Rapidity evolution of Wilson lines at the next-to-leading order: Balitsky-JIMWLK equation at NLO

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Outline

- High-energy QCD scattering processes and Wilson lines.
- High-energy Operator Product Expansion: factorization in rapidity space.
- Evolution equation and background field method.
- NLO BK equation.
- Hierarchy of Wilson lines evolution at NLO: The Balitsky-JIMWLK evolution equation at NLO.
- Conclusions.
High-energy scattering in quantum mechanics and QED

- High-energy: $E \gg V(x)$  
  WKB approximation.
- Replace the exact wave function by the semi-classical wave function.
- $\Psi(\vec{r}, t) = e^{-\frac{i}{\hbar}(Et-kx)} e^{-\frac{i}{\hbar} \int_{-\infty}^{z} dz' V(z')}$

At high-energy $\Psi = \Psi_{\text{free}} \times$ phase factor ordered along the line parallel to $\vec{v}$. The scattering amplitude is proportional to $\Psi(t = -\infty)$

$$U(x_\perp) = e^{\frac{-i}{\hbar} \int_{-\infty}^{+\infty} dz' V(z' + x_\perp)}$$

In QED

$$U(x_\perp) = e^{\frac{-ie}{\hbar} \int_{-\infty}^{+\infty} dt \dot{x}_\mu A^\mu(x(t))}$$
High-energy scattering in QCD

phase factor for the high-energy scattering: Wilson-line operator

\[ U(x_\perp, v) = e^{\frac{-ig}{\hbar} \int_{-\infty}^{+\infty} dt \, \dot{x}_\mu A^\mu(x(t))} \]

\[ Pe^{\int_{-\infty}^{+\infty} dt A(t)} = 1 + \int_{-\infty}^{+\infty} dt \, A(t) + \int_{-\infty}^{+\infty} dt \, A(t) \int_{-\infty}^{t} dt' \, A(t') + \ldots \]
Each path is weighted with the gauge factor $P e^{ig \int dx \mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.
Propagation in the shock wave: Wilson line (Spectator frame)

\[ \left[ z', z \right] = P e^{i g \int_0^1 du (z'-z)^\mu A_\mu (uz'+(1-u)z)} \]

\[ U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp] \]

\[ p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu, \quad p_1^\mu = \sqrt{s/2}(1, 0, 0, 1), \quad p_2^\mu = \sqrt{s/2}(1, 0, 0, -1) \]

s center-of-mass energy.
Propagation in the shock wave: Wilson line (Spectator frame)

Each path is weighted with the gauge factor $P e^{i g \int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction ⇒ we can replace the gauge factor along the actual path with the one along the straight-line path.
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Each path is weighted with the gauge factor $P e^{i g \int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

$Y > \eta$

$Y < \eta$
\[ \langle B | T \{ \hat{j}_\mu (x) \hat{j}_\nu (y) \} | B \rangle \simeq \int d^2 z_1 d^2 z_2 I_{\mu\nu}^{LO} (z_1, z_2; x, y) \langle B | \text{tr} \{ U_1^\eta U_2^{\dagger \eta} \} | B \rangle \]

\[ + \frac{\alpha_s}{\pi} \int d^2 z_1 d^2 z_2 d^2 z_3 I_{\mu\nu}^{NLO} (z_1, z_2, z_3; x, y) \langle B | \text{tr} \{ U_1^\eta U_2^{\dagger \eta} \} \text{tr} \{ U_3^\eta U_2^{\dagger \eta} \} | B \rangle \]

\[ \eta = \ln \frac{1}{x_B} \]

\[ |B\rangle \text{ Target state} \]
Leading Order

\[ \left[ \langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A \right]^{LO} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} \ I_{\mu\nu}^{LO}(x, y; z_1, z_2) \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \]

\[ \langle B|T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|B \rangle = \int \frac{d^2z_1 d^2z_2}{z_{12}^4} \ I_{\mu\nu}^{LO}(x, y; z_1, z_2) \langle B|\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}|B \rangle + \ldots \]

- If we use a model to evaluate \( \langle B|\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}|B \rangle \) we can calculate the DIS cross-section.
- If we want to include energy dependence to the DIS cross-section, we need to find the evolution of \( \langle B|\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}|B \rangle \) with respect to the rapidity parameter \( \eta \).
Regularization of the rapidity divergence

Matrix elements of Wilson lines: \( \langle \text{tr}\{U(x)U^\dagger(y)\}\rangle_A \) are divergent

For light-like Wilson lines loop integrals are divergent in the longitudinal direction

\[
\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty
\]

Regularization by: slope

\[
U^\eta(x_\perp) = \text{Pexp}\left\{ ig \int_{-\infty}^\infty du \, n_\mu \, A^\mu(un + x_\perp) \right\}
\]

\[
n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu
\]

At NLO the regularization by rigid cut-off is more convenient.
To get the evolution equation, consider the dipole with the rapidities up to $\eta_1$ and integrate over the gluons with rapidity $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidity up to $\eta_2$).

In the frame $||$ to $\eta_1$ the gluons with $\eta < \eta_1$ are seen as pancake.

Particles with different rapidity perceive each other as Wilson lines.
Separate fields in quantum and classical according to low and large rapidity. Formally we may write:

\[
\langle B | \mathcal{O}^{n_1} | B \rangle \rightarrow \langle \mathcal{O}^{n_1} \rangle_A \rightarrow \langle \mathcal{O}^{n_2} \otimes \mathcal{O}^{n_1} \rangle_A
\]

Integrate over the quantum fields and get one-loop rapidity evolution of the operator \( \mathcal{O} \)

\[
\langle \mathcal{O}^{n_1} \rangle_A = \alpha_s (\eta_1 - \eta_2) K_{\text{evol}} \otimes \langle \mathcal{O}^{n_2} \rangle_A
\]

Where in principle \( \mathcal{O} \) and \( \mathcal{O}' \) are different operators.
Non-linear evolution equation

- **Linear case** \( \mathcal{O}^{\eta_1} = \alpha_s \Delta \eta \ K_{\text{evol}} \otimes \mathcal{O}^{\eta_2} \)
Non-linear evolution equation

- **Linear case** \( \mathcal{O}^{\eta_1} = \alpha_s \Delta \eta \, K_{\text{evol}} \otimes \mathcal{O}^{\eta_2} \)

- **Non-linear case** \( \mathcal{O}^{\eta_1} = \alpha_s \Delta \eta \, K_{\text{evol}} \otimes \{\mathcal{O}^{\eta_2} \mathcal{O}^{\eta_2}\} \)
Non-linear evolution equation

- **Linear case**
  \[ \mathcal{O}^\eta_1 = \alpha_s \Delta \eta \ K_{\text{evol}} \otimes \mathcal{O}^\eta_2 \]

- **Non-linear case**
  \[ \mathcal{O}^\eta_1 = \alpha_s \Delta \eta \ K_{\text{evol}} \otimes \{ \mathcal{O}^\eta_2 \mathcal{O}^\eta_2 \} \]

\[
\langle \{ U^\eta_1 \}_{ij} \rangle_A = \frac{\alpha_s}{2\pi^2} \Delta \eta \int \frac{d^2z_\perp}{(x - z)^2} \left[ \langle \text{tr} \{ U^\eta_2 U^\eta_2 \dagger \} \{ U^\eta_2 \}_{ij} \rangle_A - \langle \frac{1}{N_c} \{ U^\eta_2 \}_{ij} \rangle_A \right]
\]

\[ \Delta = \eta_1 - \eta_2 \]

\[ \{ U^\dagger_1 \eta_1 \}_{ij}, \quad \{ U^\eta_1 U^\eta_1 \}_{ij}, \quad \{ U^\eta_1 U^\dagger_1 \eta_1 \}_{ij}, \quad \{ U^\dagger_1 \eta_1 U^\dagger_1 \eta_1 \}_{ij} \]
Non-linear evolution equation

- Linear case
  \[ \mathcal{O}^\eta_1 = \alpha_s \Delta \eta \ K_{\text{evol}} \otimes \mathcal{O}^\eta_2 \]

- Non-linear case
  \[ \mathcal{O}^\eta_1 = \alpha_s \Delta \eta \ K_{\text{evol}} \otimes \{\mathcal{O}^\eta_2 \mathcal{O}^\eta_2\} \]

\[ \mathcal{O}^\eta_1 \mathcal{O}^\eta_2 \]

\[ \langle \{U^\eta_1\}_x \rangle_A = \frac{\alpha_s}{2\pi^2} \Delta \eta \int \frac{d^2z_\perp}{(x - z)^2} \left[ \langle \text{tr}\{U^\eta_2 U_2^\eta 2^\dagger\}\{U^\eta_2\}_z \rangle_A - \langle \frac{1}{N_c} \{U^\eta_2\}_x \rangle_A \right] \]

\[ \Delta = \eta_1 - \eta_2 \]

\[ \{U^\eta_1\}_x, \{U^\eta_1 U^\eta_1\}_x, \{U^\eta_1 U^\eta_1\}_y, \{U^\eta_1 U^\eta_1\}_y \]

Obtain a set of rules that allow one to get the LO evolution of any trace or product of traces of Wilson lines
Leading order: BK equation

\[
\frac{d}{d\eta} \text{tr}\{\hat{U}_x \hat{U}^\dagger_y\} = K_{\text{LO}} \text{tr}\{\hat{U}_x \hat{U}^\dagger_y\} + \ldots \Rightarrow \\
\frac{d}{d\eta} \langle \text{tr}\{\hat{U}_x \hat{U}^\dagger_y\}\rangle_{\text{shockwave}} = \langle K_{\text{LO}} \text{tr}\{\hat{U}_x \hat{U}^\dagger_y\}\rangle_{\text{shockwave}}
\]

\[
x_\bullet = \sqrt{\frac{s}{2}} x^-
\]

\[
x_* = \sqrt{\frac{s}{2}} x^+
\]
Non-linear evolution equation: BK equation

\[ U_{z}^{ab} = 2 \text{tr}\{t^{a}U_{z}t^{b}U_{z}^{\dagger}\} \Rightarrow (U_{x}U_{y}^{\dagger})^{\eta_1} \rightarrow (U_{x}U_{y}^{\dagger})^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_{x}U_{z}^{\dagger}U_{z}U_{y}^{\dagger})^{\eta_2} \]
Non-linear evolution equation: BK equation

\[ U_{z}^{ab} = 2 \text{tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_2} + \alpha_s (\eta_1 - \eta_2) (U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2} \]

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{tr}\{\hat{U}(x_{\perp})\hat{U}^\dagger(y_{\perp})\} \]


\[ \frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2z \ (x - y)^2}{(x - z)^2(y - z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\} \]

Non-linear evolution equation: BK equation

\[ U_{z}^{ab} = 2 \text{tr} \{ t^{a} U_{z} t^{b} U_{z}^{\dagger} \} \Rightarrow (U_{x} U_{y}^{\dagger})^{\eta_{1}} \to (U_{x} U_{y}^{\dagger})^{\eta_{2}} + \alpha_{s} (\eta_{1} - \eta_{2}) (U_{x} U_{z}^{\dagger} U_{z} U_{y}^{\dagger})^{\eta_{2}} \]

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_{c}} \text{tr} \{ \hat{U}(x_{\perp}) \hat{U}^{\dagger}(y_{\perp}) \} \]


\[
\frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_{s} N_{c}}{2\pi^{2}} \int \frac{d^{2}z \ (x - y)^{2}}{(x - z)^{2} (y - z)^{2}} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z) \hat{U}(z, y) \right\}
\]


LLA for DIS in pQCD \Rightarrow BFKL (LLA: \( \alpha_{s} \ll 1, \alpha_{s} \eta \sim 1 \))
Non linear evolution equation: BK equation

\[ U_{z}^{ab} = 2 \text{tr} \{ t^a U_z t^b U_z^\dagger \} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_2} + \alpha_s (\eta_1 - \eta_2) (U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2} \]

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{tr} \{ \hat{U}(x_\perp) \hat{U}^\dagger(y_\perp) \} \]


\[
\frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z (x - y)^2}{(x - z)^2 (y - z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z) \hat{U}(z, y) \right\}
\]


LLA for DIS in pQCD ⇒ BFKL

LLA for DIS in sQCD ⇒ BK eqn

(\text{LLA: } \alpha_s \ll 1, \alpha_s \eta \sim 1)

(\text{LLA: } \alpha_s \ll 1, \alpha_s \eta \sim 1, \alpha_s^2 A^{1/3} \sim 1)

(s for semi-classical)
How to take higher-order corrections into account (either for BFKL or non-linear evolution equation).

Higher-order corrections are needed to improve phenomenology:
- Determine the argument of the coupling constant.
- Gives precision of LO.

Check conformal invariance (in $\mathcal{N}=4$ SYM)
\[
\frac{d}{d \eta} \text{Tr} \{ U_x U_y^\dagger \} = \\
\int \frac{d^2 z}{2\pi^2} \left( \alpha_s \frac{(x - y)^2}{(x - z)^2(z - y)^2} + \alpha_s^2 K_{NLO}(x, y, z) \right) \left[ \text{Tr} \{ U_x U_z^\dagger \} \text{Tr} \{ U_z U_y^\dagger \} - N_c \text{Tr} \{ U_x U_y^\dagger \} \right] + \\
\alpha_s^2 \int d^2 z d^2 z' \left( K_4(x, y, z, z') \{ U_x, U_{z'}^\dagger, U_z, U_y^\dagger \} + K_6(x, y, z, z') \{ U_x, U_{z'}^\dagger, U_{z'}^\dagger, U_z, U_y^\dagger \} \right)
\]

\( K_{NLO} \) is the next-to-leading order correction to the dipole kernel and \( K_4 \) and \( K_6 \) are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

- We need to calculate some diagrams analytically (pen and paper).
Diagrams of the NLO gluon contribution

Diagrams with 2 gluons interaction
Diagrams of the NLO gluon contribution

Diagrams with 2 gluons interaction
Diagrams of the NLO gluon contribution

Diagrams with 2 gluons interaction

(XXXI)  (XXXII)  (XXXIII)  (XXXIV)
"Running coupling" diagrams
Diagrams of the NLO gluon contribution

1 → 2 dipole transition diagrams
\[
\frac{d}{d\eta} \left[ \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^\dagger\} \right]^{\text{conf}} = \frac{\alpha_s}{2\pi^2} \int d^2 z_3 \left( \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_2}^\dagger\} - N_c \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^\dagger\} \right)^{\text{conf}} \\
\times \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ 1 + \frac{\alpha_s N_c}{4\pi} (b \ln z_{12}^2 \mu^2 + b \frac{z_{13}^2 - z_{23}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{13}^2}{z_{23}^2} + \frac{67}{9} - \frac{\pi^2}{3}) \right] \\
+ \frac{\alpha_s}{4\pi^2} \int \frac{d^2 z_4}{z_{34}^2} \left\{ \left[ -2 + \frac{z_{14}^2 z_{23}^2 + z_{24}^2 z_{13}^2 - 4z_{12}^2 z_{34}^2}{2(z_{14}^2 z_{23}^2 - z_{24}^2 z_{13}^2)} \ln \frac{z_{14}^2 z_{23}^2}{z_{24}^2 z_{13}^2} \right] \\
\times \left[ \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} \left( \text{tr}\{\hat{U}_{z_4} \hat{U}_{z_2}^\dagger\} - \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^\dagger\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\hat{U}_{z_2}^\dagger\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} - (z_4 \to z_3) \right) \right] \\
+ \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2} \left[ 2 \ln \frac{z_{12}^2 z_{34}^2}{z_{14}^2 z_{23}^2} + \left( 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right) \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right] \\
\times \left[ \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_3}^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} \text{tr}\{\hat{U}_{z_4} \hat{U}_{z_2}^\dagger\} - \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^\dagger\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\hat{U}_{z_2}^\dagger\hat{U}_{z_3} \hat{U}_{z_4}^\dagger\} - (z_4 \to z_3) \right] \right\} \right] \\
= \frac{11}{3} N_c - \frac{2}{3} n_f \]

I. Balitsky and G.A.C

\( K_{NLO \ BK} = \text{Running coupling part} + \text{Conformal "non-analytic" (in j) part} + \text{Conformal analytic (}N = 4\text{) part} \)

Linearized \( K_{NLO \ BK} \) reproduces the known result for the forward NLO BFKL kernel Fadin and Lipatov (1998).
In proton-Nucleus and Nucleus-Nucleus collisions there are also quadrupole Wilson line operators $\text{tr}\{U_x U_y^\dagger U_w U_z^\dagger\}$.

$\Rightarrow$ Need NLO Balitsky-JIMWLK evolution equation.
Sample of diagrams: a), b) are self-interactions; c), d) are pairwise interactions; e), f) are triple interactions.
\[ \frac{d}{d\eta}(U_1)_{ij} = \frac{\alpha_s^2}{8\pi^4} \int \frac{d^2z_4d^2z_5}{z_{45}^2} \left\{ U^d_4 (U^e_5' - U^{ee'}_4) \right\} \]

\[ \times \left( \left[ 2I_1 - \frac{4}{z_{45}^2} \right] f^{ade} f^{bd'e'} (t^a U_1 t^b)_{ij} + \left( \frac{z_{14}, z_{15}}{z_{14}^2z_{15}^2} \right) \ln \frac{z_{14}^2}{z_{15}^2} \left[ \frac{1}{3} \ln z_{14}^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] \times \frac{1}{z_{14}^2} \right) \]

\[ + \frac{\alpha_s^2 N_c}{4\pi^3} \int d^2z_4 \left( U^a_4 - U^a_1 \right) (t^a U_1 t^b)_{ij} \times \frac{1}{z_{14}^2} \left[ \frac{11}{3} \ln z_{14}^2 \mu^2 + \frac{67}{9} - \frac{\pi^2}{3} \right] \]

\[ I_1 \equiv I(z_1, z_4, z_5) = \frac{\ln z_{14}^2/z_{15}^2}{z_{14}^2 - z_{15}^2} \left[ \frac{z_{14}^2 + z_{15}^2}{z_{45}^2} - \frac{z_{14} z_{15}}{z_{14}^2} - \frac{z_{14} z_{15}}{z_{15}^2} - 2 \right] \]
\[
\frac{d}{d\eta}(U_1)_{ij}(U_2^\dagger)_{kl} = \frac{\alpha_s^2}{8\pi^4} \int d^2 z_4 d^2 z_5 (A_1 + A_2 + A_3) + \frac{\alpha_s^2 N_c}{8\pi^3} \int d^2 z_4 (B_1 + N_c B_2)
\]
\[ \mathcal{A}_1 = \left[ (t^a U_1)_{ij} (U_2 t^b)_{kl} + (U_1 t^b)_{ij} (t^a U_2)_{kl} \right] \times \left[ f^{ade} f^{bd' e'} U_4^{dd'} (U_5^{ee'} - U_4^{ee'}) \left( -K - \frac{4}{z_4^2} + \frac{I_1}{z_4^2} + \frac{I_2}{z_4^2} \right) \right] \]

\( K \) is the NLO BK kernel for \( \mathcal{N}=4 \) SYM

\[ \mathcal{A}_2 = 4(U_4 - U_1)^{dd'} (U_5 - U_2)^{ee'} \]

\[ \left\{ i \left[ f^{ad'e'} (t^d U_1 t^a)_{ij} (t^e U_2)_{kl} - f^{ade} (t^a U_1 t^{d'})_{ij} (U_2 t^{e'})_{kl} \right] J_{1245} \ln \frac{z_2^{14}}{z_2^{15}} \right\} \]

\[ + i \left[ f^{ad'e'} (t^d U_1)_{ij} (t^e U_2 t^a)_{kl} - f^{ade} (U_1 t^{d'})_{ij} (t^a U_2 t^{e'})_{kl} \right] J_{2154} \ln \frac{z_2^{24}}{z_2^{25}} \]

\[ J_{1245} \equiv J(z_1, z_2, z_4, z_5) = \frac{(z_1 z_2 z_5)}{z_2^2 z_2^{12} z_2 z_4} - 2 \frac{(z_1 z_2 z_5)(z_1 z_2 z_5)}{z_2^2 z_2^{12} z_2 z_4} + 2 \frac{(z_2 z_4 z_5)}{z_2^2 z_2^{12} z_2 z_4}, \]
\[ A_3 = 2U_{4d'}^{dd'} \left\{ i [f^{ad'e'}(U_1 t^a)_{ij} (t^d t^e U_2)_{kl} - f^{ade'}(t^a U_1)_{ij} (U_2 t^e' t^{d'})_{kl}] \right\} \]

\[ \times \left[ \mathcal{J}_{1245} \ln \frac{z_{24}^2}{z_{15}^2} + (J_{2145} - J_{2154}) \ln \frac{z_{24}^2}{z_{25}^2} \right] (U_5 - U_2)^{ee'} \]

\[ + \ i [f^{ad'e'}(t^d t^e U_1)_{ij} (U_2 t^a)_{kl} - f^{ade'}(U_1 t^e' t^{d'})_{ij} (t^a U_2)_{kl}] \]

\[ \times \left[ \mathcal{J}_{2145} \ln \frac{z_{24}^2}{z_{25}^2} + (J_{1245} - J_{1254}) \ln \frac{z_{14}^2}{z_{15}^2} \right] (U_5 - U_1)^{ee'} \right\} \]

\[ \mathcal{J}_{1245} \equiv \mathcal{J}(z_1, z_2, z_4, z_5) \]

\[ = \frac{(z_{24}, z_{25})}{z_{24}^2 z_{25}^2 z_{45}^2} - \frac{2(z_{24}, z_{45})(z_{15}, z_{25})}{z_{24}^2 z_{25}^2 z_{45}^2 z_{15}^2 z_{45}^2} + \frac{2(z_{25}, z_{45})(z_{14}, z_{24})}{z_{14}^2 z_{24}^2 z_{25}^2 z_{45}^2} - \frac{2(z_{14}, z_{24})(z_{15}, z_{25})}{z_{14}^2 z_{15}^2 z_{24}^2 z_{25}^2} \]
\[ \mathcal{B}_1 = 2 \ln \frac{z_{14}^2}{z_{12}^2} \ln \frac{z_{24}^2}{z_{12}^2} \times \left\{ (U_4 - U_1)^{ab} i \left[ f^{bde} (t^a U_1 t^d)_{ij} (U_2 t^e)_{kl} + f^{ade} (t^e U_1 t^b)_{ij} (t^d U_2)_{kl} \right] \left[ \frac{(z_{14}, z_{24})}{z_{14}^2 z_{24}^2} - \frac{1}{z_{14}^2} \right] \right\} \]

\[ + (U_4 - U_2)^{ab} i \left[ f^{bde} (U_1 t^e)_{ij} (t^a U_2 t^d)_{kl} + f^{ade} (t^d U_1)_{ij} (t^e U_2 t^b)_{kl} \right] \left[ \frac{(z_{14}, z_{24})}{z_{14}^2 z_{24}^2} - \frac{1}{z_{24}^2} \right] \right\} \]
\[ J_{12345} \equiv J(z_1, z_2, z_3, z_4, z_5) = -\frac{2(z_{14}, z_{34})(z_{25}, z_{35})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{35}^2} \]

\[ -\frac{2(z_{14}, z_{45})(z_{25}, z_{35})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{45}^2} + \frac{2(z_{25}, z_{45})(z_{14}, z_{34})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{45}^2} + \frac{(z_{14}, z_{25})}{z_{14}^2 z_{25}^2 z_{45}^2} \]
\[
\frac{d}{d\eta} (U_1)_{ij} (U_2)_{kl} (U_3)_{mn} \\
= i \frac{\alpha_s^2}{2\pi^4} \int d^2 z_4 d^2 z_5 \left\{ \mathcal{J}_{12345} \ln \frac{z_{34}^2}{z_{35}^2} \right. \\
\times f^{cde} \left[ (t^a U_1)_{ij} (t^b U_2)_{kl} (t^c U_3)_{mn} (U_4 - U_1)^{ad} (U_5 - U_2)^{be} \\
- (U_1 t^a)_{ij} (U_2 t^b)_{kl} (t^c U_3)_{mn} (U_4 - U_1)^{da} (U_5 - U_2)^{eb} \right] \\
+ \mathcal{J}_{32145} \ln \frac{z_{14}^2}{z_{15}^2} \\
\times f^{ade} \left[ (U_1 t^a)_{ij} (t^b U_2)_{kl} (t^c U_3)_{mn} (U_4 - U_3)^{cd} (U_5 - U_2)^{be} \\
- (t^a U_1)_{ij} \otimes (U_2 t^b)_{kl} (t^c U_3)_{mn} (U_4^{dc} - U_3^{dc}) (U_5^{eb} - U_2^{eb}) \right] \\
+ \mathcal{J}_{13245} \ln \frac{z_{24}^2}{z_{25}^2} \\
\times f^{bde} \left[ (t^a U_1)_{ij} (U_2 t^b)_{kl} (t^c U_3)_{mn} (U_4 - U_1)^{ad} (U_5 - U_3)^{ce} \\
- (U_1 t^a)_{ij} (t^b U_2)_{kl} (t^c U_3)_{mn} (U_4 - U_1)^{da} (U_5 - U_3)^{ec} \right]. \\
\right\}
\]
Dynamics of QCD at high-energy is non-linear.

Scattering amplitudes at high-energy and high-density QCD are factorized in rapidity space using the high-energy OPE in Wilson lines.

BK and B-JIMWLK evolution equations include the energy dependence to scattering amplitude at high energy.

NLO BK and NLO Balitsky hierarchy of evolution equation (NLO B-JIMWLK) has been presented.