Complex heavy quark potential at high temperature from lattice QCD

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Abstract

A precise definition of the $Q\bar{Q}$ potential at high temperature was obtained in the framework of effective field theories. The potential was calculated in hard thermal loop resummed perturbation theory and happens to be complex [1]. In ref. [2] the definition was adapted to Euclidean lattice simulations but difficulties associated with the infinite real time limit required to extract the potential were encountered. We will discuss how to disantangle precisely the short and long time physics from which the potential is defined [3] and present a new method to perform the analytic continuation from Euclidean to real time [4]. After these improvements, the procedure to extract the potential produces precise results [5] and its application to quenched lattice QCD data [4] gives us an estimate of both real and imaginary part of the non-perturbative complex $Q\bar{Q}$ potential across the phase transition. The final results are very encouraging since the precision obtained for the real part of the potential is below the percent level at small distance. Even if only a rough estimate of the imaginary part is obtained, we show that it could also be settled by more precise Euclidean data.

Introduction

Already in 1986, Matsui and Satz [6] proposed the melting of the J/Ψ as signal for the deconfinement transition in heavy-ion collisions. Whereas experiments [7, 8, 9, 10] observe this phenomena for various charmonium and bottomonium states, the theoretical methods allowing to put Matsui and Satz idea on a solid theoretical footing only appeared recently within the framework of effective field theories.
First derived for perturbation theory [1], the results obtained from effective field theories can now be applied to lattice simulations.

Potential description

In fact the effective field theory framework can be used thanks to a hierarchy of scales [11, 12, 13]:

$$\Lambda_{QCD} \ll m_Q, \quad T \ll m_Q. \tag{1}$$

In this limit, the propagation of a heavy quark pair can be described by a rectangular temporal Wilson loop of time extent t and spatial separation r

$$W_\square(t,r) = \frac{1}{N_c}\mathcal{P}\mathrm{Tr}[\exp[-ig\oint_\square dx^\mu A_\mu(x)]]. \tag{2}$$

It satisfies a Schrödinger equation

$$i\partial_t W_\square(t,r) = \Phi(t,r) W_\square(t,r), \tag{3}$$

where, at late times, the function $\Phi(t,r)$ becomes time independent and defines the static potential:

$$V(r) = \lim_{t\to\infty} \Phi(r,t). \tag{4}$$

Consequences of the potential description

Lattice simulation are performed in Euclidean space-time

- How can we compute $\Phi(r,t)$ and its infinite real time limit from lattice data?
- Let's look at the consequences of the potential description on other observables:

Following. [3], we rewrite the time dependent potential as $\Phi(r,t) = V(r) + \phi(r,t)$. Supposing that a potential description exists at late times, i.e. $\phi(r,t) \to 0$ when $t \to \infty$ we can solve the Schrödingier equation (3) as

$$W(r,t) = \exp[-i(\mathrm{Re}V(r)t + \mathrm{Re}\sigma(r,t)) - |\mathrm{Im}V(r)|t + \mathrm{Im}\sigma(r,t)], \tag{5}$$

where $\sigma(r,t) = \int_0^t \phi(r,t)\, dt$ and $\sigma_\infty(r) = \sigma(r,\infty)$.
Inverting equation (8) and using $W(r,-t) = W^*(r,t)$ we get

$$\rho_\square(r,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ \exp[i(\omega - \mathrm{Re}V(r))t - i\mathrm{Re}\sigma(r,|t|)\mathrm{sign}(t) - |\mathrm{Im}V(r)||t| + \mathrm{Im}\sigma(r,|t|)]. \tag{6}$$

From this expression, the short time physics (variations of $\phi(r,t)$, which are supposed to vanish at some t_∞) can be separated form the large time physics. The latter can be integrated analytically, whereas the former can be expanded supposing that $(\mathrm{Re}V(r) - \omega)t_\infty$ is small:

$$\rho_\square(r,\omega) = \frac{1}{\pi}e^{\mathrm{Im}[\sigma_\infty](r)}\frac{|\mathrm{Im}[V](r)|\cos[\mathrm{Re}[\sigma_\infty](r)] - (\mathrm{Re}[V](r) - \omega)\sin[\mathrm{Re}[\sigma_\infty](r)]}{\mathrm{Im}[V](r)^2 + (\mathrm{Re}[V](r) - \omega)^2} + c_0(r) + c_1(r)(\mathrm{Re}[V](r) - \omega) + \cdots \tag{7}$$

We see that $\rho(r,\omega)$ contains a skewed Breit-Wigner peak containing all the information about the potential. Hence it is in fact enough to fit the lowest lying peak of the reconstructed spectrum with this functional form to get $V(r)$.

From the Euclidean correlator to the potential

Till now our discussion involved real time observables but lattice measurements provides us only Euclidean correlators. To extract the potential we have to [2, 3]:

- Calculate the Wilson loop $W_\square(r,\tau)$ for all possible values of the imaginary time $\tau \in [0,\beta]$.
- Note that the real-time Wilson loop is connected to the Euclidean one using the spectral representation:

$$W_\square(r,t) = \int d\omega\ e^{-i\omega t}\ \rho_\square(r,\omega) \quad \Leftrightarrow \quad W_\square(r,\tau) = \int d\omega\ e^{-\omega\tau}\ \rho_\square(r,\omega). \tag{8}$$

- Extract the spectrum $\rho_\square(r,\omega)$ by inverting eq. (8) (with MEM or else).
- In principle $V(r)$ can be obtained by direct Fourier transform of the full $\rho_\square(r,\omega)$ but this requires a very precise and complete spectrum. It is easier to use the properties of the spectral function and fit the lowest peak with the functional form (7).

Difficulties in the analytical continuation

In general terms the aim is to invert an integral equation of the form

$$W(\tau) = \int d\omega\, K(\omega,\tau)\rho(\omega), \quad (\text{here } K(\omega,\tau) = e^{-\omega\tau}) \tag{9}$$

i.e get $\rho(\omega)$ knowing some data points $D_n = W(\tau_n), n = 0..N_\tau$ to some accuracy.
Discretizing the frequency space with n_ω points spaced by $\Delta\omega_l$ and denoting $\rho(\omega_l) = \rho_l, l = 1..n_\omega$, equation (9) becomes

$$D_i^\rho = \sum_{l=1}^{n_\omega} \Delta\omega_l K_{il}\rho_l. \tag{10}$$

Till now it looks like we just have to invert the matrix $\Delta\omega_l K_{il}$. However this is not so simple. First the data contains errors, so that the ρ_l's might not be completely fixed. Secondly the spectral function contains sharp peaks, which shape is of interest, so that in practice one needs a fine discretization for ω so that n_ω is in fact much larger than the number of data points N_τ. That said the inversion is still not hopeless because ρ is positive definite, which constrain massively the space of solutions. This last point in fact turns the linear problem (10) into a non-linear one.
The most common method used in this field is the usual or extended MEM [14, 15], however:

- It can't resolve the width of the peak [5] $\Rightarrow$ Imaginary part of the potential far off, real part not precise.
- The peak does not have a Lorentzian shape.
- Marginal improvements with better data.
- Numerically too expensive to deal with very good data.

We derived a new Bayesian method solving problems of MEM [4]

- New S functional without asymptotically flat directions.
- Hyperparameters integrated explicitly (no Gaussian approximation).

New Bayesian reconstruction method: Short summary

A way to give a meaning to the inversion (10) goes through a minimization procedure. Starting with the guess $\rho_l, l = 1..n_\omega$, we try to minimize the distance to the data

$$L = \frac{1}{2}\sum_{ij}(D_i - D_i^\rho)C_{ij}(D_j - D_j^\rho), \tag{11}$$

with C_{ij} the covariance matrix of the data. In fact we know that the data points have errors and hence the correct result should have $L \sim N_\tau$, so that the probability of having the data D_i given some spectrum is

$$P(D|\rho) = \exp(-L - \gamma(L - N_\tau)^2) \tag{12}$$

and the limit $\gamma \to \infty$ is taken numerically. This is of course not yet enough to define a unique solution if $n_\omega > N_\tau$. In addition we will suppose that $\rho(\omega)$ is a smooth function and that part of it might be known at least to some accuracy (prior function $m(\omega)$), which we will encode in a probability function for ρ itself:

$$P(\rho|m,\alpha) = \exp[S] = \exp\left[\alpha\sum_{l=0}^{n_\omega}\Delta\omega_l\left(1 - \frac{\rho_l}{m_l} - \log\frac{\rho_l}{m_l}\right)\right]. \tag{13}$$

Now, for any given α, there is a unique solution $\omega_l,\ l = 0..n_\omega$ which minimize

$$P(\rho|D,m,\alpha) = \frac{P(D|\rho)P(\rho|m,\alpha)}{P(D|m,\alpha)}. \tag{14}$$

The final result $P(\rho|D,m)$ is obtained by integrating out α (see ref.[4] for details),

$$P(\rho|D,m) \propto P(D|\rho)\int d\alpha P(\rho|m,\alpha). \tag{15}$$

Unlike for the MEM, we first integrate over α explicitly and then find the spectrum that minimize the expression obtained:

$$\frac{\delta}{d\rho}P(\rho|D,m) = 0. \tag{16}$$

This new method (and the MEM) will be tested starting from the HTL Euclidean correlator.

Test of the reconstructed spectra

Using leading order hard thermal loop perturbation theory one can test the method as all quantities can be calculated to high precision. We reconstruct the spectral function with:

- MEM where we have chosen to give the exact Euclidean data (best case).
- The new method where we added relative gaussian errors (10^{-2} to 10^{-5}) to the data.

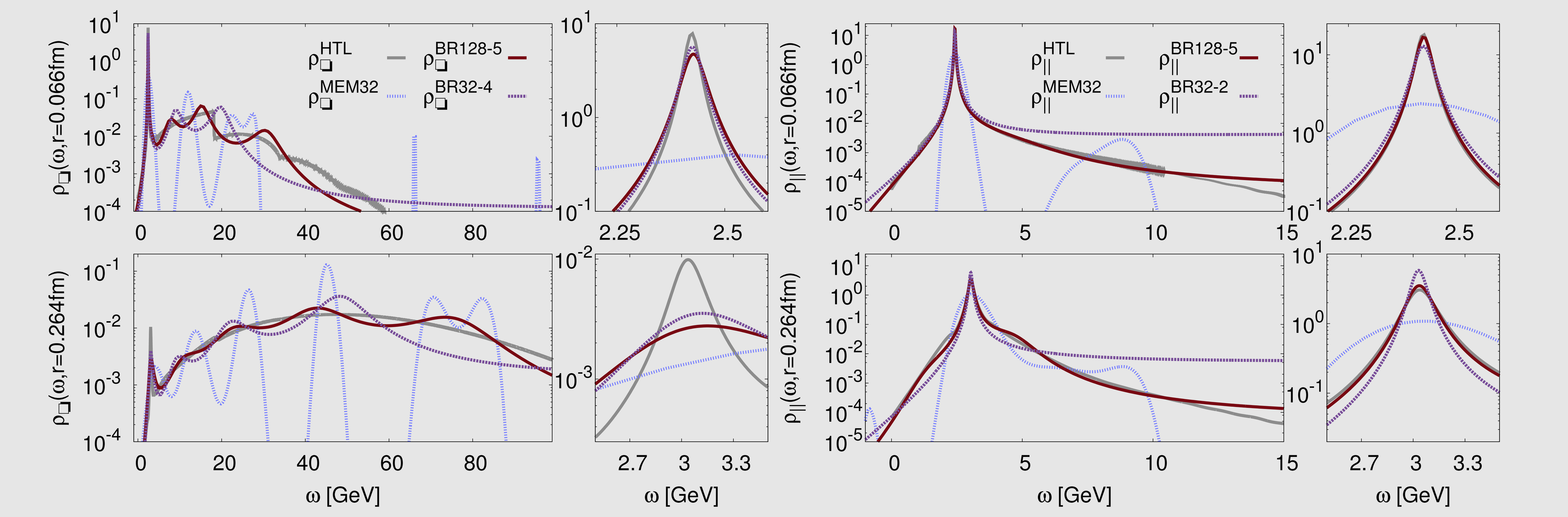

Figure: Comparison of the spectra: exact HTL result, MEM reconstruction with 32 points (no errors added), our new method (BR) with 32 and 128 points and errors added. We show two distances, $r = 0.06$fm and $r = 0.26$fm (top) and two observables: Wilson loop (respectively 10^{-4}, 10^{-5} errors), (bottom): Wilson lines (respectively 10^{-2}, 10^{-5} errors).

- The Wilson loop contains cusp divergences and is harder to reconstruct.

Test of the reconstructed potential

We can now fit the potential from the spectra for the HTL's and quenched lattice data:

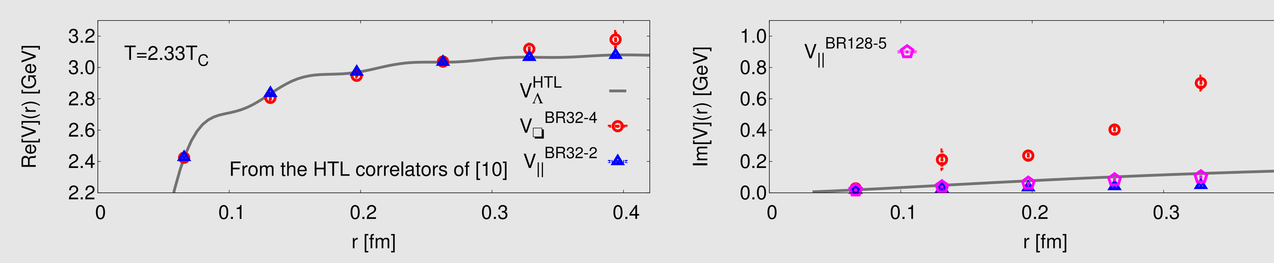

Figure: Reconstruction of the HTL ($T = 2.33T_C$) potential (solid line) based on the Euclidean HTL Wilson Loop $V_\square(r)$ (circle) and the HTL Wilson line correlator $V_{||}(r)$ (triangle). (left): $\mathrm{Re}[V](r)$(right):$\mathrm{Im}[V](R)$ requires $N_\tau = 128$ and 10^{-5} errors for a reliable determination

Temperature dependence of the potential

We generated new quenched anisotropic ($\xi = 3.5, \beta = 7$) lattice data of size $32^3 \times N_\tau$ with $N_\tau = 32, 40, 48, 64, 72, 80, 96$ corresponding to $T = 629..209MeV$ and show the dependence of the potential with respect to the temperature.

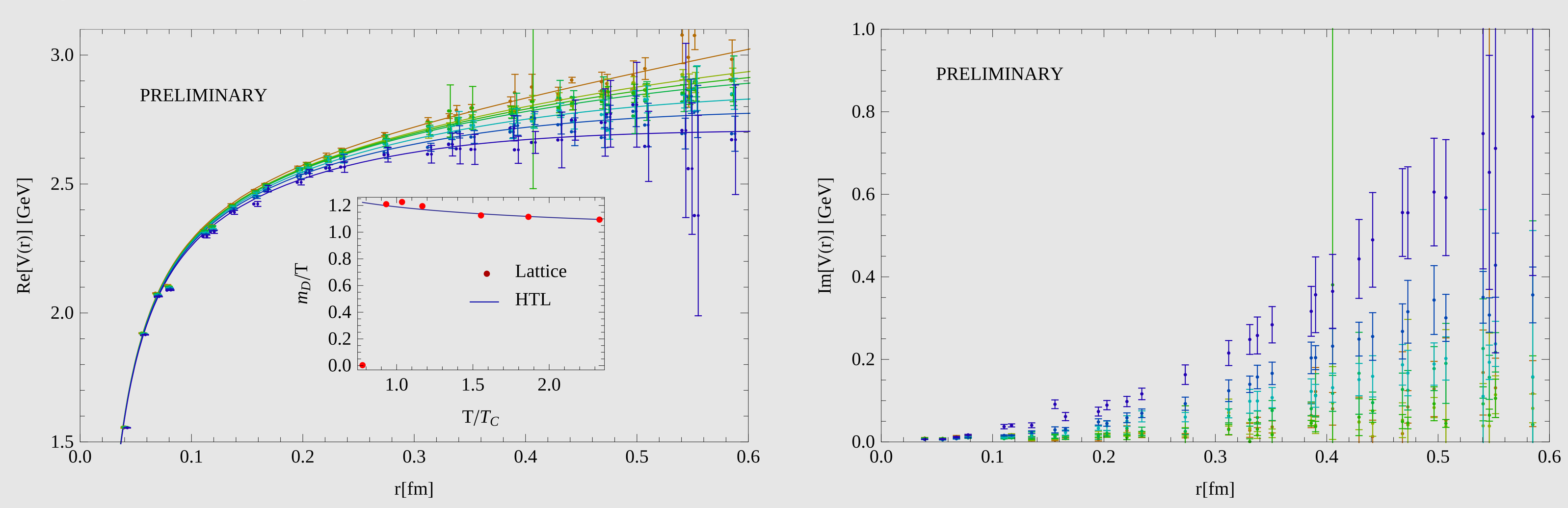

Figure: Lattice results and fits for the potential at different temperatures, (insert): fitted Debye masses (see ref. [16]) together with HTL ($m_D = g(4\pi T)T$).

Conclusion

The procedure of ref. [2, 3] was applied to quenched lattice calculations:

- The main difficulty is the reconstruction of the spectral function from Euclidean data.
  - The extended MEM finds the potential peak but fails to capture its Lorentzian structure.
  - The new Bayesian approach captures the Lorentzian shape of the peak and gives a good reconstruction of the background. We get the real part precisely as well as the imaginary part in the case of precise data.
  - Whereas MEM shows inconsistent and slow improvement, the new method clearly shows more stable results with better data and is significantly faster.
- To extract the potential, it is enough to fit the lowest spectral peak of the reconstructed spectra.
- The real part of the potential can be extracted from standard simulations.
- The imaginary part requires extensive computations.
- More lattice data and improved reconstruction soon.

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