Analytical solutions of 2nd order conformal hydrodynamics

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Introduction

- Second order conformal hydrodynamics (where $\varepsilon = 3p$ and bulk scalar $\Pi = 0$) contains many terms in the equations of motion for the temperature T, flow velocity u^μ, and shear stress tensor $\pi^{\mu\nu}$.
- We have derived the first analytical (and semi-analytical) solutions of 2nd order conformal hydro equations [1,2].
- These solutions are found by mapping a nontrivial flow in flat spacetime to a trivial (static) flow in curved spacetime [3] via a Weyl transformation: $g_{\mu\nu} \to \Lambda^2 \hat{g}_{\mu\nu}$.

2nd Order Conformal Hydrodynamics

Energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$ (∇_μ is the space-time covariant derivative):

$$D\epsilon + (\epsilon + p)\vartheta + \pi^{\mu\nu}\sigma_{\mu\nu} = 0 \qquad (\epsilon + p)Du^\mu + \Delta^{\mu\alpha}\nabla_\alpha p + \Delta^\mu_\nu \nabla_\alpha \pi^{\alpha\nu} = 0\,, \tag{1}$$

where we defined the comoving derivative $D \equiv u^\mu \nabla_\mu$ and fluid expansion rate $\vartheta \equiv \nabla_\mu u^\mu$, while

$$\sigma^{\mu\nu} \equiv \nabla^{\langle\mu}u^{\nu\rangle} \equiv \left(\frac{1}{2}(\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\right)\nabla_\alpha u_\beta\,, \tag{2}$$

is the shear tensor. We employ the result of Denicol et al. [4] and generalize it to curved spacetimes [1,2] taking into account the constraints from conformal symmetry [5] to find the most general equation for $\pi^{\mu\nu}$

$$\begin{aligned}\pi^{\mu\nu} = & -2\eta\sigma^{\mu\nu} - \tau_\pi\left(\Delta^\mu_\alpha\Delta^\nu_\beta D\pi^{\alpha\beta} + \frac{4}{3}\pi^{\mu\nu}\vartheta\right) + \lambda_2\pi^{\langle\mu}_{\ \lambda}\Omega^{\nu\rangle\lambda} \\ & + \lambda_1\pi^{\langle\mu}_{\ \lambda}\pi^{\nu\rangle\lambda} + \lambda_3\Omega^{\langle\mu}_{\ \lambda}\Omega^{\nu\rangle\lambda} - \tau_{\pi\pi}\sigma^{\langle\mu}_{\ \lambda}\pi^{\nu\rangle\lambda} - \tilde{\eta}_3\sigma^{\langle\mu}_{\ \lambda}\sigma^{\nu\rangle\lambda} - \tilde{\eta}_4\sigma^{\langle\mu}_{\ \lambda}\Omega^{\nu\rangle\lambda} \\ & + \tau_\sigma\left(\Delta^\mu_\alpha\Delta^\nu_\beta D\sigma^{\alpha\beta} + \frac{1}{3}\sigma^{\mu\nu}\vartheta\right) + \kappa\left(\mathcal{R}^{\langle\mu\nu\rangle} - 2u_\alpha\mathcal{R}^{\alpha\langle\mu\nu\rangle\beta}u_\beta\right)\end{aligned} \tag{3}$$

where $\Omega^{\mu\nu} \equiv \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(\nabla_\alpha u_\beta - \nabla_\beta u_\alpha)$ is the vorticity tensor and η is the shear viscosity that appears in the first-order (Navier-Stokes) theory.

Gubser flow in Israel-Stewart theory

Gubser generalized Bjorken's solution by including a nontrivial $x_\perp$-dependence while retaining boost invariance [3] by performing the Weyl transformation of the metric

$$d\hat{s}^2 \equiv \frac{ds^2}{\tau^2} = \frac{-d\tau^2 + dx_\perp^2 + x_\perp^2 d\phi^2}{\tau^2} + d\eta^2 = -d\varrho^2 + \cosh^2\varrho(d\Theta^2 + \sin^2\Theta d\phi^2) + d\eta^2\,, \tag{4}$$

where

$$\sinh\varrho = -\frac{L^2 - \tau^2 + x_\perp^2}{2L\tau}\,, \qquad \tan\Theta = \frac{2Lx_\perp}{L^2 + \tau^2 - x_\perp^2}\,. \tag{5}$$

Eq. (4) shows that Minkowski space is conformal to $dS_3 \times \mathbb{R}$ (dS_3 is the 3-dimensional de Sitter space) up to a Weyl rescaling factor $\Lambda^2 = \tau^2$. The parameter L has dimension of length and is identified with the 'radius' of dS_3. The flow velocity of the static fluid in $dS_3 \times \mathbb{R}$ is simply $\hat{u}_\varrho = (-1, 0, 0, 0)$. The flow 4-vector in Minkowski space (Milne coordinates) is (with $u_\eta = u_\phi = 0$)

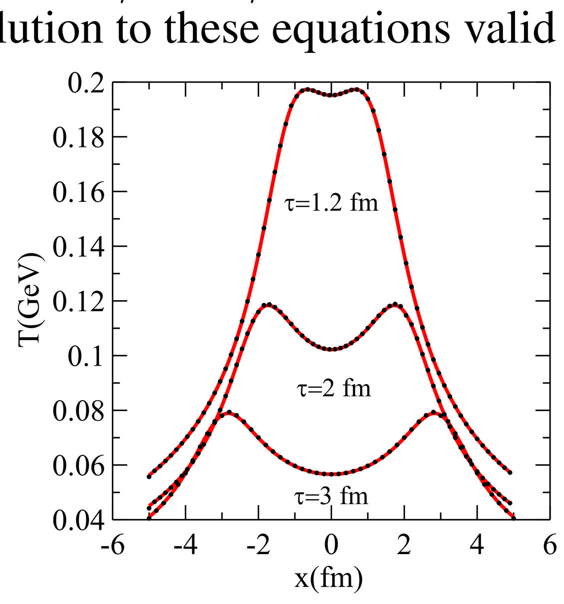
$$u_\tau = -\cosh\left[\tanh^{-1}\frac{2\tau x_\perp}{L^2 + \tau^2 + x_\perp^2}\right]\,, \qquad u_\perp = \sinh\left[\tanh^{-1}\frac{2\tau x_\perp}{L^2 + \tau^2 + x_\perp^2}\right]\,. \tag{6}$$

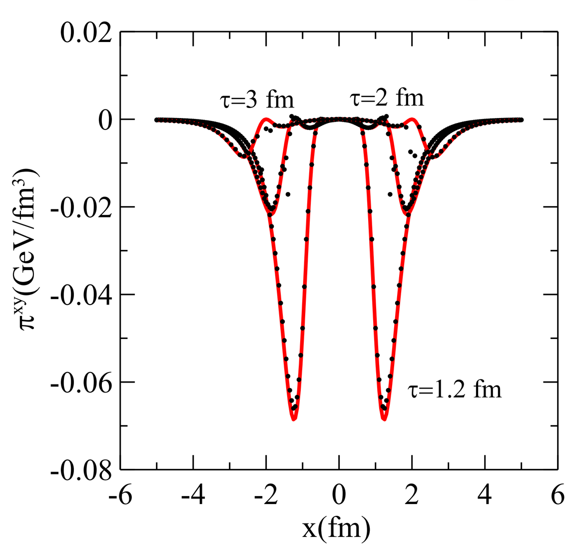
Comparing semi-analytical solution to MUSIC ($\eta/s = 0.2$)

We take the simplest conformal theory in which only η and $\tau_\pi = 5(\eta/s)/T$ are nonzero. The equations drastically simplify in $dS_3 \times \mathbb{R}$ and we have

$$\frac{1}{\hat{T}}\frac{d\hat{T}}{d\rho} + \frac{2}{3}\tanh\rho = \frac{1}{3}\bar{\pi}^\eta_\eta(\rho)\,\tanh\rho \qquad \frac{5}{\hat{T}}\frac{\eta}{s}\left[\frac{d\bar{\pi}^\eta_\eta}{d\rho} + \frac{4}{3}\left(\bar{\pi}^\eta_\eta\right)^2\tanh\rho\right] + \bar{\pi}^\eta_\eta = \frac{4}{3}\frac{\eta}{s\hat{T}}\tanh\rho\,, \tag{7}$$

where $\bar{\pi}^\eta_\eta \equiv \hat{\pi}^\eta_\eta/(\hat{T}\hat{s})$. These equations can then be solved numerically. An analytical solution to these equations valid at low temperatures can be found in [1].





First full analytical solution of 2nd order hydrodynamics

In [2], several solutions of the 2nd order hydro equations were found using the same idea: nontrivial flow in flat spacetime is taken, via a Weyl transformation, to a trivial flow in curved spacetime. The transformation is

$$\begin{aligned} d\hat{s}^2 \equiv \frac{ds^2}{x_\perp^2} &= \frac{-dt^2 + dz^2 + dx_\perp^2}{x_\perp^2} + d\phi^2 \\ &= -\cosh^2\rho\, dT^2 + d\rho^2 + \sinh^2\rho d\bar{\Theta}^2 + d\phi^2\,. \end{aligned} \tag{8}$$

where

$$\tan T = \frac{L^2 + r^2 - t^2}{2Lt}\,, \qquad \cosh\rho = \frac{1}{2Lx_\perp}\sqrt{(L^2 + (r+t)^2)(L^2 + (r-t)^2)}\,. \tag{9}$$

The metric in Eq. (8) is that of $AdS_3 \times S^1$ where AdS_3 is 3-dimensional Anti-de Sitter space. In this space the 4-flow is again trivial but in Minkowski space

$$u_t = -\frac{L^2 + r^2 + t^2}{\sqrt{(L^2 + (r+t)^2)(L^2 + (r-t)^2)}}\,, \qquad \vec{u} = \frac{2t\vec{r}}{\sqrt{(L^2 + (r+t)^2)(L^2 + (r-t)^2)}}\,. \tag{10}$$

An interesting property of this flow is that $\hat{\sigma}_{\mu\nu} = \hat{\Omega}_{\mu\nu} = 0$ and $\hat{\vartheta} = 0$. Therefore, the full equation for the shear stress tensor $\hat{\pi}^{\mu\nu}$ simplifies to

$$\hat{\pi}^{\mu\nu} = \frac{\lambda_1}{\hat{\epsilon}}\hat{\pi}^{\langle\mu}_{\ \lambda}\hat{\pi}^{\nu\rangle\lambda}\,. \tag{11}$$

Note that this equation has a trivial solution ($\hat{\pi}^{\mu\nu} = 0$) but there are other nontrivial solutions as well. Assuming $\hat{\pi}^{\mu\nu}$ is diagonal, we find the non-perturbative solutions in λ_1

$$(\hat{\pi}^{\rho\rho}, \sinh^2\rho\,\hat{\pi}^{\Theta\Theta}, \hat{\pi}^{\phi\phi}) = \frac{\hat{\epsilon}}{\lambda_1} \times \begin{cases} (-1, -1, 2)\,, \\ (-1, 2, -1)\,, \\ (2, -1, -1)\,. \end{cases} \tag{12}$$

The corresponding energy density in Minkowski space reads [2]

$$\epsilon \propto \begin{cases} \frac{1}{(L^2+(t+r)^2)^2(L^2+(t-r)^2)^2}\left(\frac{4L^2x_\perp^2}{(L^2+(t+r)^2)(L^2+(t-r)^2)}\right)^{\frac{9}{2(\lambda_1-3)}}\,, \\ \frac{1}{(L^2+(t+r)^2)^2(L^2+(t-r)^2)^2}\left(1 - \frac{4L^2x_\perp^2}{(L^2+(t+r)^2)(L^2+(t-r)^2)}\right)^{\frac{9}{2(\lambda_1-3)}}\,, \\ \frac{1}{(L^2+(t+r)^2)^2(L^2+(t-r)^2)^2}\left(\frac{4L^2x_\perp^2\left((L^2+(t+r)^2)(L^2+(t-r)^2) - 4L^2x_\perp^2\right)}{(L^2+(t+r)^2)^2(L^2+(t-r)^2)^2}\right)^{-\frac{9}{2(\lambda_1+6)}}\,. \end{cases} \tag{13}$$

Note that ϵ is even under time reversal $t \to -t$ even though $\pi^{\mu\nu}$ is nonzero !!!

Conclusions and Outlook

- These analytical (and semi-analytical) solutions of 2nd order viscous conformal hydrodynamics can be readily used to check the precision of existing numerical codes that solve the (2+1) and (3+1) relativistic hydrodynamic equations.
- The MUSIC code from the McGill group has been shown to reproduce these analytical (and semi-analytical) solutions to a good approximation in the case of Gubser flow.
- Full analytical solutions of the 2nd order conformal hydrodynamic equations, which include nonzero vorticity, can be found using the method shown here (for details see [2]).
- Due to non-linear structure of the equation of motion for $\pi^{\mu\nu}$ (particulary, the term $\frac{\lambda_1}{\epsilon}\hat{\pi}^{\langle\mu}_{\ \lambda}\hat{\pi}^{\nu\rangle\lambda}$), one can see that there are solutions in 2nd order hydrodynamics in which $\pi_{\mu\nu} \neq 0$ but time reversal invariance is not broken.
- It would be interesting to see how these solutions may appear in other approaches such as kinetic theory and the fluid/gravity correspondence.

References

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