A unified picture of parton multiple scattering in the small-x regime and forward physics at RHIC and LHC

Yang-Ting Chien
Los Alamos National Laboratory

Based on work with Z. Kang, J. Qiu, I. Vitev, H. Xing, and S. Yoshida
Forward physics: two QCD based formalisms

**CGC**

![CGC Graph](image1.png)

Albacete, Marquet, 2010,
Kang, Vitev, Xing, 2014

**pQCD+CNM**

![pQCD+CNM Graph](image2.png)

Qiu, Vitev, 2006

Single hadron

- Albacete, Marquet, 2010,
- Kang, Vitev, Xing, 2011

Dihadron

- Albacete, Marquet, 2011
- Kang, Vitev, Xing, 2011
Major difference between pp and pA

The major ingredient in these two formalisms are the multiple scattering

- CGC: multiple scattering are resummed in the Wilson lines (unintegrated gluon distribution)
- pQCD+CNM (multiple scattering (high-twist) expansion approach
  - Most important part – “dynamic shadowing”, multiple scattering are described by power suppressed corrections, in the high-twist correlation functions
  - Include additional scattering order by order: cross section = single scattering + double scattering + ...
Phase diagram for gluon density and relevance to parton multiple scattering (PMS)

- **Dilute region (x large):** leading-twist pQCD works, PMS not important
  Collinear factorization formalism

- **Relatively dense region (x relatively small):** PMS starts to become important and additional scattering is power suppressed by $Q_s^2 / Q^2$
  Multiple scattering (high-twist) expansion formalism

- **Saturation region (x extremely small):** all the additional PMS becomes equally important, all power terms $(Q_s^2 / Q^2)^n$ have to be resummed
  Small-x (CGC) formalism

\[ Y = \ln \frac{1}{x} \]

\[ \ln Q^2 \]

\[ \sigma \sim \sigma_0 \left[ 1 + \left( \frac{Q_s^2}{Q^2} \right)^1 + \left( \frac{Q_s^2}{Q^2} \right)^2 + \cdots \right] \]
Both CGC and multiple scattering expansion formalisms describe the same type of physics (parton multiple scattering), what are the connections between them?

Step I: from small-$x$ (CGC) formalism to collinear factorization formalism

Saturation region $\rightarrow$ dilute region
An example: direct photon production

- Quark scatters off the classical gluon field to produce a photon
  - Scattering before and after: initial-state and final-state multiple scattering

- Differential cross section
  \[ E_\gamma \frac{d\sigma}{d^3p_\gamma} = \frac{\alpha_{em}}{2\pi^2} \sum_q e_q^2 \int dx' q(x') \left[ 1 + (1 - \xi)^2 \right] \int d^2 k_\perp F(x_A, k_\perp) H(p_\perp, k_\perp) \]

- Unintegrated gluon distribution (UGD)

\[ F(x_A, k_\perp) = \int \frac{d^2 x_\perp d^2 y_\perp}{(2\pi)^2} e^{-ik_\perp \cdot (x_\perp - y_\perp)} \frac{1}{N_c} \langle \text{Tr} (U(x_\perp)U^\dagger(y_\perp)) \rangle_{x_A} \]

\[ U(x_\perp) = \mathcal{P} \exp \left\{ ig_s \int_{-\infty}^{+\infty} dx^+ A^- (x^+, x_\perp) \right\} \]

\[ H(p_\perp, k_\perp) = \frac{\xi^2 k_\perp^2}{p_\perp^2 (p_\perp - \xi k_\perp)^2} \]

Kopeliovich, Raufeisen, Tarasov 01, Baier, Mueller, Schiff 04, Gelis, Jalilian, Marian, 02, 03, Stasto, Xiao, Zaslavsky, 2012
Dilute limit: perform $k_T$ expansion

When $k_\perp \sim Q_s \ll p_\gamma$ of photon, one can perform the $k_T$ expansion

- Start from
  
  \[ F(x_A, k_\perp) H(p_\perp, k_\perp) = F(x, k_\perp) H(p_\perp, k_\perp = 0) + \frac{1}{2!} k_\perp^\alpha k_\perp^\beta \frac{\partial}{\partial k_\perp^\alpha \partial k_\perp^\beta} [F(x_A, k_\perp) H(p_\perp, k_\perp)]|_{k_\perp \to 0} + \cdots \]

- Realize $H(p_\perp, k_\perp = 0) = 0$

\[
\frac{\partial}{\partial k_\perp^\alpha \partial k_\perp^\beta} [F(x_A, k_\perp) H(p_\perp, k_\perp)]|_{k_\perp \to 0} = F,_{\alpha \beta} H + F,_{\alpha} H,_{\beta} + F,_{\beta} H,_{\alpha} + FH,_{\alpha \beta}
\]

- Do not forget gluon momentum fraction $x_A$ also depends on $k_\perp$

\[
x_A = \frac{k^-}{P_A} = \frac{1}{\xi (1 - \xi) x's} [p_\perp^2 - 2\xi k_\perp \cdot p_\perp + \xi k_\perp^2]
\]

- Finally using $H,_{\alpha} = \frac{\partial}{\partial k_\perp^\alpha} H(p_\perp, k_\perp)|_{k_\perp \to 0} = 0$

\[
\frac{\partial}{\partial k_\perp^\alpha \partial k_\perp^\beta} [F(x_A, k_\perp) H(p_\perp, k_\perp)]|_{k_\perp \to 0} = FH,_{\alpha \beta}
\]
Dilute limit: no PMS at all

- Collect the 2\textsuperscript{nd} order $k_T$ expansion

\[ E_{\gamma} \frac{d\sigma}{d^3p_{\gamma}} = \frac{\alpha_{em}}{2\pi^2} \sum_q e_q^2 \int dx' q(x') \left[ 1 + (1 - \xi)^2 \right] \frac{\xi^2}{(p_{\perp}^2)^2} \int d^2k_{\perp} k_{\perp}^2 F(x_A, k_{\perp}) \big|_{x_A \to x = p_{\perp}^2/\xi(1-\xi)x'} \]

- Then use the following relation: relate the UGD to the integrated gluon distribution

\[ \int d^2k_{\perp} k_{\perp}^2 F(x, k_{\perp}) = \frac{2\pi^2\alpha_s}{N_c} xg(x) \]

- Thus we have

\[ E_{\gamma} \frac{d\sigma}{d^3p_{\gamma}} = \frac{\alpha_{em}\alpha_s}{N_c} \sum_q e_q^2 \int dx' q(x') xg(x) \left[ 1 + (1 - \xi)^2 \right] \frac{\xi^2}{(p_{\perp}^2)^2} \]

- On the other hand, from photon production in collinear factorization formalism with (only $q+g$ channel)

\[ \hat{s} = \frac{p_{\perp}^2}{\xi(1-\xi)}, \quad \hat{t} = -\frac{p_{\perp}^2}{\xi}, \quad \hat{u} = -\frac{p_{\perp}^2}{1-\xi}, \]

- Two results agree with each other
Step II: from small-x (CGC) formalism to multiple scattering/high-twist expansion formalism

Saturation region → relatively dense region
Multiple scattering (high-twist) formalism

- First nontrivial term - double scattering for direct photon production
- Consider q+g channel in the forward region

- Four gluon correlation function

\[
T_{g/A}(x) = \frac{1}{xp^+} \int \frac{dy^-}{2\pi} e^{ixp^+y^-} \int \frac{dy_1^-dy_2^-}{2\pi} \theta(y_2^-) \theta(y_1^- - y^-) \\
\times \langle A|F_{\alpha^+}(0)F_{\sigma^+}(y_2^-)F_{\sigma}(y_1^-)F^{\alpha}(y^-)|A\rangle.
\]

- Final result

\[
E_\gamma \frac{d\sigma^{(D)}}{d^3P_\gamma} = \left(\frac{4\pi^2\alpha_s}{N_c}\right) \frac{\alpha_s^2}{s} \sum_q e_q^2 \int \frac{dx'}{x'} q(x') \int \frac{dx}{x} \delta(\hat{s} + \hat{t} + \hat{u}) \\
\times \left[x^2 \frac{\partial^2 T_{g/A}(x)}{\partial x^2} - x \frac{\partial T_{g/A}(x)}{\partial x} + T_{g/A}(x)\right] \left(-\frac{1}{\hat{t}} - \frac{1}{\hat{s}}\right) \frac{1}{N_c} \left[-\frac{\hat{s}}{\hat{t}} - \frac{\hat{t}}{\hat{s}}\right]
\]

- Replace Mandelstam variables

\[
E_\gamma \frac{d\sigma^{(D)}}{d^3P_\gamma} \sim \left(\frac{4\pi^2\alpha_s^2\alpha_{em}}{N_c^2}\right) \sum_q e_q^2 \int dx' q(x') \left[x^2 \frac{\partial^2 T_{g/A}(x)}{\partial x^2} - x \frac{\partial T_{g/A}(x)}{\partial x} + T_{g/A}(x)\right] \left[1 + (1 - \xi)^2\right] \frac{\xi^4}{(p_{\perp}^2)^3}
\]
From the side of small-x formalism

- Starting from the small-x formalism, we should study the next nontrivial expansion
- Double scattering contribution is related to four gluon correlation function, and thus suggest us perform $k_T$-expansion to the 4th order

\[
F(x_A, k_\perp) H(p_\perp, k_\perp) = F(x, k_\perp) H(p_\perp, k_\perp = 0) + \frac{1}{2!} k_\perp^\alpha k_\perp^\beta \left. \frac{\partial}{\partial k_\perp^\alpha \partial k_\perp^\beta} [F(x_A, k_\perp) H(p_\perp, k_\perp)] \right|_{k_\perp \to 0} \\
+ \frac{1}{4!} k_\perp^\alpha k_\perp^\beta k_\perp^\gamma k_\perp^\delta \left. \frac{\partial}{\partial k_\perp^\alpha \partial k_\perp^\beta \partial k_\perp^\gamma \partial k_\perp^\delta} [F(x_A, k_\perp) H(p_\perp, k_\perp)] \right|_{k_\perp \to 0} + \cdots
\]

- Use the following formula

\[ \frac{\partial}{\partial k_\perp^\alpha \partial k_\perp^\beta \partial k_\perp^\gamma \partial k_\perp^\delta} [F(x_A, k_\perp) H(p_\perp, k_\perp)] = \frac{\partial^2 F}{\partial x^2} [x,\alpha x,\beta H,\gamma \delta + x,\alpha x,\gamma H,\beta \delta + x,\alpha x,\delta H,\beta \gamma + x,\gamma x,\delta H,\alpha \beta + x,\beta x,\delta H,\alpha \gamma + x,\beta x,\gamma H,\alpha \delta] \\
+ \frac{\partial F}{\partial x} [x,\alpha \beta H,\gamma \delta + x,\alpha \gamma H,\beta \delta + x,\alpha \delta H,\beta \gamma + x,\gamma \delta H,\alpha \beta + x,\beta \delta H,\alpha \gamma + x,\beta \gamma H,\alpha \delta] \\
+ x,\alpha H,\beta \gamma \delta + x,\beta H,\alpha \gamma \delta + x,\gamma H,\alpha \beta \delta + x,\delta H,\alpha \beta \gamma] \\
+ FH,\alpha \beta \gamma \delta
\]

- Short-hand notation

\[ x,\alpha = \frac{\partial x_A}{\partial k_\perp^\alpha} \quad H,\alpha \beta \gamma \delta = \frac{\partial H(p_\perp, k_\perp)}{\partial k_\perp^\alpha \partial k_\perp^\beta \cdots} \]
The correlation function part

- Correlation functions could have three different types, characterized by different theta-functions

\[
F(x_A, k_\perp) = \int \frac{d^2 x_\perp d^2 y_\perp}{(2\pi)^2} e^{-ik_\perp \cdot (x_\perp - y_\perp)} \frac{1}{N_c} \left\langle \text{Tr} \left( U(x_\perp) U^\dagger(y_\perp) \right) \right\rangle_{x_A}
\]

\[
U(x_\perp) = \mathcal{P} \exp \left\{ ig_s \int_{-\infty}^{+\infty} dx^+ A^-(x^+, x_\perp) \right\}
\]

\[
= \sum_{n=0}^{\infty} (ig_s)^n \int_{-\infty}^{\infty} d\lambda_1^+ A^-(\lambda_1) \int_{\lambda_1^+}^{\infty} d\lambda_2^+ A^-(\lambda_2) \cdots \int_{\lambda_{n-1}^+}^{\infty} d\lambda_n^+ A^-(\lambda_n) |_{\lambda_i^- = 0, \lambda_i^\perp = x_\perp}
\]

\[
\int_{\lambda_1^+}^{\infty} d\lambda_2^+ A^-(\lambda_2) = \int_{-\infty}^{\infty} d\lambda_2^+ A^-(\lambda_2) \theta(\lambda_2^+-\lambda_1^+)
\]

- Expand both \(U(x)\) and \(U^\dagger(y)\) to \(O(g^2)\): **central cut**; Expand \(U(x)\) to \(O(g)\) and \(U^\dagger(y)\) to \(O(g^3)\): **right cut**; Expand \(U(x)\) to \(O(g^3)\) and \(U^\dagger(y)\) to \(O(g)\): **left cut**

- They correspond to the relevant terms in the twist-4 formalism
Some expansion details

- The contraction of indices

\[
\int d^2 k_\perp k_\perp^\alpha k_\perp^\beta k_\perp^\gamma k_\perp^\delta F(x, k_\perp) \sim \int d^2 k_\perp k_\perp^4 \left[ g_\perp^{\alpha\beta} g_\perp^{\gamma\delta} + g_\perp^{\alpha\gamma} g_\perp^{\beta\delta} + g_\perp^{\alpha\delta} g_\perp^{\beta\gamma} \right] F(x, k_\perp)
\]

- After a tedious but straightforward calculation, the small-x formalism could be written as

\[
E_\gamma \frac{d\sigma^{(D)}}{d^3 P_\gamma} \sim \left( \frac{4\pi^2 \alpha_s^2 \alpha_{em}}{N_c^2} \right) \sum_q e_q^2 \int dx' q(x') \left[ x^2 \frac{\partial^2}{\partial x^2} - x \frac{\partial}{\partial x} + 1 \right] \int d^2 k_\perp k_\perp^4 F(x, k_\perp) \left[ 1 + (1 - \xi)^2 \right] \frac{\xi^4}{(p_\perp^2)^3}
\]

\[
\int d^2 k_\perp k_\perp^4 F(x, k_\perp)|_{\text{twist-4}} \sim T_{g/A}(x)
\]

- This is the same as what was obtained in the high-twist formalism for the double scattering contribution
They agree with each other

\[ Y = \ln \frac{1}{x} \]

- Small-\(x/\text{CGC formalism} \)
- Multiple scattering/high-twist formalism
- Collinear factorization formalism

\[ \ln Q^2 \]
For a fixed rapidity, in the small $p_T$ region where $p_T \lesssim Q_s$, PMS is very important and has to be resummed – small-x/CGC formalism should work.

When $p_T \gtrsim Q_s$, PMS is still important but first term (double scattering) should be enough – high-twist formalism.

When $p_T \gg Q_s$, PMS is not important, only single scattering is relevant – usual collinear factorization formalism.

In the overlap region, they agree with each other.
Summary

- Multiple scattering (high-twist) expansion and small-x (CGC) formalism are two closely related formalisms, which both deal with parton multiple scattering.
- In small-x gluon saturation region, CGC formalism resum all order multiple scattering to Wilson lines.
- In relatively small-x region, high-twist formalism describe the coherent multiple scattering as power suppressed correction to the cross section, which agrees with CGC formalism after expansion.
- We have shown these connections mathematically.
- This connection should put strong constrain on the phenomenology.

\[ Y = \ln \frac{1}{x} \]