Analytical solutions of 2nd order conformal hydrodynamics

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Introduction

- Second order conformal hydrodynamics (where $\varepsilon = 3p$ and bulk scalar $\Pi = 0$) contains many terms in the equations of motion for the temperature $T$, flow velocity $u$, and shear stress tensor $\sigma_{\mu\nu}$.
- We have derived the first analytical (and semi-analytical) solutions of 2nd order conformal hydrodynamic equations [1,2].
- These solutions are found by mapping a nontrivial flow in flat spacetime to a trivial (static) flow in curved spacetime [3] via a Weyl transformation: $g_{\mu\nu} \rightarrow \Lambda^2 g_{\mu\nu}$.

2nd Order Conformal Hydrodynamics

Energy-momentum conservation $\nabla_{\mu} T^{\mu\nu} = 0$ (the space-time covariant derivative):

$$\nabla_{\mu} T^{\mu\nu} = 0 \quad (\nabla^{\mu} u_{\mu})$$

where we defined the comoving derivative $D = \nabla^\mu u_{\mu}$ and fluid expansion rate $\theta = \nabla^\mu u_\mu$.

2nd order conformal hydrodynamics can be readily used to check the precision of existing numerical codes that solve the (2+1) and (3+1) relativistic hydrodynamic equations.

Gubser flow in Israel-Stewart theory

Gubser generalized Bjorken’s solution by including a nontrivial $r_z$-dependence while retaining boost invariance [3] by performing the Weyl transformation of the metric

$$\frac{\partial^2 s^\mu}{\partial^2 T} = -\frac{\partial^2 x^\mu}{\partial^2 T} = \frac{\partial^2 x^\mu}{\partial^2 T} = d^2 - d^2 + x^\mu d^2 + d^2 - d^2 + \cos \theta d^3 + \sin \theta d^3 + d^2,$$

where $\sin \theta = \frac{L^2 - r^2 - z^2}{2Lr}$, $\tan \theta = \frac{2Lz}{L^2 - r^2 - z^2}$.

Comparing semi-analytical solution to MUSIC ($\theta/s = 0.2$)

We take the simplest conformal theory in which only $\eta$ and $\tau_z = 5\eta/s$/$T$ are nonzero. The equations drastically simplify in $dS_3 \times R$ and we have

$$\frac{\partial^2 s^\mu}{\partial^2 T} = -\frac{\partial^2 x^\mu}{\partial^2 T} + \frac{2}{T} \tan \rho \left( \frac{\alpha}{T} \right) \tan \rho \left( \frac{\beta}{T} \right)$$

where $\alpha = 1/5s/T(s)$, $\beta = 1/2s/T(s)$. These equations can then be solved numerically. An analytical solution to these equations valid at low temperatures can be found in [1].

First full analytical solution of 2nd order hydrodynamics

In [2], several solutions of the 2nd order hydro equations were found using the same idea: nontrivial flow in flat spacetime is taken, via a Weyl transformation, to a trivial flow in curved spacetime. The transformation is

$$\frac{ds^2}{\Lambda^2} = \frac{d^2 x^2}{x^2} + d\phi^2$$

where

$$\tan T = \frac{L^2 + r^2 - z^2}{2L^2}, \quad \cos \rho = \frac{1}{2Lz} \sqrt{(L^2 + (r + t)^2)(L^2 + (r - t)^2)}.$$

The metric in Eq. (8) is that of $AdS_3 \times S^2$ where $AdS_3$ is 3-dimensional Anti-de Sitter space. In this space the 4-flow is again trivial but in Minkowski space

$$u_\mu = \frac{L^2 + r^2 + z^2}{\sqrt{(L^2 + (r + t)^2)(L^2 + (r - t)^2)}}, \quad \tilde{u}_\mu = \frac{2L^2}{\sqrt{(L^2 + (r + t)^2)(L^2 + (r - t)^2)}}$$

An interesting property of this flow is that $\sigma_{\mu\nu} = 0$ and $\theta = 0$. Therefore, the full equation for the shear stress tensor $\sigma_{\mu\nu}$ simplifies to

$$\tilde{\sigma}_{\mu\nu} = \frac{4}{\epsilon_{\mu\nu}} \left( \frac{\epsilon_{\rho\nu}}{\epsilon_{\rho\nu}} \right) \left( \frac{\epsilon_{\tau\mu}}{\epsilon_{\tau\mu}} \right)$$

Note that this equation has a trivial solution ($\epsilon_{\mu\nu} = 0$) but there are other nontrivial solutions as well. Assuming $\epsilon_{\mu\nu} = 0$ is diagonal, we find the non-perturbative solutions in $\lambda_1$

$$\epsilon_{\mu\nu} = \frac{1}{\lambda_1} \left( \frac{1}{\lambda_1} \right)$$

The corresponding energy density in Minkowski space reads [2]

$$\epsilon = \frac{L^2 + r^2 + z^2}{(L^2 + (r + t)^2)(L^2 + (r - t)^2)} \frac{\lambda_1 + 1}{\lambda_1}$$

Note that $\epsilon$ is even under time reversal $t \rightarrow -t$ even though $\epsilon_{\mu\nu}$ is nonzero !!!