about its chromodynamic properties like collective excitations or screening lengths. The polarization tensor of QCD is obtained by summing up four contributions corresponding to the graphs



After subtracting the vacuum effect, the polarization tensor equals

$$\Pi^{\mu\nu}_{ab}(k)=g^2\delta^{ab}\int\frac{d^3p}{(2\pi)^3}\frac{f(\mathbf{p})}{E_p}\frac{g^{\mu\nu}(k\cdot p)^2-(k^\mu p^\nu+p^\mu k^\nu)(k\cdot p)+k^2p^\mu p^\nu}{(k\cdot p+i0^+)^2},$$

where $f(\mathbf{p})\equiv n_q(\mathbf{p})+\bar{n}_q(\mathbf{p})+2N_c n_g(\mathbf{p})$. As seen, the tensor is symmetric with respect to Lorentz indices $\Pi^{\mu\nu}_{ab}(k)=\Pi^{\nu\mu}_{ab}(k)$ and transverse $k_\mu\Pi^{\mu\nu}_{ab}(k)=0$, as required by the gauge invariance.

5 Conclusions

The transversality of the polarization tensor, which is not assumed but appears automatically, clearly shows that the derived Green's functions of ghosts work properly. This opens a possibility to perform other real-time calculations in the Feynman gauge which are usually much simpler than those in physical gauges like the Coulomb one.

[1] A. Czajka and St. Mrówczyński, Phys. Rev. D **89**, 085035 (2014).

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