The Binder cumulant in O(N)-models

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The continuous phase transition in QCD

The exact nature of the chiral phase transition at $\mu = 0$ is still under investigation.

Expectations:

- QCD, $N_f = 2$: O(4) scaling behavior [1]
- Lattice QCD: O(2) (staggered fermions) or O(4) scaling behavior for $N_f = 2$ or $N_f = (2+1)$ [2, 3, 4]

However, lattice simulations results are not yet conclusive!

Complications appearing in lattice QCD:

- finite quark masses $\rightarrow$ no chiral limit in lattice simulations
- finite simulation volumes affect the scaling behavior

$\Rightarrow$ Scaling analysis is difficult

Possible extension of the standard scaling analysis in lattice QCD:

- Investigate **higher order critical fluctuations** in finite volume by means of the **Binder cumulant**

The Binder cumulant in O(N)-models

Make use of universality close to the critical point:

- Investigate the behavior of the Binder cumulant in the most simple model from the same universality class as QCD is assumed to fall in
- Consider O(N)-symmetric continuous ϕ-model

For an O(N)-model with N scalar degrees of freedom, $\vec{\phi} = (\sigma, \pi_1, \ldots, \pi_{N-1})^T$, the Binder cumulant of the fourth order is defined as [5]:

$$B_4 = \langle \vec{\phi}^4 \rangle / \langle \vec{\phi}^2 \rangle^2$$

B_4 is dimensionless and therefore a finite-size-scaling function:

$$B_4 = Q_B(tL^{1/\nu}, L^{-\omega})$$

with $t = (T - T_C)/T_0$ and ν, ω critical exponents.
It can be expanded as:

$$Q_B(tL^{1/\nu}, L^{-\omega}) = a_0 + a_1 t L^{1/\nu} + a_2 L^{-\omega} + \ldots$$

At $T = T_C$ and for $L \rightarrow \infty$:

B_4 exhibits a **universal value**

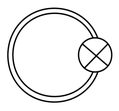
The **universal value of** B_4 obtained using O(N)-models is **applicable to lattice QCD!**

Model and Method

To control critical physics use **Functional Renormalization Group (FRG)** in its particular form given by the **Wetterich's flow equation** [6]:

$$\partial_k \Gamma_k[\Phi_k] = \frac{1}{2} \mathrm{Tr} k \partial_k R_k \left(\Gamma_k^{(2)}[\Phi_k] + R_k \right)^{-1} =$$

$$= \frac{1}{2}$$



$$\lim_{k \rightarrow \Lambda} \Gamma_k[\phi_k] = S[\phi] \qquad \lim_{k \rightarrow 0} \Gamma_k[\phi_k] = \Gamma[\phi]$$

Γ_k: scale-dependent effective action
R_k: IR cut-off function (here: $R_k = (k^2 - \vec{p}^2)\Theta(k^2 - \vec{p}^2)$ [7])

$\Rightarrow \Gamma$ includes **all quantum corrections**

For O(4)- and O(2)-models we use the following ansatz for Γ_Λ:

$$\Gamma_\Lambda[\phi] = \int d^d x \left(\tfrac{1}{2} \tilde{Z}_\phi (\partial_\mu \phi)^2 + U_\Lambda(\phi^2) \right)$$

with $\vec{\phi}(x) = (\sigma, \vec{\pi})^T \in \mathbb{R}^N$ and $N \in \{2, 4\}$

- $U_\Lambda(\phi^2)$ is O(N)-symmetric
- Explicit symmetry breaking introduced by term $-H\sigma$ (symmetry breaking only in the direction of σ)
- three-dimensional theory

We use the **Local potential approximation (LPA)**:

- No spatial dependence of the expectation value of the scalar fields
- $\tilde{Z}_\phi = 1$ justified by small anomalous dimension ($\eta \approx 0$) [8]

Results for the Binder cumulant, Part I

Analytic results in the limit $H \rightarrow 0$, $L \rightarrow \infty$:

- $T \ll T_C$: system becomes strongly ordered

$$\Rightarrow \langle \vec{\phi}^4 \rangle \rightarrow \langle \sigma \rangle^4 \text{ and } B_4 = 1$$

- $T \gg T_C$: fluctuations gain importance (B_4 depends on N)

$$B_4 = \frac{N+2}{N}$$

These predictions are in agreement with numerical results:

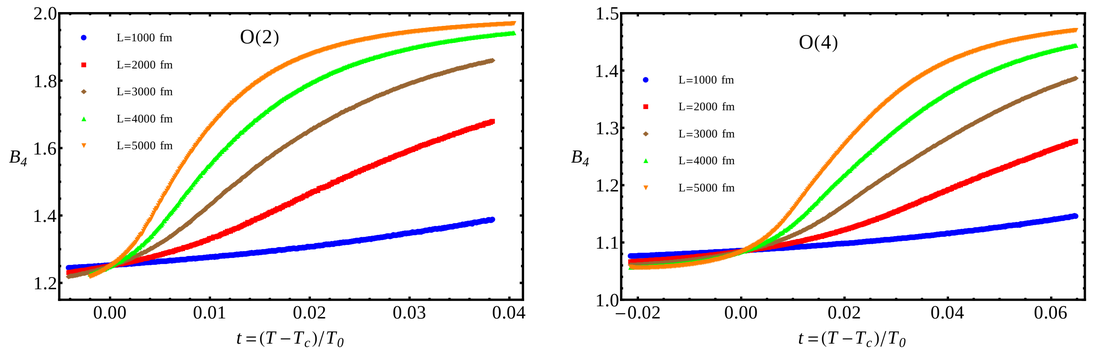


Figure: Binder cumulant B_4 for very large volumes at almost vanishing symmetry-breaking field as a function of temperature. We observe the correct asymptotic behavior.

Our numerical results for $B_4(T_C)$ are in a perfect agreement with spin-model lattice simulations:

B_4 at T_C	FRG in LPA*	Spin-model Lattice simulations	discrepancy
O(2)	1.2491(39)	1.242(2) [9]	0.6 %
O(4)	1.0836(10)	1.092(3) [10]	0.8 %

Results for the Binder cumulant, Part II

In order to apply our results for $B_4(T_C)$ to lattice QCD analysis we need to study finite-size and finite explicit symmetry breaking effects.

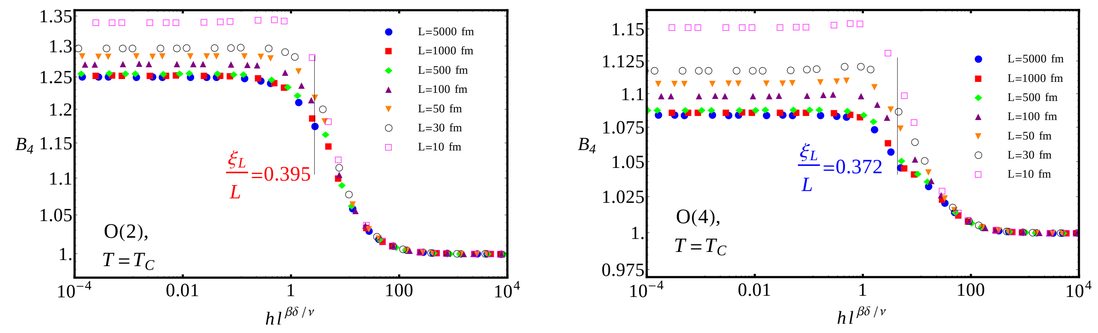


Figure: Rescaled results for $B_4(T_C)$ for different volumes and symmetry breaking fields. We observe finite-size and finite explicit symmetry breaking effects.

- Finite-size effects along the B_4-axis can be perfectly explained by the corrections due to the first irrelevant operator in the RG-flow
- Finite explicit symmetry breaking, together with finite-size effects, translates into the finite-size (*left part in both plots*) and infinite-size scaling regions (*right part in both plots*). These regions can be described by the universal value of ξ_L/L or $m_\pi L$

The major result

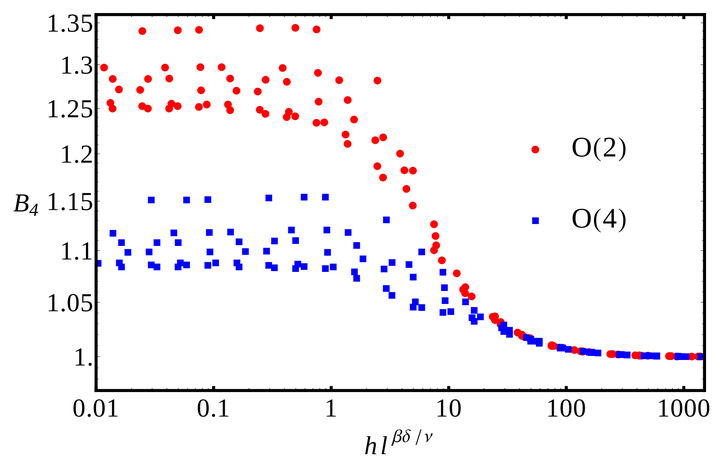


Figure: The same data as above but in one plot.

$B_4(T_C)$ can be applied to determine the universality class of the chiral phase transition in $N_f = 2$ or $N_f = (2+1)$ lattice QCD even in the presence of non-universal finite-size corrections (as long as $\xi_L/L \gtrsim 0.4$ or $m_\pi L \lesssim 2$).

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