Exact solutions of kinetic equation in the relaxation time approximation

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Abstract

We solve exactly the one-dimensional boost-invariant Boltzmann kinetic equation for a relativistic massive system of partons with the collision term treated in the relaxation time approximation. Using exact solutions we numerically establish correct forms of the kinetic coefficients of a massive system. Subsequently, we compare the proper-time evolution of the bulk viscous pressure obtained from the exact solution with results obtained from the dynamical equations of second-order viscous hydrodynamics and leading-order anisotropic hydrodynamics. We show that, unlike the viscous hydrodynamics, the anisotropic hydrodynamics approach qualitatively describe kinetic theory results.

The Boltzmann equation in relaxation time approximation

We consider the *relativistic Boltzmann equation*

$$p^\mu \partial_\mu f = C[f]$$

with the collision term treated in the *relaxation time approximation*

$$C[f] = \frac{p^\mu u_\mu}{\tau_{\rm eq}}(f_{\rm eq} - f).$$

We attempt to describe the early-time evolution of matter produced in ultra-relativistic heavy-ion collisions, thus, at first approximation, we consider the system as a *transversely homogeneous boost-invariant* one. Using convenient boost-invariant variables the kinetic equation may be written in the simple form

$$\frac{\partial f}{\partial \tau} = \frac{f_{\rm eq} - f}{\tau_{\rm eq}}.$$

Its solution has the general form

$$f(\tau, w, p_T) = D(\tau,\tau_0) f_0(w,p_T) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm eq}(\tau')} D(\tau,\tau')\, f_{\rm eq}(\tau', w, p_T),$$

where we have introduced the damping function

$$D(\tau_2,\tau_1) = \exp\left[-\int_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_{\rm eq}(\tau'')}\right].$$

Enforcing *energy and momentum conservation* one gets the *dynamical Landau matching condition* for energy densities

$$2m^2 T(\tau)^2 \left[3K_2\left(\frac{m}{T(\tau)}\right) + \frac{m}{T(\tau)} K_1\left(\frac{m}{T(\tau)}\right)\right] =$$

$$D(\tau,\tau_0)\Lambda_0^4 \tilde{\mathcal{H}}_2\left[\frac{\tau_0}{\tau\sqrt{1+\xi_0}}, \frac{m}{\Lambda_0}\right] + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm eq}} D(\tau,\tau') T^4(\tau') \tilde{\mathcal{H}}_2\left[\frac{\tau'}{\tau}, \frac{m}{T(\tau')}\right].$$

This is an integral equation for the effective temperature $T(\tau)$ which we *solve exactly* using the *iterative method* up to the arbitrary numerical precision. Having proper-time dependence of temperature one can calculate the bulk properties of matter such as energy density ε and transverse $\mathcal{P}_T$ and longitudinal $\mathcal{P}_L$ pressure by performing momentum-space integrals.

Shear and bulk viscous pressures from kinetic theory

Kinetic estimate of the *bulk and shear viscous pressures* may be obtained from the exact solution of Boltzmann kinetic equation by computing

$$\Pi_\eta^{\rm kin}(\tau) = \frac{2}{3}\left[\mathcal{P}_T(\tau) - \mathcal{P}_L(\tau)\right],$$

$$\Pi_\zeta^{\rm kin}(\tau) = \frac{1}{3}\left[\mathcal{P}_L(\tau) + 2\mathcal{P}_T(\tau) - 3P_{\rm eq}(\tau)\right].$$

Kinetic coefficients in the relaxation time approximation

We compare the shear and bulk viscosities from the literature

$$\eta(T) = \frac{\tau_{\rm eq} P_{\rm eq}(T)}{15}\hat{m}^3\left[\frac{3}{\hat{m}^2}\frac{K_3}{K_2} - \frac{1}{\hat{m}} + \frac{K_1}{K_2} - \frac{K_{i,1}}{K_2}\right]$$

$$\zeta(T) = \tau_{\rm eq} P_{\rm eq}\frac{\hat{m}^2}{3}\left[-\frac{\hat{m}K_2}{3(3K_3 + \hat{m}K_2)} + \frac{\hat{m}}{3}\left(\frac{K_1}{K_2} - \frac{K_{i,1}}{K_2}\right)\right]$$

obtained by Anderson & Witting and Sasaki & Redlich respectively with those extracted from the exact solution by considering the late-time near-equilibrium form of the solutions

$$\Pi_\eta(\tau) \approx \frac{4\eta(T)}{3\tau}, \qquad \Pi_\zeta(\tau) \approx -\frac{\zeta(T)}{\tau}.$$

At late times the exact solution is well-approximated by the near-equilibrium shear and bulk viscosity above, thus these formulas are the correct forms of the kinetic coefficients for massive systems.

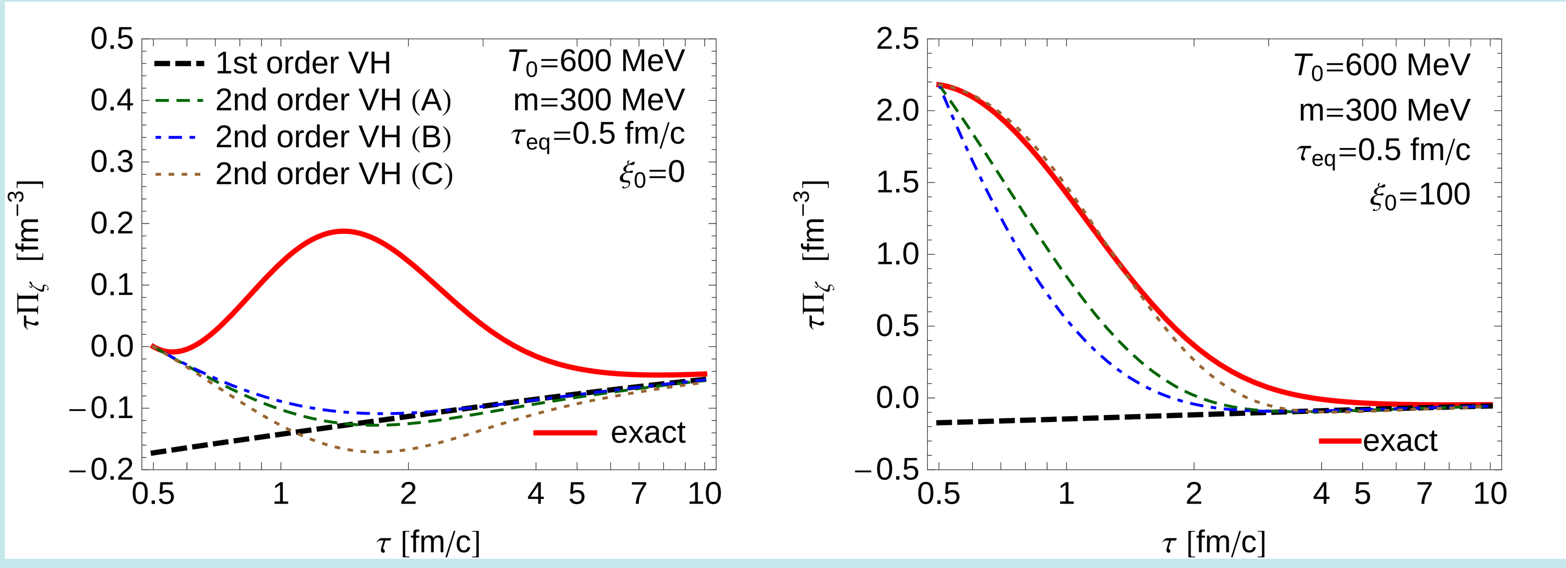
Bulk viscous pressure evolution in 2nd order viscous hydrodynamics

The exact computation of the proper-time dependence of bulk viscous pressure can be compared with *2nd order viscous hydrodynamic* predictions. We consider three possibilities for the evolution equation which appear in the literature

$$\tau_{\rm eq}\frac{d\Pi_\zeta^{\rm hyd}}{d\tau} + \Pi_\zeta^{\rm hyd} = -\frac{\zeta}{\tau} - \frac{\tau_{\rm eq}\Pi_\zeta^{\rm hyd}}{2}\left(\frac{1}{\tau} - \frac{1}{\zeta}\frac{d\zeta}{d\tau} - \frac{1}{T}\frac{dT}{d\tau}\right), \qquad (A)$$

$$\tau_{\rm eq}\frac{d\Pi_\zeta^{\rm hyd}}{d\tau} + \Pi_\zeta^{\rm hyd} = -\frac{\zeta}{\tau} - \frac{4\tau_{\rm eq}\Pi_\zeta^{\rm hyd}}{3\tau}, \qquad (B)$$

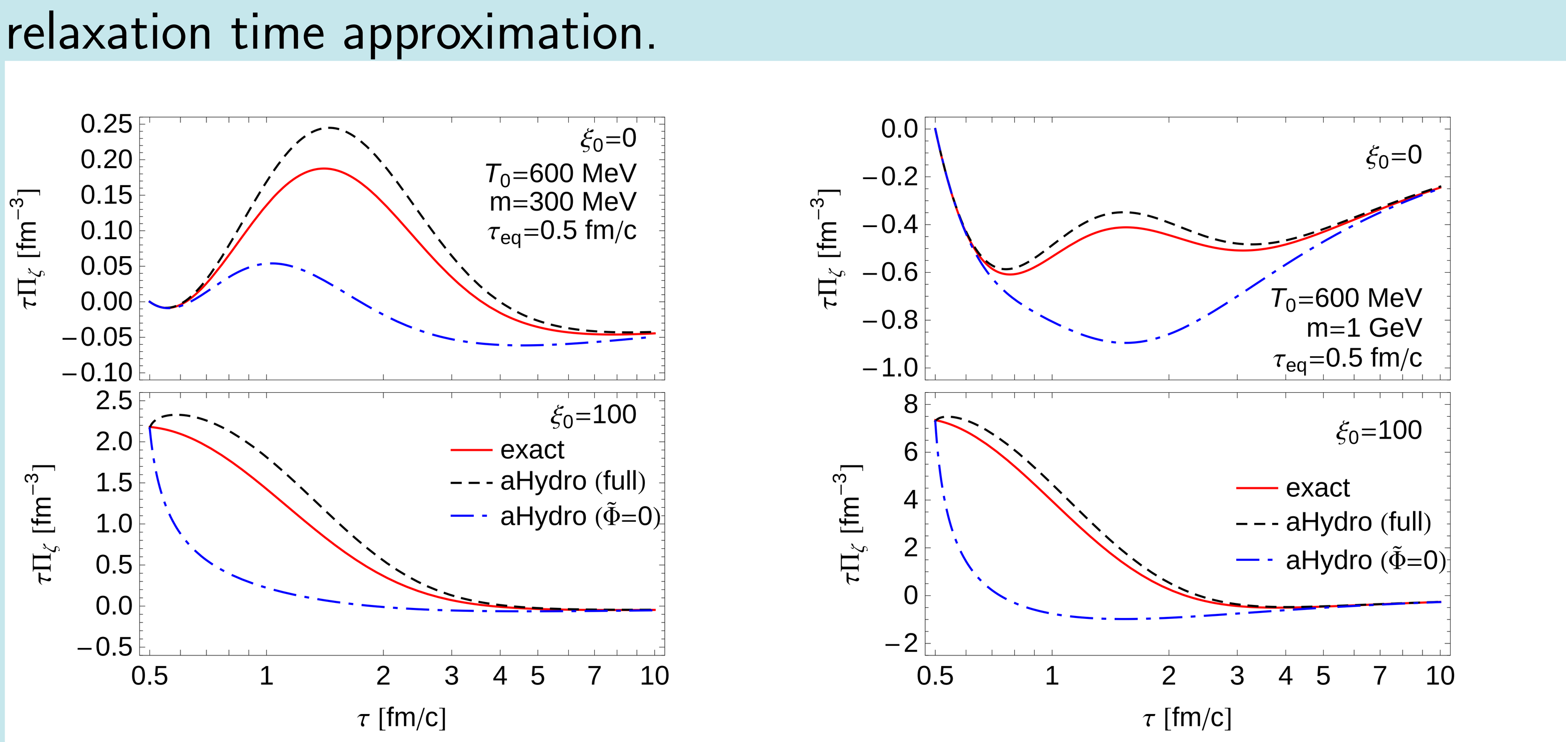
$$\tau_{\rm eq}\frac{d\Pi_\zeta^{\rm hyd}}{d\tau} + \Pi_\zeta^{\rm hyd} = -\frac{\zeta}{\tau}. \qquad (C)$$



All 2nd order viscous hydrodynamics frameworks tend toward the 1st order solution at late times. However, *none of the formulations seems to accurately describe the early-time evolution of the bulk viscous pressure* in all cases. These results indicate that there may be something incomplete in the manner in which 2nd order viscous hydrodynamics treats the bulk pressure.

Bulk viscous pressure evolution in anisotropic hydrodynamics

In *anisotropic hydrodynamics* framework one can introduce an explicit degree of freedom that can be associated with the bulk pressure of the system. The resulting form generalizes the ellipsoidal one-particle distribution function appropriate for massless particles to massive particles. Using 0th, 1st and 2nd moments of the generalized form of distribution function one obtains a system of ordinary differential equations that can be solved numerically. We compare the resulting anisotropic hydrodynamics solutions with the exact solution of the (0+1)d Boltzmann equation in the relaxation time approximation.



Conclusions

We found that the standard expressions available in the literature for the mass and temperature dependence of the shear and bulk viscosities describe the evolution of the system well for $\tau \gg \tau_{\rm eq}$. Furthermore, we found that none of the 2nd order viscous hydrodynamics equations describe the early evolution of bulk pressure correctly. We find also that the explicit inclusion of the bulk degree of freedom in anisotropic hydrodynamics approach allows for correct description of proper-time dependence of bulk pressure resulting from the exact solution for a massive gas.

References

[1] W.Florkowski, R.R., M.Strickland, Nucl.Phys.A916 (2013) 249
[2] W.Florkowski, R.R., M.Strickland, Phys.Rev.C88 (2013) 024903
[3] W.Florkowski, E.Maksymiuk, R.R., M.Strickland, Phys.Rev.C (2014)
[4] M.Nopoush, M.Strickland, R.R., arXiv:1405.1355

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