Rapidity evolution of Wilson lines at the next-to-leading order: Balitsky-JIMWLK equation at NLO

Giovanni Antonio Chirilli

The Ohio State University

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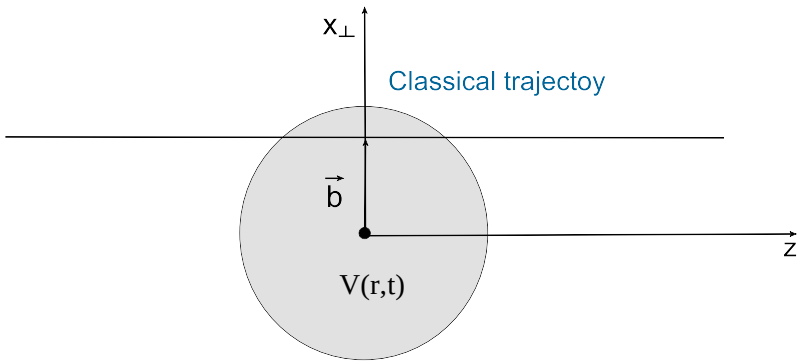
Outline

- High-energy QCD scattering processes and Wilson lines.
- High-energy Operator Product Expansion: factorization in rapidity space.
- Evolution equation and background field method.
- NLO BK equation.
- Hierarchy of Wilson lines evolution at NLO: The Balitsky-JIMWLK evolution equation at NLO.
- Conclusions.

High-energy scattering in quantum mechanics and QED



- High-energy: $E >> V(x)$ WKB approximation.
- Replace the exact wave function by the semi-classical wave function.
- $\Psi(\vec{r},t) = e^{-\frac{i}{\hbar}(Et-kx)} e^{-\frac{i}{\hbar}\int_{-\infty}^{z} dz' V(z')}$

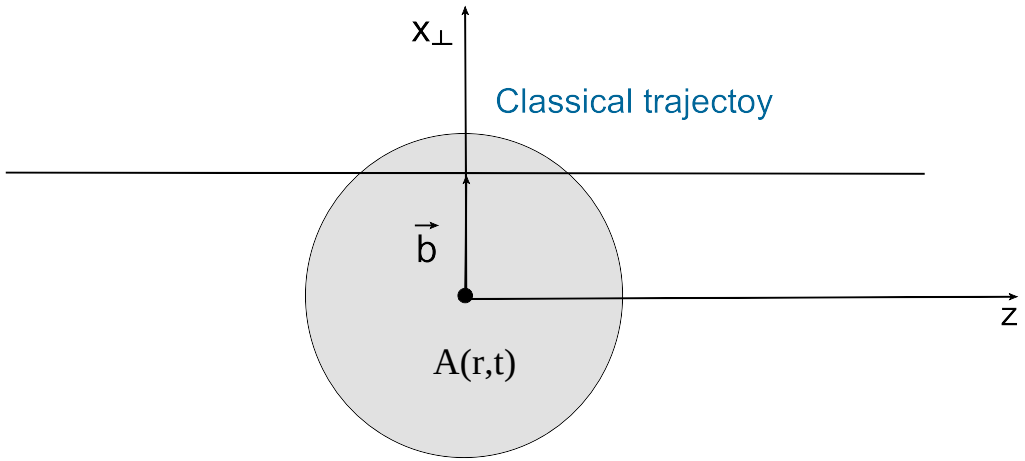
At high-energy $\Psi = \Psi_{\text{free}} \times$ phase factor ordered along the line parallel to $\vec{v}$. The scattering amplitude is proportional to $\Psi(t=-\infty)$

$$U(x_\perp) = e^{\frac{-i}{v\hbar}\int_{-\infty}^{+\infty} dz' V(z'+x_\perp)}$$

In QED

$$U(x_\perp) = e^{\frac{-ie}{c\hbar}\int_{-\infty}^{+\infty} dt \dot{x}_\mu A^\mu(x(t))}$$

High-energy scattering in QCD



phase factor for the high-energy scattering: Wilson-line operator

$$U(x_\perp, v) = \mathrm{P}e^{\frac{-ig}{c\hbar}\int_{-\infty}^{+\infty} dt\ \dot{x}_\mu A^\mu(x(t))}$$

$$\mathrm{P}e^{\int_{-\infty}^{+\infty} dt A(t)} = 1 + \int_{-\infty}^{+\infty} dt\ A(t) + \int_{-\infty}^{+\infty} dt\ A(t) \int_{-\infty}^{t} dt'\ A(t') + \ldots$$

Propagation in the shock wave: Wilson line (Spectator frame)

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

Propagation in the shock wave: Wilson line (Spectator frame)

$$[z', z] = Pe^{ig\int_0^1 du(z'-z)^\mu A_\mu(uz'+(1-u)z)} \qquad U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]$$

$$p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu, \qquad p_1^\mu = \sqrt{s/2}(1,0,0,1), \qquad p_2^\mu = \sqrt{s/2}(1,0,0,-1)$$

s center-of-mass energy.

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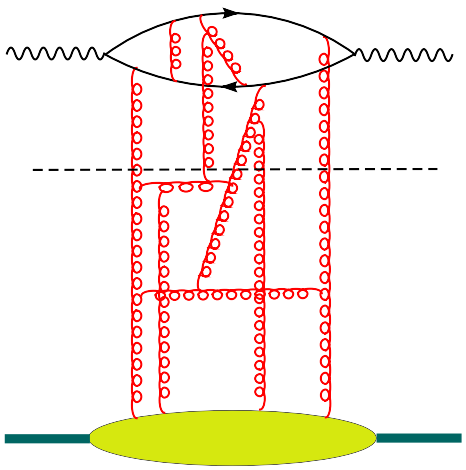

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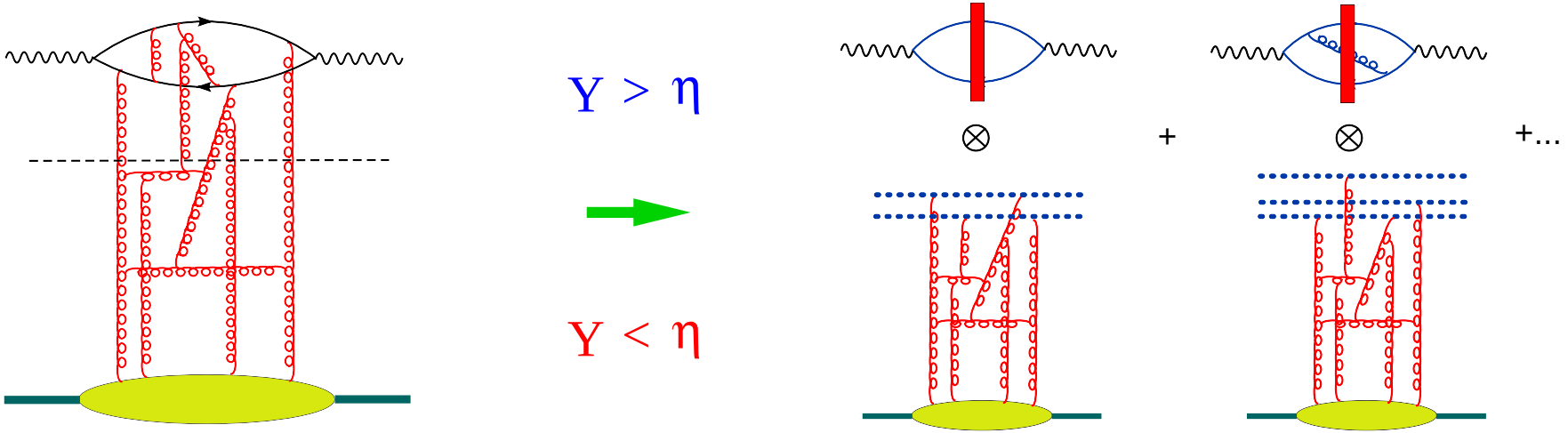
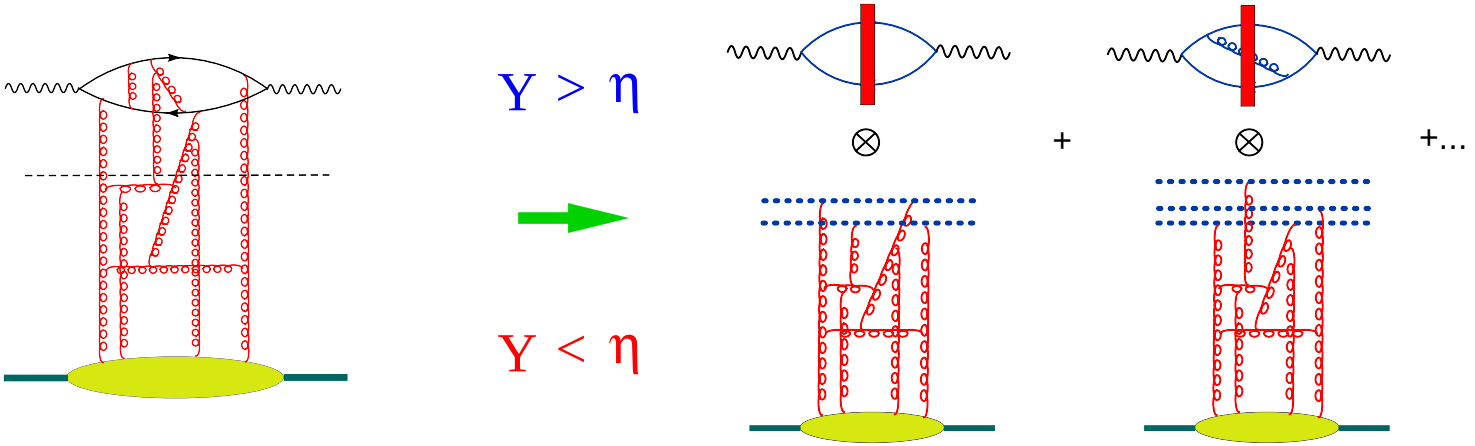


High-energy Operator Product Expansion



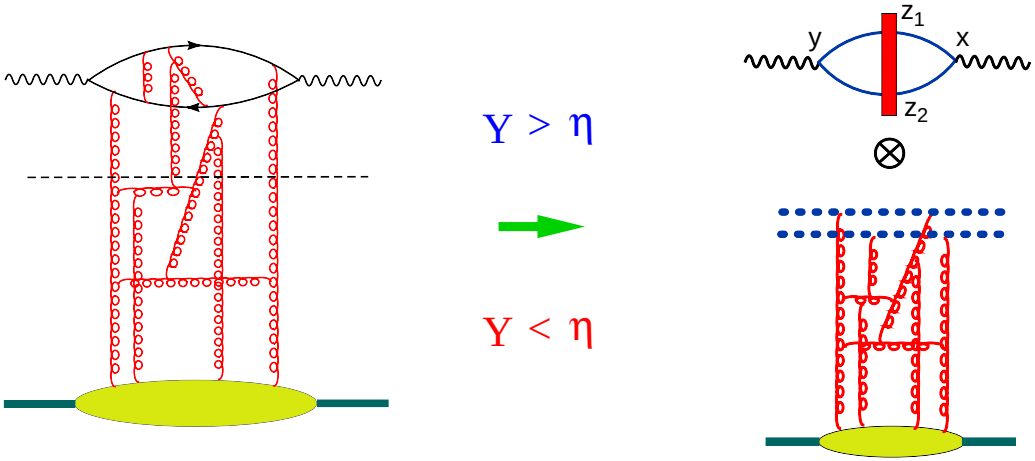
$$\langle B|\mathrm{T}\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|B\rangle \simeq \int d^2z_1 d^2z_2\; I^{LO}_{\mu\nu}(z_1,z_2;x,y)\langle B|\mathrm{tr}\{U^\eta_{z_1}U^{\dagger\,\eta}_{z_2}\}|B\rangle$$

$$+\frac{\alpha_s}{\pi}\int d^2z_1 d^2z_2 d^2z_3\; I^{NLO}_{\mu\nu}(z_1,z_2,z_3;x,y)\langle B|\mathrm{tr}\{U^\eta_{z_1}U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{U^\eta_{z_3}U^{\dagger\eta}_{z_2}\}|B\rangle$$

- $\eta = \ln\frac{1}{x_B}$
- $|B\rangle$ Target state

Leading Order

$$\left[\langle T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}\rangle_A\right]^{\rm LO} = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\,{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$

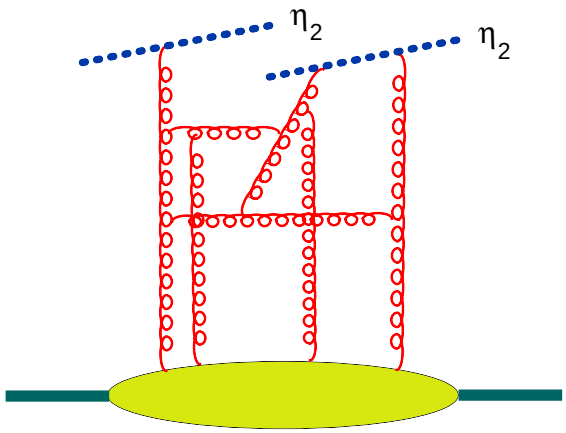


$$\langle B|T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\}|B\rangle = \int \frac{d^2z_1 d^2z_2}{z_{12}^4}\, I^{\rm LO}_{\mu\nu}(x,y;z_1,z_2)\langle B|{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|B\rangle + \ldots$$

- If we use a model to evaluate $\langle B|{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|B\rangle$ we can calculate the DIS cross-section.
- If we want to include energy dependence to the DIS cross section, we need to find the evolution of $\langle B|{\rm tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}|B\rangle$ with respect to the rapidity parameter η.

Regularization of the rapidity divergence

Matrix elements of Wilson lines: $\langle \mathrm{tr}\{U(x)U^\dagger(y)\}\rangle_A$ are divergent



For light-like Wilson lines loop integrals are divergent in the longitudinal direction

$$\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty$$

Regularization by: slope

$$U^\eta(x_\perp) = \mathrm{Pexp}\left\{ig \int_{-\infty}^{\infty} du\, n_\mu\, A^\mu(un + x_\perp)\right\} \qquad n^\mu = p_1^\mu + e^{-2\eta} p_2^\mu$$

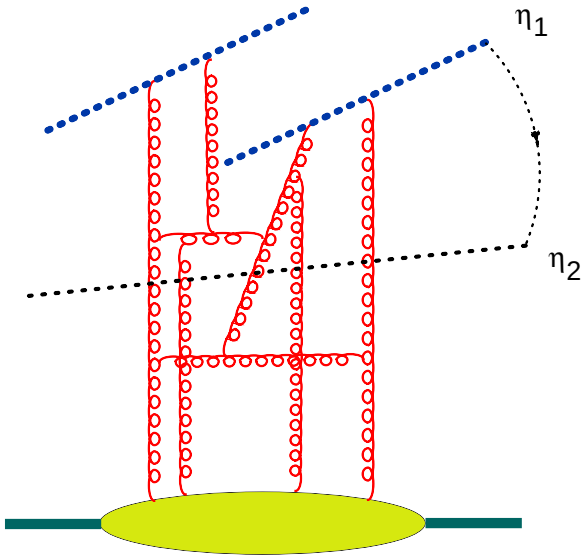
- At NLO the regularization by rigid cut-off is more convenient.

Evolution Equation

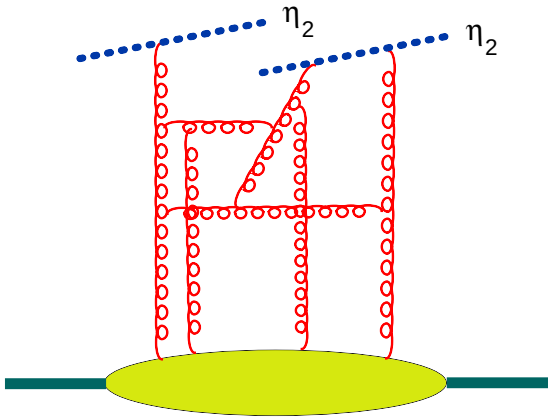
$$\frac{d}{d\eta}\mathrm{tr}\{\hat{U}_x\hat{U}_y^\dagger\} \quad \Rightarrow \quad \frac{d}{d\eta}\langle \mathrm{tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle$$

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidity $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidity up to η_2).

In the frame $||$ to η_1 the gluons with $\eta < \eta_1$ are seen as pancake.

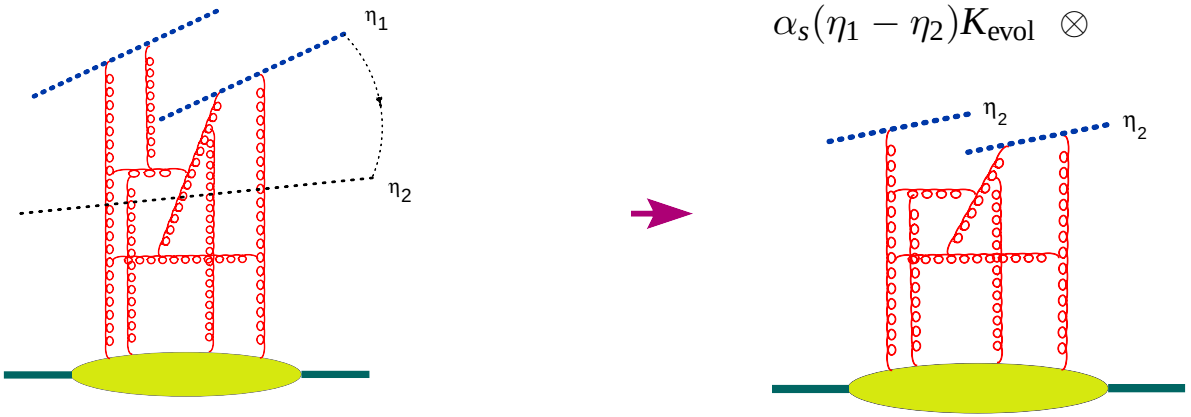


$$\alpha_s(\eta_1 - \eta_2)K_{\mathrm{evol}} \quad \otimes$$



Particles with different rapidity perceive each other as Wilson lines.

Evolution Equation



- Separate fields in quantum and classical according to low and large rapidity. Formally we may write:

$$\langle B|\mathcal{O}^{\eta_1}|B\rangle \to \langle \mathcal{O}^{\eta_1}\rangle_A \to \langle \mathcal{O}'^{\eta_2} \otimes \mathcal{O}'^{\eta_1}\rangle_A$$

- Integrate over the quantum fields and get one-loop rapidity evolution of the operator $\mathcal{O}$

$$\langle \mathcal{O}^{\eta_1}\rangle_A = \alpha_s(\eta_1 - \eta_2) K_{\text{evol}} \otimes \langle \mathcal{O}'^{\eta_2}\rangle_A$$

- Where in principle $\mathcal{O}$ and $\mathcal{O}'$ are different operators.

Non-linear evolution equation

- Linear case $\mathcal{O}^{\eta_1} = \alpha_s \Delta\eta \, K_{\rm evol} \otimes \mathcal{O}^{\eta_2}$

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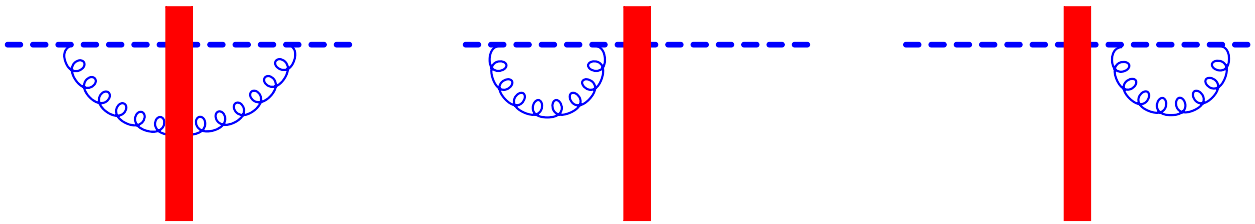

$$\langle \{U_x^{\eta_1}\}_{ij} \rangle_A = \frac{\alpha_s}{2\pi^2} \Delta\eta \int \frac{d^2 z_\perp}{(x-z)_\perp^2} \left[\langle {\rm tr}\{U_x^{\eta_2} U_z^{\eta_2 \dagger}\} \{U_z^{\eta_2}\}_{ij} \rangle_A - \langle \frac{1}{N_c} \{U_x^{\eta_2}\}_{ij} \rangle_A \right]$$

$\Delta = \eta_1 - \eta_2$

$$\{U_x^{\dagger \eta_1}\}_{ij}, \quad \{U_x^{\eta_1} U_y^{\eta_1}\}_{ij}, \quad \{U_x^{\eta_1} U_y^{\dagger \eta_1}\}_{ij}, \quad \{U_x^{\dagger \eta_1} U_y^{\dagger \eta_1}\}_{ij}$$

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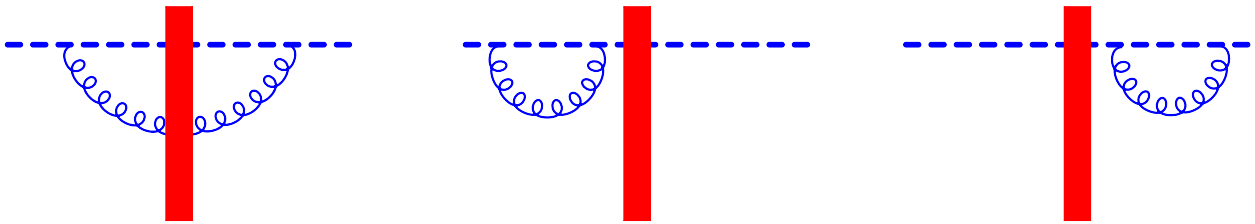

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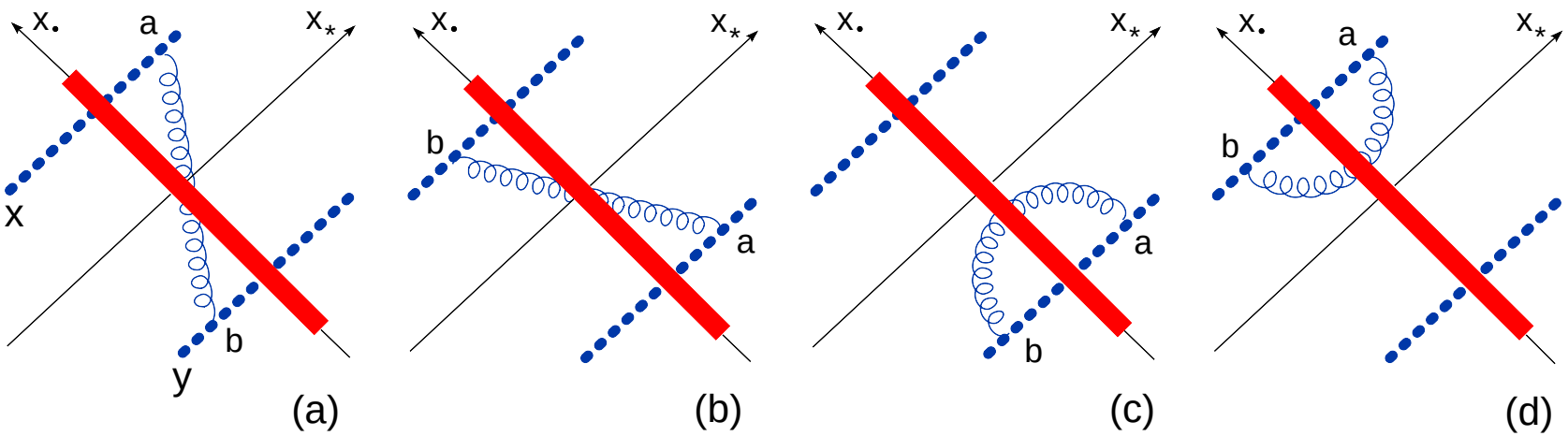
Obtain a set of rules that allow one to get the LO evolution of any trace or product of traces of Wilson lines

Leading order: BK equation

$$\frac{d}{d\eta}\mathrm{tr}\{\hat{U}_x\hat{U}_y^\dagger\} = K_{\mathrm{LO}}\mathrm{tr}\{\hat{U}_x\hat{U}_y^\dagger\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle \mathrm{tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{tr}\{\hat{U}_x\hat{U}_y^\dagger\}\rangle_{\mathrm{shockwave}}$$



$x_\bullet = \sqrt{\frac{s}{2}}x^-$ $\qquad$ $x_* = \sqrt{\frac{s}{2}}x^+$

Non-linear evolution equation: BK equation

$$U_z^{ab} = 2\mathrm{tr}\{t^a U_z t^b U_z^\dagger\} \quad \Rightarrow (U_x U_y^\dagger)^{\eta_1} \to (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}$$

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$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

BK equation: **Ian Balitsky (1996), Yu. Kovchegov (1999)**

$$\frac{d}{d\eta}\hat{\mathcal{U}}(x,y) = \frac{\alpha_s N_c}{2\pi^2}\int \frac{d^2z\,(x-y)^2}{(x-z)^2(y-z)^2}\left\{\hat{\mathcal{U}}(x,z) + \hat{\mathcal{U}}(z,y) - \hat{\mathcal{U}}(x,y) - \hat{\mathcal{U}}(x,z)\hat{\mathcal{U}}(z,y)\right\}$$

Alternative approach: JIMWLK (1997-2000)

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Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

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Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1,\ \alpha_s\eta \sim 1$)

LLA for DIS in sQCD $\Rightarrow$ BK eqn (LLA: $\alpha_s \ll 1,\ \alpha_s\eta \sim 1,\ \alpha_s^2 A^{1/3} \sim 1$)

(s for semi-classical)

Motivation: Why NLO correction?

- How to take higher-order corrections into account (either for BFKL or non-linear evolution equation).
- Higher-order corrections are needed to improve phenomenology:
  - Determine the argument of the coupling constant.
  - Gives precision of LO.
- Check conformal invariance (in $\mathcal{N}$=4 SYM)

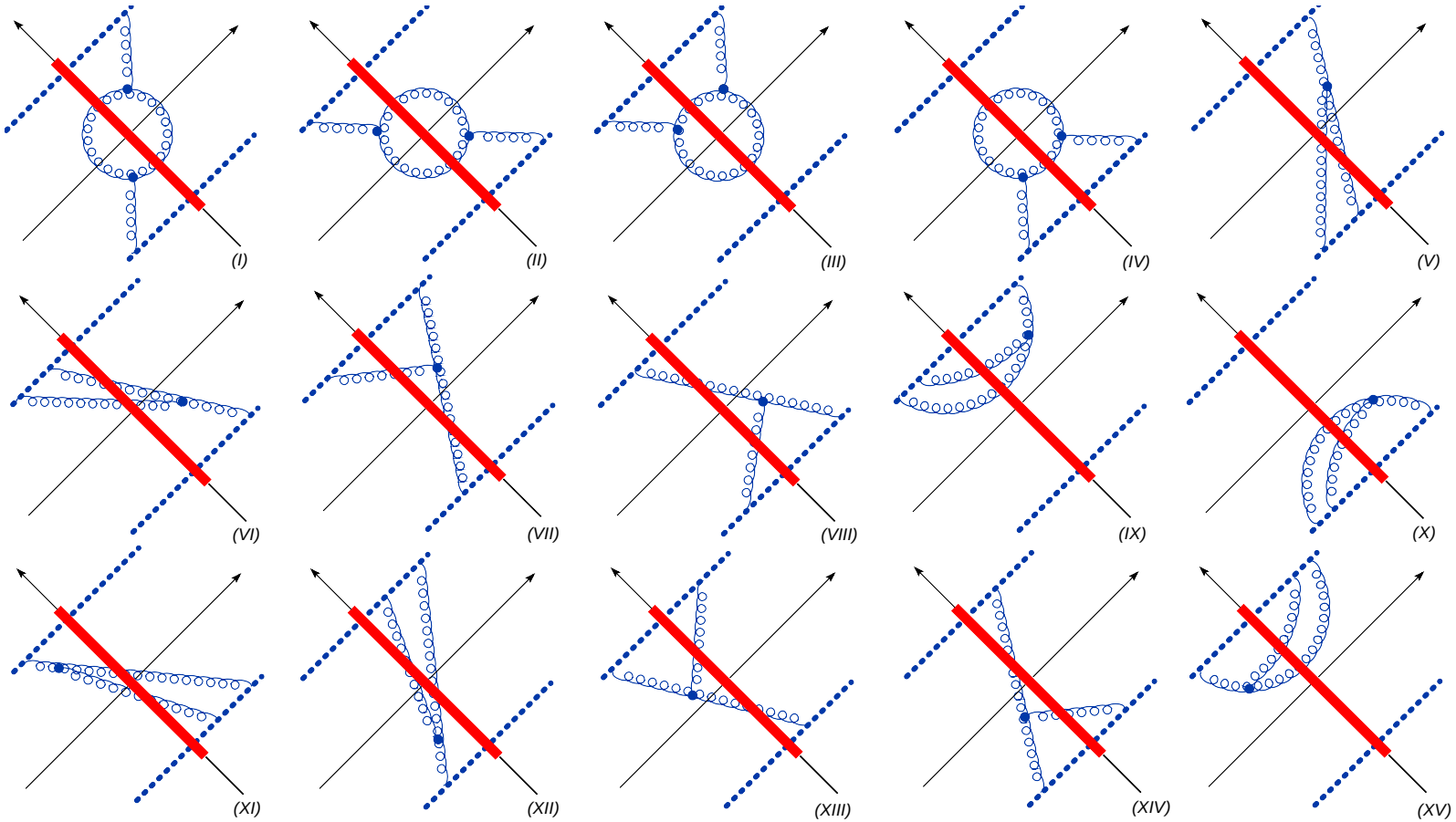
$$\frac{d}{d\eta}Tr\{U_x U_y^\dagger\} =$$
$$\int \frac{d^2z}{2\pi^2}\left(\alpha_s \frac{(x-y)^2}{(x-z)^2(z-y)^2} + \alpha_s^2 K_{NLO}(x,y,z)\right)[Tr\{U_x U_z^\dagger\}Tr\{U_z U_y^\dagger\} - N_c Tr\{U_x U_y^\dagger\}] +$$
$$\alpha_s^2 \int d^2z d^2z' \left(K_4(x,y,z,z')\{U_x, U_{z'}^\dagger, U_z, U_y^\dagger\} + K_6(x,y,z,z')\{U_x, U_{z'}^\dagger, U_{z'}, U_z, U_z^\dagger, U_y^\dagger\}\right)$$

K_{NLO} is the next-to-leading order correction to the dipole kernel and K_4 and K_6 are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.

- We need to calculate some diagrams analytically (pen and paper).

Diagrams of the NLO gluon contribution

Diagrams with 2 gluons interaction



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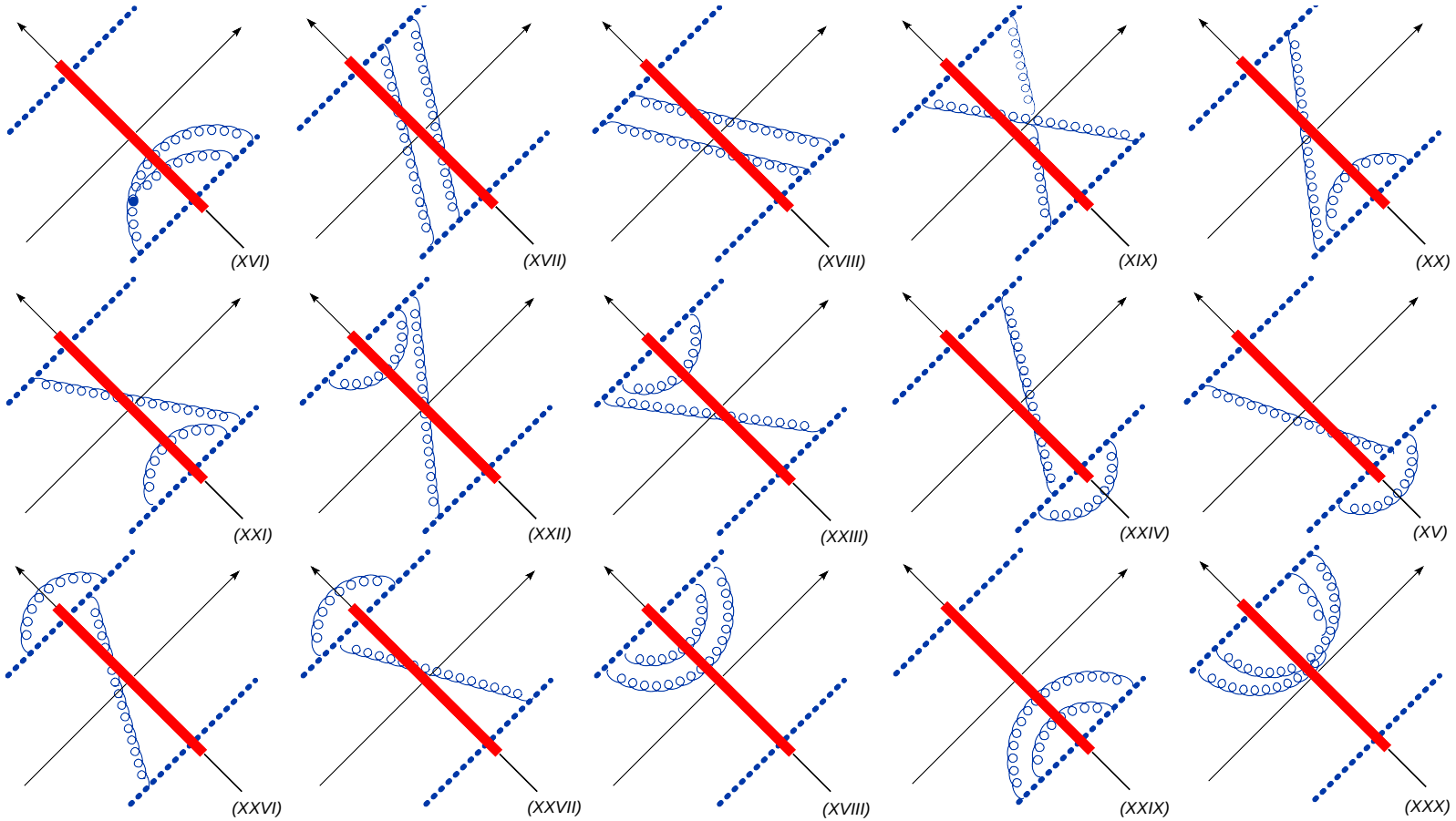

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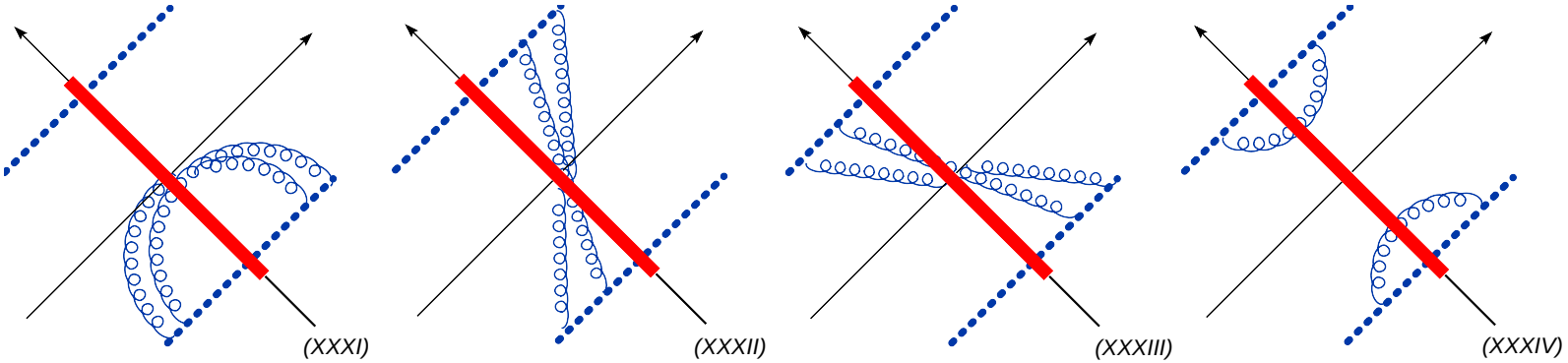
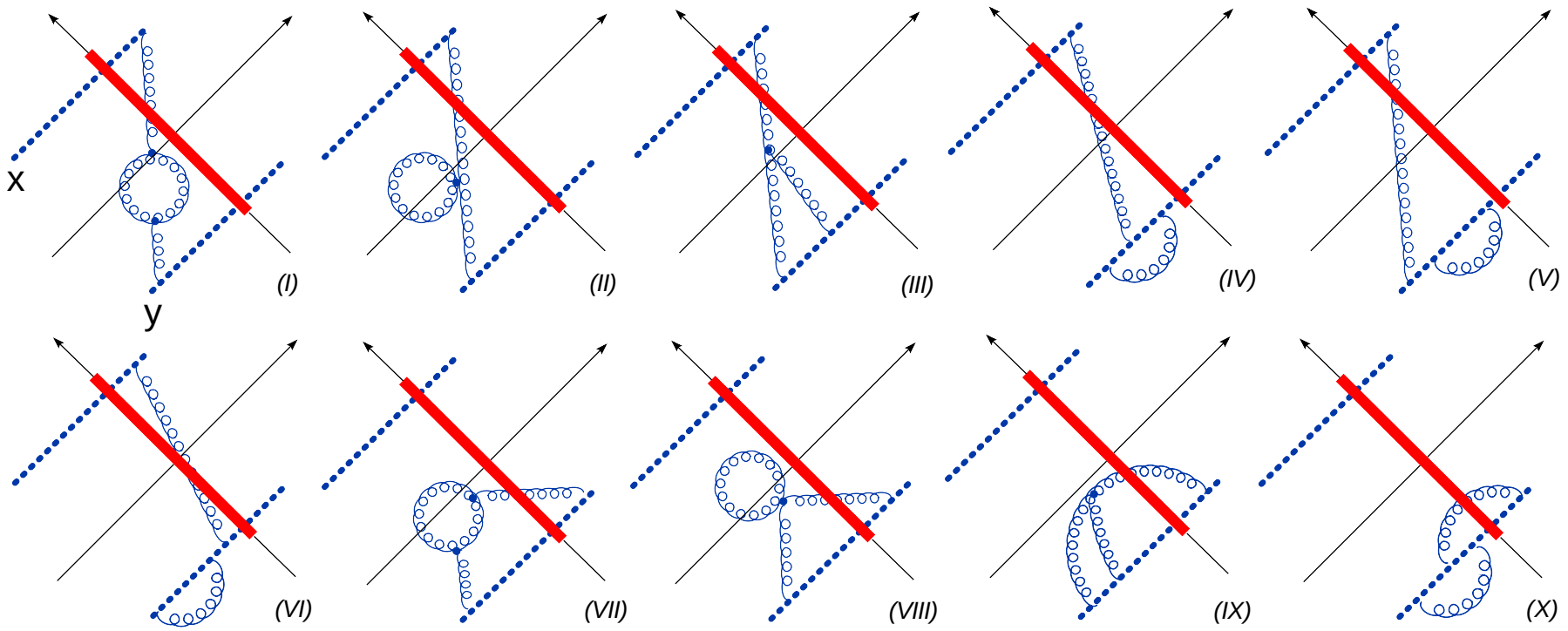


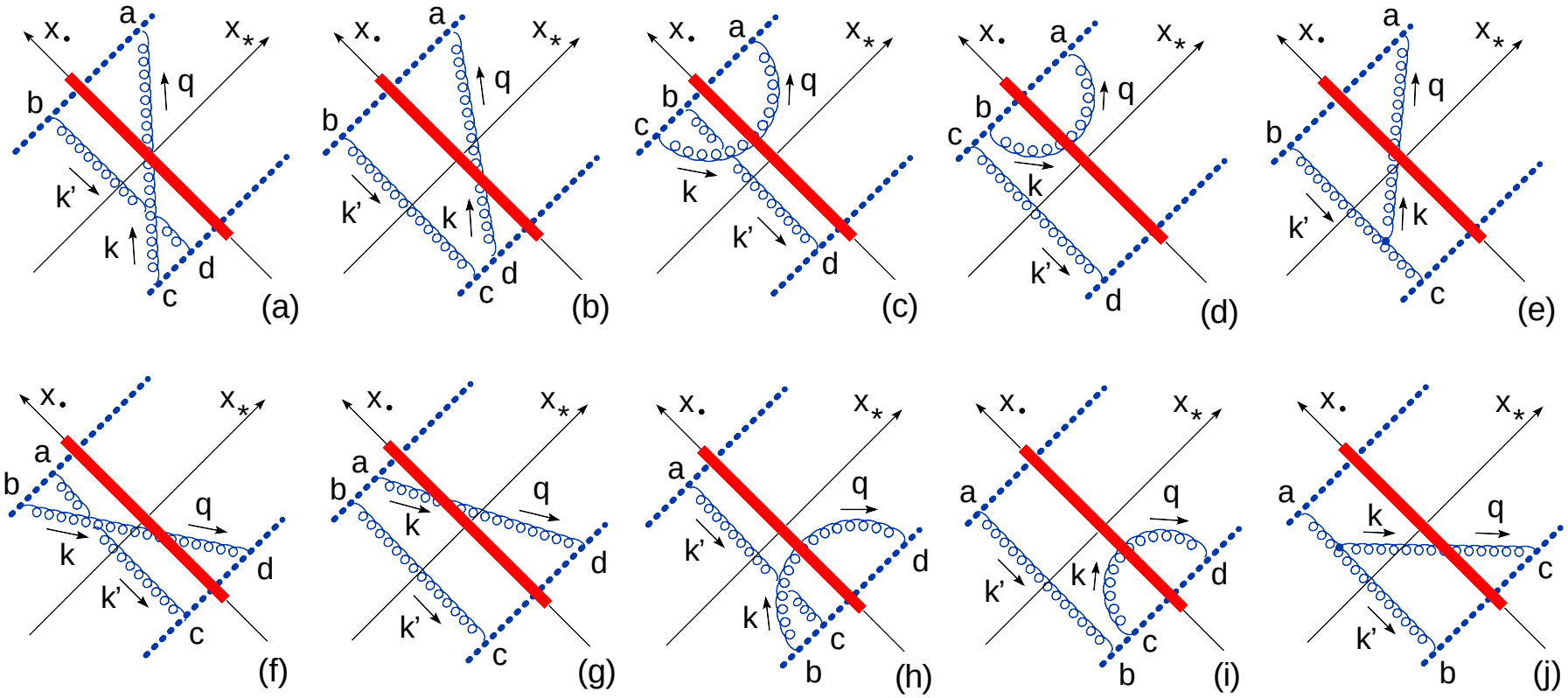
Diagrams of the NLO gluon contribution

"Running coupling" diagrams



Diagrams of the NLO gluon contribution

$1 \to 2$ dipole transition diagrams



NLO evolution of composite "conformal" dipoles in QCD

$$\frac{d}{d\eta}[\mathrm{tr}\{\hat{U}_{z_1}U^\dagger_{z_2}\}]^{\mathrm{conf}} = \frac{\alpha_s}{2\pi^2}\int d^2z_3\,\Big([\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]^{\mathrm{conf}}$$

$$\times\frac{z_{12}^2}{z_{13}^2z_{23}^2}\Big[1+\frac{\alpha_sN_c}{4\pi}\Big(b\ln z_{12}^2\mu^2 + b\frac{z_{13}^2-z_{23}^2}{z_{13}^2z_{23}^2}\ln\frac{z_{13}^2}{z_{23}^2}+\frac{67}{9}-\frac{\pi^2}{3}\Big)\Big]$$

$$+\frac{\alpha_s}{4\pi^2}\int\frac{d^2z_4}{z_{34}^4}\Big\{\Big[-2+\frac{{z_{14}}^2z_{23}^2+{z_{24}}^2z_{13}^2-4z_{12}^2z_{34}^2}{2({z_{14}}^2z_{23}^2-{z_{24}}^2z_{13}^2)}\ln\frac{{z_{14}}^2z_{23}^2}{{z_{24}}^2z_{13}^2}\Big]$$

$$\times[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_4}\}\{\hat{U}_{z_4}\hat{U}^\dagger_{z_2}\} - \mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\hat{U}_{z_4}\hat{U}^\dagger_{z_2}\hat{U}_{z_3}\hat{U}^\dagger_{z_4}\} - (z_4\to z_3)]$$

$$+\frac{z_{12}^2z_{34}^2}{z_{13}^2{z_{24}}^2}\Big[2\ln\frac{z_{12}^2z_{34}^2}{z_{14}^2z_{23}^2}+\Big(1+\frac{z_{12}^2z_{34}^2}{z_{13}^2z_{24}^2-z_{14}^2z_{23}^2}\Big)\ln\frac{z_{13}^2{z_{24}}^2}{{z_{14}}^2z_{23}^2}\Big]$$

$$\times[\mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_4}\}\mathrm{tr}\{\hat{U}_{z_4}\hat{U}^\dagger_{z_2}\} - \mathrm{tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_4}\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\hat{U}_{z_4}\hat{U}^\dagger_{z_3}\} - (z_4\to z_3)]\Big\}$$

$$b = \tfrac{11}{3}N_c - \tfrac{2}{3}n_f$$

I. Balitsky and G.A.C

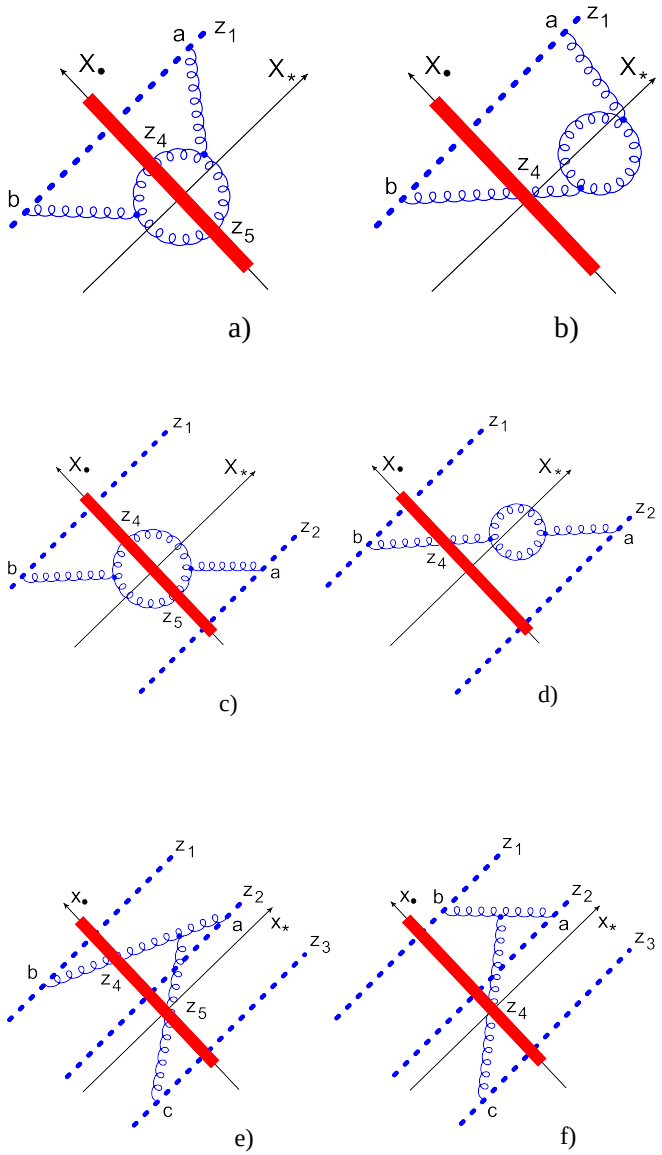
$K_{\mathrm{NLO\ BK}}$ = Running coupling part + Conformal "non-analytic" (in j) part + Conformal analytic ($\mathcal{N}=4$) part

Linearized $K_{\mathrm{NLO\ BK}}$ reproduces the known result for the forward NLO BFKL kernel Fadin and Lipatov (1998).

NLO Balitsky-JIMWLK evolution equation

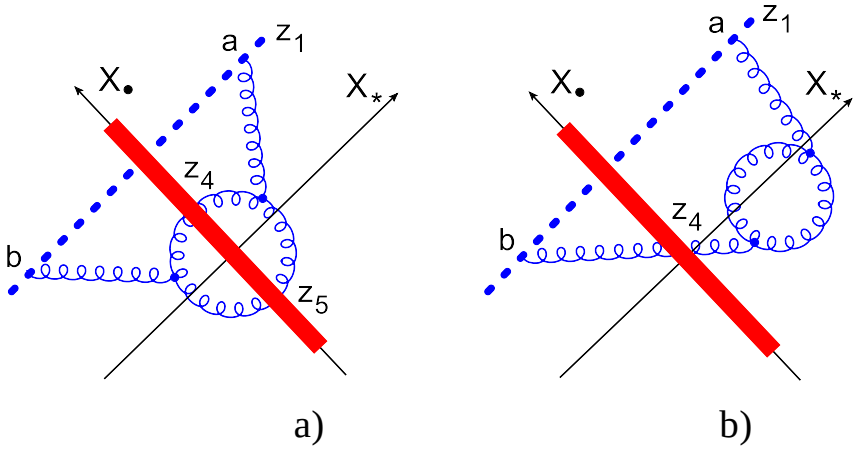
- In proton-Nucleus and Nucleus-Nucleus collisions there are also quadrupole Wilson line operators $\mathrm{tr}\{U_x U_y^\dagger U_w U_z^\dagger\}$.
- $\Rightarrow$ Need NLO Balitsky-JIMWLK evolution equation.

NLO Balitsky-JIMWLK evolution equation



Sample of diagrams: a), b) are self-interactions; c), d) are pairwise interactions; e), f) are triple interactions.

Self interaction at NLO I. Balitsky and G.A.C. (2013)



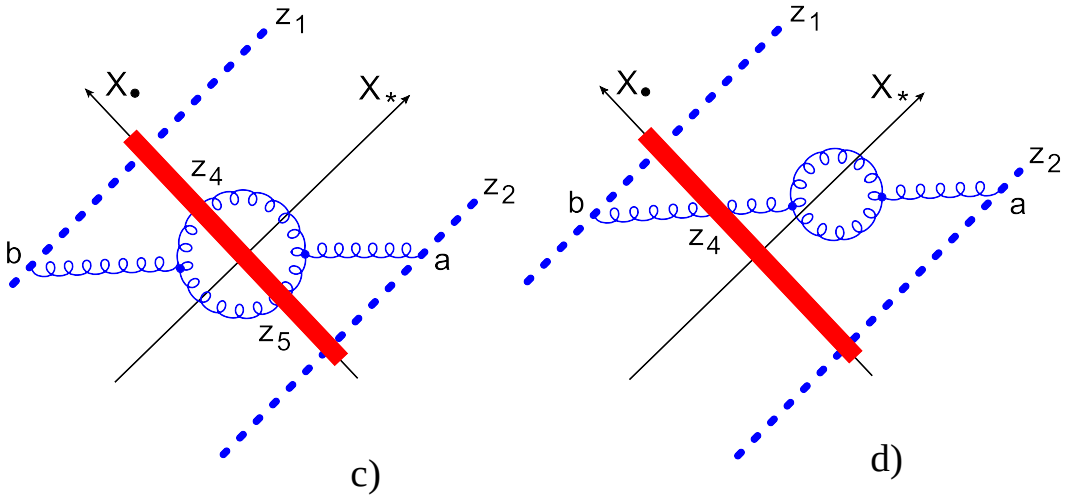
$$\frac{d}{d\eta}(U_1)_{ij} = \frac{\alpha_s^2}{8\pi^4}\int \frac{d^2z_4 d^2z_5}{z_{45}^2}\Big\{U_4^{dd'}(U_5^{ee'} - U_4^{ee'})$$

$$\times\Big(\Big[2I_1 - \frac{4}{z_{45}^2}\Big]f^{ade}f^{bd'e'}(t^aU_1t^b)_{ij} + \frac{(z_{14},z_{15})}{z_{14}^2 z_{15}^2}\ln\frac{z_{14}^2}{z_{15}^2}\big[if^{ad'e'}(\{t^d,t^e\}U_1t^a)_{ij} - if^{ade}(t^aU_1\{t^{d'},t^{e'}\})_{ij}\big]\Big)$$

$$+\frac{\alpha_s^2 N_c}{4\pi^3}\int d^2z_4\,(U_4^{ab} - U_1^{ab})(t^aU_1t^b)_{ij}\times\frac{1}{z_{14}^2}\Big[\frac{11}{3}\ln z_{14}^2\mu^2 + \frac{67}{9} - \frac{\pi^2}{3}\Big]$$

$$I_1 \equiv I(z_1,z_4,z_5) = \frac{\ln z_{14}^2/z_{15}^2}{z_{14}^2 - z_{15}^2}\Big[\frac{z_{14}^2 + z_{15}^2}{z_{45}^2} - \frac{(z_{14},z_{15})}{z_{14}^2} - \frac{(z_{14},z_{15})}{z_{15}^2} - 2\Big]$$

Pairwise Interaction at NLO I. Balitsky and G.A.C. (2013)



$$\frac{d}{d\eta}(U_1)_{ij}(U_2^\dagger)_{kl} = \frac{\alpha_s^2}{8\pi^4}\int d^2z_4 d^2z_5(\mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3) + \frac{\alpha_s^2 N_c}{8\pi^3}\int d^2z_4(\mathcal{B}_1+N_c\mathcal{B}_2)$$

$$\mathcal{A}_1 = \left[(t^a U_1)_{ij}(U_2 t^b)_{kl} + (U_1 t^b)_{ij}(t^a U_2)_{kl}\right]$$
$$\times\left[f^{ade} f^{bd'e'} U_4^{dd'}(U_5^{ee'} - U_4^{ee'})\left(-K - \frac{4}{z_{45}^4} + \frac{I_1}{z_{45}^2} + \frac{I_2}{z_{45}^2}\right)\right]$$

K is the NLO BK kernel for $\mathcal{N}$=4 SYM

$$\mathcal{A}_2 = 4(U_4 - U_1)^{dd'}(U_5 - U_2)^{ee'}$$
$$\Big\{ i\left[f^{ad'e'}(t^d U_1 t^a)_{ij}(t^e U_2)_{kl} - f^{ade}(t^a U_1 t^{d'})_{ij}(U_2 t^{e'})_{kl}\right] J_{1245} \ln\frac{z_{14}^2}{z_{15}^2}$$
$$+ i\left[f^{ad'e'}(t^d U_1)_{ij}(t^e U_2 t^a)_{kl} - f^{ade}(U_1 t^{d'})_{ij}(t^a U_2 t^{e'})_{kl}\right] J_{2154} \ln\frac{z_{24}^2}{z_{25}^2}\Big\}$$

$$J_{1245} \equiv J(z_1, z_2, z_4, z_5) = \frac{(z_{14}, z_{25})}{z_{14}^2 z_{25}^2 z_{45}^2} - 2\frac{(z_{15}, z_{45})(z_{15}, z_{25})}{z_{14}^2 z_{15}^2 z_{25}^2 z_{45}^2} + 2\frac{(z_{25}, z_{45})}{z_{14}^2 z_{25}^2 z_{45}^2},]$$

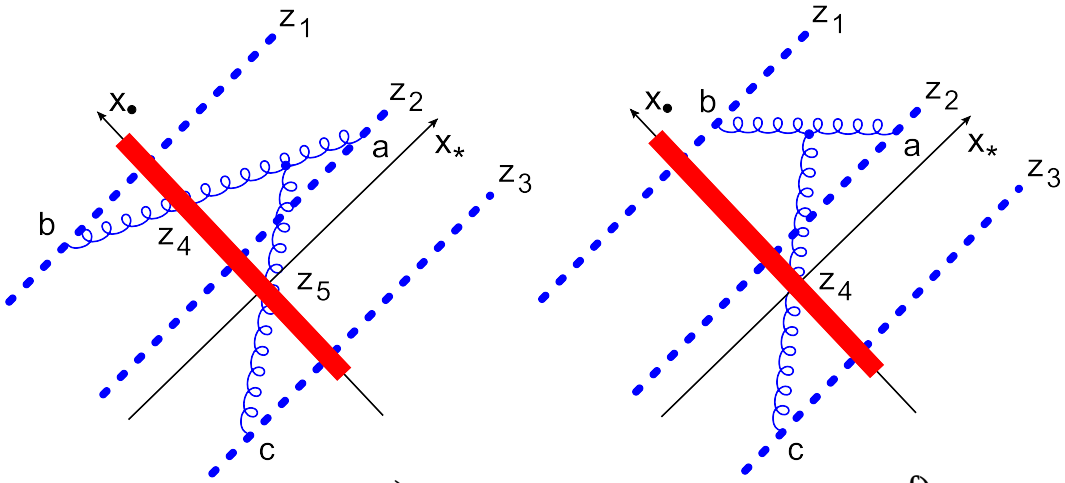
$$
\begin{aligned}
\mathcal{A}_3 \;=\; & 2U_4^{dd'}\Big\{ i\big[f^{ad'e'}(U_1t^a)_{ij}(t^dt^eU_2)_{kl} - f^{ade}(t^aU_1)_{ij}(U_2t^{e'}t^{d'})_{kl}\big] \\
& \times \Big[\mathcal{J}_{1245}\ln\frac{z_{14}^2}{z_{15}^2} + (J_{2145}-J_{2154})\ln\frac{z_{24}^2}{z_{25}^2}\Big](U_5-U_2)^{ee'} \\
& + i\big[f^{ad'e'}(t^dt^eU_1)_{ij}(U_2t^a)_{kl} - f^{ade}(U_1t^{e'}t^{d'})_{ij}(t^aU_2)_{kl}\big] \\
& \times \Big[\mathcal{J}_{2145}\ln\frac{z_{24}^2}{z_{25}^2} + (J_{1245}-J_{1254})\ln\frac{z_{14}^2}{z_{15}^2}\Big](U_5-U_1)^{ee'}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_{1245} \;\equiv\; & \mathcal{J}(z_1,z_2,z_4,z_5) \\
= \; & \frac{(z_{24},z_{25})}{z_{24}^2z_{25}^2z_{45}^2} - \frac{2(z_{24},z_{45})(z_{15},z_{25})}{z_{24}^2z_{25}^2z_{15}^2z_{45}^2} + \frac{2(z_{25},z_{45})(z_{14},z_{24})}{z_{14}^2z_{24}^2z_{25}^2z_{45}^2} - 2\frac{(z_{14},z_{24})(z_{15},z_{25})}{z_{14}^2z_{15}^2z_{24}^2z_{25}^2}
\end{aligned}
$$

$$\mathcal{B}_1 = 2\ln\frac{z_{14}^2}{z_{12}^2}\ln\frac{z_{24}^2}{z_{12}^2}$$
$$\times\Big\{(U_4-U_1)^{ab}i\big[f^{bde}(t^aU_1t^d)_{ij}(U_2t^e)_{kl}+f^{ade}(t^eU_1t^b)_{ij}(t^dU_2)_{kl}\big]\Big[\frac{(z_{14},z_{24})}{z_{14}^2z_{24}^2}-\frac{1}{z_{14}^2}\Big]$$
$$+(U_4-U_2)^{ab}i\big[f^{bde}(U_1t^e)_{ij}(t^aU_2t^d)_{kl}+f^{ade}(t^dU_1)_{ij}(t^eU_2t^b)_{kl}\big]\Big[\frac{(z_{14},z_{24})}{z_{14}^2z_{24}^2}-\frac{1}{z_{24}^2}\Big]\Big\}$$

$$\mathcal{B}_2 = \big[2U_4^{ab}-U_1^{ab}-U_2^{ab}\big]\big[(t^aU_1)_{ij}(U_2t^b)_{kl}+(U_1t^b)_{ij}(t^aU_2)_{kl}\big]$$
$$\times\Big\{\frac{(z_{14},z_{24})}{z_{14}^2z_{24}^2}\Big[\frac{11}{3}\ln z_{12}^2\mu^2+\frac{67}{9}-\frac{\pi^2}{3}\Big]+\frac{11}{3}\Big(\frac{1}{2z_{14}^2}\ln\frac{z_{24}^2}{z_{12}^2}+\frac{1}{2z_{24}^2}\ln\frac{z_{14}^2}{z_{12}^2}\Big)\Big\}$$

Triple interaction I. Balitsky and G.A.C. (2013)



$$\mathcal{J}_{12345} \equiv \mathcal{J}(z_1, z_2, z_3, z_4, z_5) = -\frac{2(z_{14}, z_{34})(z_{25}, z_{35})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{35}^2}$$
$$- \frac{2(z_{14}, z_{45})(z_{25}, z_{35})}{z_{14}^2 z_{25}^2 z_{35}^2 z_{45}^2} + \frac{2(z_{25}, z_{45})(z_{14}, z_{34})}{z_{14}^2 z_{25}^2 z_{34}^2 z_{45}^2} + \frac{(z_{14}, z_{25})}{z_{14}^2 z_{25}^2 z_{45}^2}$$

$$
\begin{aligned}
&\frac{d}{d\eta}(U_1)_{ij}(U_2)_{kl}(U_3)_{mn} \\
&= i\frac{\alpha_s^2}{2\pi^4}\int d^2z_4 d^2z_5 \left\{ \mathcal{J}_{12345} \ln \frac{z_{34}^2}{z_{35}^2} \right. \\
&\times f^{cde}\left[(t^aU_1)_{ij}(t^bU_2)_{kl}(U_3t^c)_{mn}(U_4-U_1)^{ad}(U_5-U_2)^{be}\right. \\
&\left.- (U_1t^a)_{ij}(U_2t^b)_{kl}(t^cU_3)_{mn}(U_4-U_1)^{da}(U_5-U_2)^{eb}\right] \\
&+ \mathcal{J}_{32145} \ln \frac{z_{14}^2}{z_{15}^2} \\
&\times f^{ade}\left[(U_1t^a)_{ij}(t^bU_2)_{kl}(t^cU_3)_{mn}(U_4-U_3)^{cd}(U_5-U_2)^{be}\right. \\
&\left.- (t^aU_1)_{ij}\otimes(U_2t^b)_{kl}(U_3t^c)_{mn}(U_4^{dc}-U_3^{dc})(U_5^{eb}-U_2^{eb})\right] \\
&+ \mathcal{J}_{13245} \ln \frac{z_{24}^2}{z_{25}^2} \\
&\times f^{bde}\left[(t^aU_1)_{ij}(U_2t^b)_{kl}(t^cU_3)_{mn}(U_4-U_1)^{ad}(U_5-U_3)^{ce}\right. \\
&\left.\left.- (U_1t^a)_{ij}(t^bU_2)_{kl}(U_3t^c)_{mn}(U_4-U_1)^{da}(U_5-U_3)^{ec}\right]\right\}
\end{aligned}
$$

Conclusions

- Dynamics of QCD at high-energy is non-linear.
- Scattering amplitudes at high-energy and high-density QCD are factorized in rapidity space using the high-energy OPE in Wilson lines.
- BK and B-JIMWLK evolution equations include the energy dependence to scattering amplitude at high energy.
- NLO BK and NLO Balitsky hierarchy of evolution equation (NLO B-JIMWLK) has been presented.
  - NLO BK equation: I. Balitsky and G.A.C. (2007).
  - Triple interaction: A. Grabovsky (2013).
  - Full (self, pairwise and triple interactions) NLO Balitsky hierarchy (NLO B-JIMWLK) from explicit calculation (confirming independently Grabovsky result): I. Balitsky and G.A.C. (2013).
  - NLO JIMWLK Hamiltonian using Grabovsky and NLO BK results: Kovner and Lublinsky (2013).