Relaxation-time approximation and relativistic viscous hydrodynamics from kinetic theory

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Introduction

Relativistic viscous hydrodynamics has been quite successful in explaining the spectra and azimuthal anisotropy of particles produced in heavy-ion collisions at the RHIC and recently at the LHC [1]. Despite its success, the formulation of a causal theory of relativistic viscous hydrodynamics from kinetic theory is based on strong assumptions and approximations [2]:

1. Use of second moment of Boltzmann equation to obtain viscous evolution.
2. Grad's 14-moment approximation for nonequilibrium distribution function.

In this poster, we present an alternative derivation of evolution equation for shear stress tensor which do not make use of both these assumptions. We iteratively solve the Boltzmann equation in relaxation time approximation to obtain the non-equilibrium phase-space distribution function, $f(x,p)$, up to second-order in gradients. Subsequently, we derive a third-order viscous evolution equation and quantify the significance of this derivation within one-dimensional scaling expansion where we demonstrate that the results obtained using third-order equations are in excellent agreement with the exact solution of Boltzmann equation as well as transport results.

Analytic Results

The equation of motion governing the hydrodynamic evolution of a relativistic system with no net conserved charges is obtained from the local conservation of energy and momentum, $\partial_\mu T^{\mu\nu}=0$. In terms of single-particle phase-space distribution function, the energy-momentum tensor of a macroscopic system can be expressed as

$$T^{\mu\nu}=\int dp\, p^\mu p^\nu\, f(x,p)=\epsilon u^\mu u^\nu - P\Delta^{\mu\nu}+\pi^{\mu\nu}, \tag{1}$$

The conservation of the energy-momentum tensor, when projected along and orthogonal to u^μ, leads to the evolution equations for ϵ and u^μ:

$$\dot\epsilon+(\epsilon+P)\theta-\pi^{\mu\nu}\nabla_{(\mu}u_{\nu)}=0 \quad\Rightarrow\quad \dot\beta=\frac{\beta}{3}\theta-\frac{\beta}{12P}\pi^{\rho\gamma}\sigma_{\rho\gamma}, \tag{2}$$
$$(\epsilon+P)\dot u^\alpha-\nabla^\alpha P+\Delta^\alpha_\nu\partial_\mu\pi^{\mu\nu}=0 \quad\Rightarrow\quad \nabla^\alpha\beta=-\beta\dot u^\alpha-\frac{\beta}{4P}\Delta^\alpha_\rho\partial_\gamma\pi^{\rho\gamma}.$$

When the system is close to local thermodynamic equilibrium, the distribution function can be written as $f=f_0+\delta f$, where $\delta f\ll f_0$ and $f_0=\exp(-\beta\, u\cdot p)$. Projecting the traceless symmetric part of Eq. (1), we obtain an expression for the shear stress tensor and its time evolution,

$$\pi^{\mu\nu}=\Delta^{\mu\nu}_{\alpha\beta}\int dp\, p^\alpha p^\beta\,\delta f, \qquad \dot\pi^{\langle\mu\nu\rangle}=\Delta^{\mu\nu}_{\alpha\beta}\int dp\, p^\alpha p^\beta\,\delta\dot f. \tag{3}$$

In the following, we determine δf and subsequently derive evolution equations for shear stress tensor.

We start from the relativistic Boltzmann equation with the relaxation-time approximation for the collision term

$$p^\mu\partial_\mu f=-(u\!\cdot\! p)\frac{\delta f}{\tau_R} \quad\Rightarrow\quad f=f_0-\frac{\tau_R}{(u\!\cdot\! p)}p^\mu\partial_\mu f, \tag{4}$$

where $f=f_0+\delta f$ and τ_R is the relaxation time. Expanding the distribution function f about its equilibrium value in powers of space-time gradients, i.e., $f=f_0+\delta f^{(1)}+\delta f^{(2)}+\cdots$ and solving Eq. (4) iteratively, we obtain [3, 4],

$$\delta f^{(1)}=-\frac{\tau_R}{u\cdot p}p^\mu\partial_\mu f_0, \qquad \delta f^{(2)}=\frac{\tau_R}{u\cdot p}p^\mu p^\nu\partial_\mu\Big(\frac{\tau_R}{u\cdot p}\partial_\nu f_0\Big). \tag{5}$$

Substituting $\delta f=\delta f^{(1)}$ in the expression for $\pi^{\mu\nu}$ in Eq. (3), performing the integrations, and retaining only first-order terms, we obtain $\pi^{\mu\nu}=2\tau_R\beta_\pi\sigma^{\mu\nu}$, where $\beta_\pi=4P/5$ [3, 4].

To obtain the second-order evolution equation for shear stress tensor, we rewrite Eq. (4) in the form $\delta\dot f=-\dot f_0-p^\gamma\nabla_\gamma f/(u\!\cdot\! p)-\delta f/\tau_R$. Using this expression for $\delta\dot f$ in Eq. (3), we obtain

$$\dot\pi^{\langle\mu\nu\rangle}+\frac{\pi^{\mu\nu}}{\tau_R}=-\Delta^{\mu\nu}_{\alpha\beta}\int dp\, p^\alpha p^\beta\Big(\dot f_0+\frac{1}{u\!\cdot\! p}p^\gamma\nabla_\gamma f\Big). \tag{6}$$

Using Eq. (5) for $\delta f^{(1)}$ and Eq. (2) for derivatives of β, and keeping terms up to quadratic order in gradients, the second-order shear evolution equation is obtained as [3, 4]

$$\dot\pi^{\langle\mu\nu\rangle}+\frac{\pi^{\mu\nu}}{\tau_\pi}=2\beta_\pi\sigma^{\mu\nu}+2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}-\frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma}-\frac{4}{3}\pi^{\mu\nu}\theta, \tag{7}$$

where $\omega^{\mu\nu}\equiv(\nabla^\mu u^\nu-\nabla^\nu u^\mu)/2$.

Similarly, substituting $\delta f=\delta f^{(1)}+\delta f^{(2)}$ in Eq. (6) and performing the integrations, we finally obtain a novel third-order evolution equation for shear stress tensor [4]:

$$\begin{aligned}\dot\pi^{\langle\mu\nu\rangle}=&-\frac{\pi^{\mu\nu}}{\tau_\pi}+2\beta_\pi\sigma^{\mu\nu}+2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}-\frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma}-\frac{4}{3}\pi^{\mu\nu}\theta-\frac{1}{3\beta_\pi}\pi_\gamma^{\langle\mu}\pi^{\nu\rangle\gamma}\theta\\ &-\frac{38}{245\beta_\pi}\pi^{\mu\nu}\pi^{\rho\gamma}\sigma_{\rho\gamma}-\frac{22}{49\beta_\pi}\pi^{\rho\langle\mu}\pi^{\nu\rangle\gamma}\sigma_{\rho\gamma}+\frac{25}{7\beta_\pi}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma}\\ &-\frac{24}{35}\nabla^{\langle\mu}\left(\pi^{\nu\rangle\gamma}\dot u_\gamma\tau_\pi\right)+\frac{12}{7}\nabla_\gamma\left(\tau_\pi\dot u^{\langle\mu}\pi^{\nu\rangle\gamma}\right)+\frac{6}{7}\nabla_\gamma\left(\tau_\pi\dot u^\gamma\pi^{\langle\mu\nu\rangle}\right)\\ &-\frac{2}{7}\nabla_\gamma\left(\tau_\pi\nabla^{\langle\mu}\pi^{\nu\rangle\gamma}\right)-\frac{1}{7}\nabla_\gamma\left(\tau_\pi\nabla^\gamma\pi^{\langle\mu\nu\rangle}\right)+\frac{4}{35}\nabla^{\langle\mu}\left(\tau_\pi\nabla_\gamma\pi^{\nu\rangle\gamma}\right)\\ &-\frac{2}{7}\tau_\pi\omega^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma}-\frac{2}{7}\tau_\pi\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\omega_{\rho\gamma}-\frac{10}{63}\tau_\pi\pi^{\mu\nu}\theta^2+\frac{26}{21}\tau_\pi\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}\theta\end{aligned} \tag{8}$$

We compare the above equation with that obtained in Ref. [5] by invoking the second law of thermodynamics,

$$\dot\pi^{\langle\mu\nu\rangle}=-\frac{\pi^{\mu\nu}}{\tau_\pi'}+2\beta_\pi'\sigma^{\mu\nu}-\frac{4}{3}\pi^{\mu\nu}\theta+\frac{5}{36\beta_\pi'}\pi^{\mu\nu}\pi^{\rho\gamma}\sigma_{\rho\gamma}-\frac{16}{9\beta_\pi'}\pi_\gamma^{\langle\mu}\pi^{\nu\rangle\gamma}\theta, \tag{9}$$

where $\beta_\pi'=2P/3$ and $\tau_\pi'=\eta/\beta_\pi'$. We notice that the right-hand-side of the above equation contains one second-order and two third-order terms compared to three second-order and fourteen third-order terms obtained in Eq. (8).

Numerical Results

In the following, we consider boost-invariant Bjorken expansion of a massless Boltzmann gas. We have solved the evolution equations with initial temperature $T_0=300$ MeV at initial time $\tau_0=0.25$ fm/c and with $T_0=500$ MeV at $\tau_0=0.4$ fm/c, corresponding to initial conditions of RHIC and LHC.

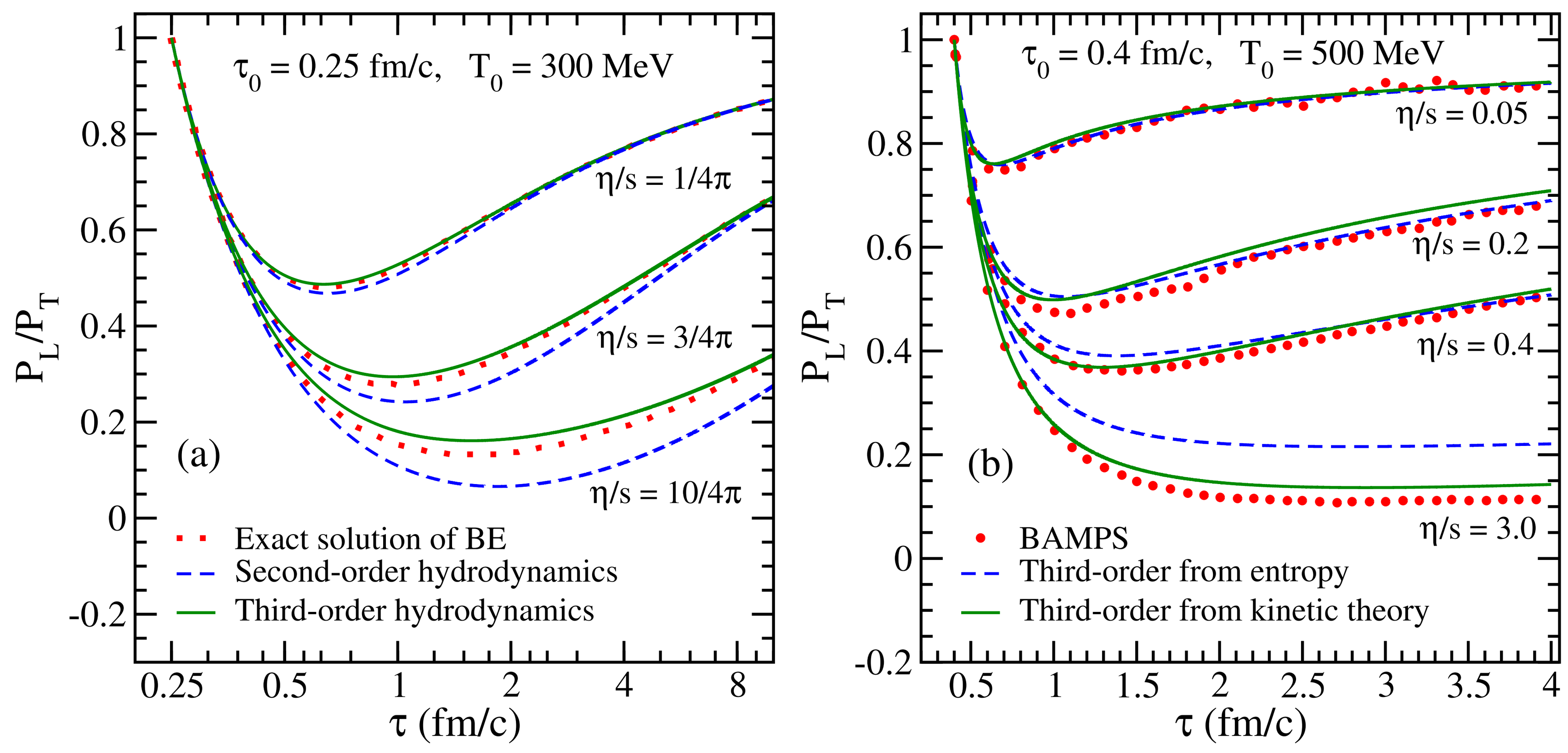

Figure 1: (a): Time evolution of P_L/P_T obtained using exact solution of Boltzmann equation (dotted line), second-order equations (dashed lines), and third-order equations (solid lines). (b): Time evolution of P_L/P_T in BAMPS (dots), third-order calculation from entropy method, Eq. (9) (dashed lines), and the present work (solid lines). Both figures are for isotropic initial pressure configuration ($\pi_0=0$) and various η/s.

Figures 1 (a) and (b) shows the proper time dependence of pressure anisotropy $P_L/P_T\equiv(P-\pi)/(P+\pi/2)$ where $\pi\equiv-\tau^2\pi^{\eta\eta}$. In Fig. 1 (a), we observe an improved agreement of third-order results (solid lines) with the exact solution of Boltzmann equation (dotted line) [6] as compared to second-order results (dashed line) suggesting the convergence of the derivative expansion. In Fig. 1 (b) we notice that while the results from entropy derivation (dashed lines) overestimate the pressure anisotropy for $\eta/s>0.2$, those obtained in the present work (solid lines) are in better agreement with the results of the parton cascade BAMPS (dots) [5].

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