1. Introduction

- Event-by-event fluctuations in high-energy heavy-ion collisions
  - Recently in heavy-ion collisions, it is actively discussed to constrain the initial-state models and the transport properties of the created QGP using the observed event-by-event fluctuations.

- Hydrodynamic fluctuation = the thermal fluctuation of hydrodynamics
  - It is already revealed that the initial-state fluctuation play an important role in the event-by-event fluctuations. However, there is still another source of event-by-event fluctuations called hydrodynamic fluctuation.

- Hydrodynamic fluctuation in an integrated dynamical model
  - The hydrodynamic fluctuation has an effect on observables such as the flow harmonics \( v_n \). In order to extract the quantitative properties of the matter, the effect of the hydrodynamic fluctuation should be investigated in dynamical models.

2. Colored noise in causal hydrodynamics

- In first-order dissipative hydrodynamics
  - \( \pi^{\mu\nu} = 2\eta(\delta^{\mu\nu}) + \delta\pi^{\mu\nu} \) Constitutive equation (CE) for shear stress

  Fluctuation-Dissipation Relation (FDR)
  - Power spectrum of hydrodynamic fluctuation
    \( \langle \delta\pi^{\mu\nu}(x)\delta\pi^{\alpha\beta}(x') \rangle \propto \delta(x-x') \) \textit{White Noise}

  \( \text{\textit{White Noise}} \) in first-order hydrodynamical systems.

- In causal dissipative hydrodynamics with non-zero relaxation times
  - \( \pi^{\mu\nu} = -\tau_x^2 D(\mu,\nu) + 2\eta(\delta^{\mu\nu}) + \cdots \)
    - Relaxation time 2, non-zero due to the causality
    - Common in current dynamical calculations

  Relaxation in a light cone

  Fluctuation-Dissipation Relation (FDR)
  - Power spectrum of hydrodynamic fluctuation
    \( \langle \delta\pi^{\mu\nu}(x)\delta\pi^{\alpha\beta}(x') \rangle = TG\pi^{\mu\nu}(x-x')^{\mu\alpha\beta} \) \textit{Colored Noise}

  \( \text{\textit{Colored Noise}} \) in causal hydrodynamical systems.

3. White noise in differential form of CE

- Causal hydrodynamic fluctuation in differential form of CE
  - For practical calculations, the differential form of CE is useful.

  \[ \mathcal{L}J = kF + \xi \]
  \[ \mathcal{L} = 1 + \frac{\eta}{\tau_x}D + \cdots \]

  Fluctuation
  \[ \xi = \mathcal{L}\delta J \]
  \[ \delta J \] Noise in integral form

  Q: Power spectrum of \( \xi \rightarrow \text{colored or white?} \)

- Using the general assumption:
  - \( L \) is truncated to finite order in derivatives
  - \( \xi \) is either retarded: \( G=0 \) if \( t<0 \)
  - \( \xi \) is either relaxing: \( G>0 \) as \( t\rightarrow \infty \)
  - \( \tau_x \): time-like: \( G=0 \) if \( x<0 \)

  \( G \) is positive semidefinite since it is a covariance matrix of the noise (FDR)

  Constraints
  \[ \mathcal{L}_\omega,k = 1 + i\omega A \xi_k \]
  \[ \langle \xi(x)\xi(x') \rangle = 2T\kappa\mathcal{O}(x-x'). \] \textit{White Noise}

  \( \text{\textit{White Noise}} \) in first-order hydrodynamical systems.

3. Hydro fluctuation in dynamical models

- We implemented hydrodynamical fluctuations in relativistic dissipative hydrodynamics and are testing the behavior of the hydrodynamic fluctuation.

- Simulations from smooth initial condition

  \[ \tau = 5.0 \text{ fm}, \text{ ideal gas} \]
  \[ \Omega = 1/4n, \tau = 5.0 \text{ fm}, \text{ ideal gas} \]

- Temperature (MeV)
  - without fluctuation
  - with hydro fluctuations
  - with hydro fluctuations coarse-grained (~0.55fm)

Adding directly the hydrodynamic fluctuation causes too large gradients for the gradient expansion of CE

- Coarse-graining scale of the fluctuation is needed (~ microscopical scale)

- Hydrodynamic fluctuations and the observables
  \[ \langle \delta\pi^{\mu\nu}/2\eta\delta^{\mu\nu} \rangle \sim 1/\sqrt{\langle n \rangle}/(V/\text{fm}^4) \]

  The effect is important in small \( n/P \) or small \( V \)
  - peripheral collisions, central pp, pA collisions
  - higher harmonics \( \nu \)

4. Summary

- Hydrodynamic fluctuation should be colored/white noise in the integral/differential form of the constitutive equation due to the causality.
- In numerical simulations, some coarse-graining scale of the fluctuation should be introduced.
- Hydrodynamic fluctuation are more important in small systems and higher orders of the flow harmonics.