# Two loop perturbative corrections to the thermal effective potential in gluodynamics

**Adrian Dumitru [†], Yun Guo [*] and Chris P. Korthals Altes [‡]**

[†] RIKEN BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973, USA ; Department of Natural Sciences, Baruch College, CUNY, 17 Lexington Avenue, New York, New York 10010, USA ; The Graduate School and University Center, The City University of New York, 365 Fifth Avenue, New York, New York 10016, USA

[*] Departamento de Fisica de Particulas, Universidade de Santiago de Compostela, E-15782 Santiago de Compostela, Galicia, Spain ; Physics Department, Guangxi Normal University, 541004 Guilin, China

[‡] Centre Physique Théorique, au CNRS Case 907, Campus de Luminy, F-13288 Marseille, France ; NIKHEF Theory Group, Science Park 105, 1098 XG Amsterdam, The Netherlands

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## Abstract

The thermodynamics of pure glue theories can be described in terms of an effective action for the Polyakov loop. This effective action is of the Landau-Ginzburg type, and its variables are the angles parametrizing the loop. We compute perturbative corrections to this action. Remarkably, two-loop corrections turn out to be proportional to the one-loop action, independent of the eigenvalues of the loop. By a straightforward generalization of the 't Hooft coupling, this surprisingly simple result holds for any of the classical and exceptional groups .

## Introduction & Motivation

- Understanding of the deconfined phase QCD at high temperature is growing as a result of the heavy ion experiments at RHIC and the LHC and theoretical and numerical lattice work.
- Matrix models [1], which typically employ the one-loop perturbative effective potential as a function of the eigenvalues of the Polyakov loop, can describe the themodynamics of the pure glue theories very well. They provide a possible way to study the deconfine phase transition.
- These models can be improved systematically, at least in regard to the perturbative component.
- The calculation of the leading perturbative correction due to interactions among gluons in the presence of the condensate can answer how radiative corrections affect the one-loop effective potential.
- It was known [2] since long that for the SU(N) groups along straight paths from the origin to the degenerate Z(N) minima, two-loop effective potential is simply a multiplicative and background independent renormalization of the one-loop result. These paths run along the edges of the SU(N) Weyl chamber.

***However***

- In general the minimum of the potential does not exactly follow a straight path as a function of temperature. We do not yet know if this simple result still holds inside the Weyl chamber.
- Lattice simulations [3] for groups without a center show that deconfinement does not require a global symmetry. Hence, aside from SU(N) it is important to perform the perturbative two-loop calculations also for all other classical gauge groups, including the exceptional group G(2).

## Basic formulas & Main results

The effective potential $\mathcal{V}(\ell)$ is simply the traditional path integral over the gauge fields subject to a constraint: the integration is done while preserving the value of the Polyakov loop at some fixed value $\ell$.

$$\exp(-V\mathcal{V}(\ell)) \equiv \int DA_\mu\, \delta(\ell - \frac{1}{N}\mathrm{Tr}\bar{\mathbf{L}})\, \exp(-\frac{1}{g^2}S(A))\,.$$

- We are interested in a loop which is constant in space, so the constraint is a delta function. $\mathrm{Tr}\bar{\mathbf{L}}$ is the spatially averaged loop
- To fix all independent phases of the loop, one has to take as many powers of the loop as there are independent phases. To avoid clutter we do not write these higher powers explicitly.
- In the large volume limit the above definition of the effective potential is equivalent to the traditional definition of the effective potential where a source term is introduced into the path integral of the partition function.

Fourier transforming the delta function constraint will introduce an auxiliary field $\epsilon$ in the path integral. We added a phase to the original gauge action and the constrained version is given by

$$S_{con}(A,\epsilon) = i\epsilon\left(\ell - \frac{1}{N}\mathrm{Tr}\bar{\mathbf{L}}\right) + S(A)\,.$$

Expand the constrained action in terms of the fluctuations of the fields, we can compute the effective potential. Up to two-loop order, in terms of the Bernoulli polynomials, it reads [4]

$$\Gamma^{(1)} = -\frac{\pi^2 T^4 d(A)}{45} + \sum_a \widehat{B}_4(q_a)\,,$$

$$\Gamma^{(2)}_f = g^2 \sum_{a,b,c} |f^{a,b,c}|^2 \widehat{B}_2(q_b)\widehat{B}_2(q_c)\,, \quad \Gamma^{(2)}_i = 2g^2 \sum_{d,b,c} f^{d,b,-b} f^{d,c,-c} \widehat{B}_1(q_b)\widehat{B}_3(q_c)\,.$$

- $\Gamma^{(1)}$ is the one-loop effective potential where the interaction vanishes. Two-loop effective potential contains two part: the free-energy contribution $\Gamma^{(2)}_f$ and the insertion contribution $\Gamma^{(2)}_i$ .
- In the free-energy contribution, the color indices run over both diagonal and off-diagonal indices, In the insertion contribution, d is the diagonal index while b and c denote off-diagonal indices.
- The structure constants can be determined from the commutation relation between the generators of the groups. If $E^b$ is some off-diagonal generator, then $E^{-b} \equiv (E^b)^\dagger$ .
- Both free-energy and insertion contributions are gauge dependent. However, the gauge dependence cancels precisely between these two contributions which ensures the gauge invariance of the effective potential.

In terms of the quadratic Casimir invariant in the adjoint representation, we verify the following important result [5]

$$\frac{\Gamma^{(2)}}{\Gamma^{(1)}} = -\frac{5g^2 C_2(A)}{16\pi^2}\,.$$

- We developed a MATHEMATICA program for all classical groups to evaluate explicitly the free-energy and insertion contributions which shows the above simple relation between the one- and two-loop effective potential.
- This result indicates that in perturbation theory the eigenvalue distribution of the loop is not affected by two-loop corrections. In particular, the expectation value of the Polyakov loop calculated at these minima remains $\ell = \pm 1$ .
- The pressure calculated from the minimum of the potential equals the known perturbative pressure calculated at vanishing background field.
- The simple relation is general for SU(N), SO(2N), SO(2N+1), Sp(2N) and G(2). It holds not only alone the edges of the Weyl chamber, but also inside the Weyl chamber.

## Generalities on the classical Lie algebras

We start with the commutation relations in the Cartan basis for any semi-simple Lie algebra

$$[\vec{H}, E_\alpha] = \vec{\alpha}\, E_\alpha\,, \quad [E_\alpha, E_{-\alpha}] = \vec{\alpha}\cdot\vec{H}\,, \quad [E_\alpha, E_\beta] = f^{\alpha,\beta,-\alpha-\beta} E_{\alpha+\beta}\,, \text{if } \alpha+\beta \text{ is a root; if not, it vanishes.}$$

We define the structure constants from the generators in the fundamental representation of the group, with the generators normalized as

$$\mathrm{Tr}(E_\alpha E_{-\alpha}) = \mathrm{Tr}(H^2_d) = 1/2\,.$$

- The components of $\vec{H}$ are the orthonormal matrices spanning the Cartan sub-algebra and they are the diagonal generators in the Cartan basis.
- The orthonormal $E_\alpha$ labeled by the roots α are vectors in Cartan space. They are the off-diagonal generators.

The roots themselves are labeled by an off-diagonal index. For a typical off-diagonal index, we have

$$[H^d, E_\alpha] = f^{d,\alpha,-\alpha} E_\alpha\,.$$

- The dth component of a root (labeled by an off-diagonal index α) is the structure constant $f^{d,\alpha,-\alpha}$ .
- Besides the structure constants involving a diagonal component, we have another kind of structure constants $f^{\alpha,\beta,-\alpha-\beta}$ which connect off-diagonal generators.
- With our normalization, the absolute values of $f^{\alpha,\beta,-\alpha-\beta}$ are all equal to $1/\sqrt{2}$ for SU(N); for SO(2N+1), SO(2N) and Sp(2N), they are $1/2$ .
- In order to determine the structure constants, one has to know the generators in the Cartan basis for each group [6] and compute their commutation relations.

## A simplified form for the insertion

The insertion diagram involves sums over diagonal indices d which can be performed quite easily as they correspond to inner products between the corresponding roots.

for SU(N):

$$\Gamma^{(2)}_i = 4g^2 \sum_{ijl} \widehat{B}_1(q_i - q_j)\widehat{B}_3(q_i - q_l)\,.$$

for SO(2N):

$$\Gamma^{(2)}_i = 2g^2 \sum_{i,j,l} \Big(\widehat{B}_1(q_i + q_j) + \widehat{B}_1(q_i - q_j)\Big)\Big(\widehat{B}_3(q_i + q_l) + \widehat{B}_3(q_i - q_l)\Big)\,.$$

for SO(2N+1):

$$\Gamma^{(2)}_i = \Gamma^{(2)}_i(SO(2N)) + 2g^2 \sum_{i,j} \Big[(\widehat{B}_1(q_i+q_j)+\widehat{B}_1(q_i-q_j))\widehat{B}_3(q_i)+(\widehat{B}_3(q_i+q_j)+\widehat{B}_3(q_i-q_j))\widehat{B}_1(q_i)\Big] + 2g^2 \sum_i \widehat{B}_1(q_i)\widehat{B}_3(q_i))\,.$$

for Sp(2N):

$$\Gamma^{(2)}_i = 2g^2 \sum_i \Big(\sum_j (\widehat{B}_1(q_i+q_j)+\widehat{B}_1(q_i-q_j)) + 2\widehat{B}_1(2q_i)\Big)\Big(\sum_l (\widehat{B}_3(q_i+q_l)+\widehat{B}_3(q_i-q_l)) + 2\widehat{B}_3(2q_i)\Big)\,.$$

- In the above equation, the constraint $i \neq j\,, i \neq l$ applies in the sum.
- Using the explicit expressions for the insertion contribution, one can compute for larger N more efficiently by using the MATHEMATICA program.
- For the free-energy contribution, if one of the indices in the structure constant is diagonal, we can also get the simplified expressions by using the properties of the roots.
- From these expressions, one can easily prove that the result for the insertion contribution is independent of the value of $\widehat{B}_1(n)$ for integer $n$ . $\widehat{B}_1(n)$ has discontinuities at integer $n$ .

## Conclusions & Outlook

- The main result of this work is that the two-loop renormalization of the effective potential is very simple: the two-loop potential is proportional to that at one loop, independent of the eigenvalues of the loop. There is nothing in the way we perform the computation that suggests such simplicity.
- This is a very general result for the classic groups which holds along the edge of the Weyl chamber and inside the Weyl chamber.
- At two-loop order in perturbation theory the minima of the perturbative action don't change. How this works out to three-loop order is something that remains to be worked out.
- The two-loop effective potential found here could be supplemented by a model for non-perturbative physics, e.g. along the lines of Refs. [1], in an attempt to understand the eigenvalue distribution of the Polyakov loop in the gauge theories.
- The disadvantage of using the MATHEMATICA program is that for large N, the calculation becomes rather time-consuming. An analytical proof of the simple relation between one- and two-loop effective potential is still needed.

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