Viscous hydrodynamics for systems undergoing strongly anisotropic expansion

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Motivation

- Large differences between the longitudinal and transverse expansion rates lead to large shear viscous effects (longitudinal/transverse pressure anisotropies) in the early stage of heavy-ion collisions.

- These cause Israel-Stewart theory to break down at early times.

- Anisotropic hydrodynamics (aHydro) deals with the large longitudinal/transverse pressure anisotropy “nonperturbatively”; this improves the performance of hydrodynamics at early times.

- But: aHydro accounts for only one of the 5 independent components of the shear stress tensor, ignoring the others \( \implies \) unreliable for the computation of elliptic flow which is sensitive to \( \pi^{xx} - \pi^{yy} \), for example.

- On the other hand: these 4 remaining components of the viscous stress tensor become never as large as the longitudinal/transverse pressure difference (with smooth initial density profiles they start out as zero, with fluctuating initial conditions they are initially small).

- \( \implies \) Idea: treat large longitudinal/transverse pressure anisotropy “nonperturbatively” with aHydro, add remaining viscous corrections “perturbatively” à la Israel-Stewart \( \implies \) vaHydro.

- Expect better performance at all times compared to both aHydro and Israel-Stewart theory.
Kinetic theory vs. hydrodynamics

Both simultaneously valid if weakly coupled and pressure gradients are small.

Hydro equations remain structurally unchanged for strongly coupled systems.

\[ p^\mu \partial_\mu f(x, p) = C(x, p) = \frac{p \cdot u(x)}{\tau_{\text{rel}}(x)} \left( f_{\text{eq}}(x, p) - f(x, p) \right) \]

in relaxation time approximation (RTA).

For conformal systems \( \tau_{\text{rel}}(x) = c/T(x) \).

Macroscopic currents:

\[ j^\mu(x) = \int_p p^\mu f(x, p) \equiv \langle p^\mu \rangle \]

\[ T^{\mu\nu}(x) = \int_p p^\mu p^\nu f(x, p) \equiv \langle p^\mu p^\nu \rangle \]

where

\[ \int_p \ldots \equiv \frac{g}{(2\pi)^3} \int \frac{d^3p}{E_p} \ldots \equiv \langle \ldots \rangle \]
Ideal fluid dynamics (I)

Ideal hydro $\iff f(x, p) = f_{\text{iso}}(x, p) \equiv f_{\text{iso}} \left( \frac{p \cdot u(x) - \mu(x)}{T(x)} \right)$

(Locally isotropic momentum distribution, not necessarily exponential or in chemical equilibrium)

If not in chemical equilibrium, then $\partial_\mu j^\mu \neq 0$.

If not exponential in $(p \cdot u(x) - \mu(x))/T(x)$, then $C(x, p) \neq 0$, but still $\int p p^\mu C = 0$ (energy-momentum conservation).

For ideal hydro

$$j_{\text{id}}^\mu(x) = n(x) u^\mu(x)$$
$$T_{\text{id}}^{\mu\nu} = e(x) u^\mu(x) u^\nu(x) - P(e(x)) \Delta^{\mu\nu}(x)$$

where $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu =$ spatial projector in l.r.f.

Write

$$p^\mu = E u^\mu + p^{(\mu)}$$

where $E \equiv u \cdot p =$ energy in l.r.f., $p^{(\mu)} \equiv \Delta^{\mu\nu} p_\nu =$ spatial momentum in l.r.f.
Ideal fluid dynamics (II)

Ideal hydro equations follow from

\[ \partial_{\mu} j^{\mu} = \frac{n_{eq} - n(x)}{\tau_{rel}(x)} \]
\[ \partial_{\mu} T^{\mu\nu} = 0 \]

which one can solve for \( n(x), \ e(x), \ u^{\mu}(x) \).

Then \( T(x), \ \mu(x), \ P(x) \) follow from the EOS.

**Note:** if system is locally isotropic but not in chemical and thermal equilibrium, this can be accounted for by non-equilibrium chemical potentials and a non-equilibrium pressure in the EOS \( P(e, n) = P(T, \mu) \). In this case one sees non-zero entropy production \( \partial_{\mu} S^{\mu} \sim 1/\tau_{rel} \neq 0 \).
Israel-Stewart viscous fluid dynamics (I)

\[ f(x,p) = f_{\text{iso}} \left( \frac{p \cdot u(x) - \mu(x)}{T(x)} \right) + \delta f(x,p) \]

Separation made unique by Landau matching:

First define l.r.f. by \( T^{\mu\nu} u_\nu = e u^\mu \) with \( u^\nu u_\nu = 1 \) \( \implies \) fixes flow vector \( u^\mu \)

Next, require

\[
\begin{align*}
  e(x) &= e_{\text{iso}}(T, \mu) \implies \langle E^2 \rangle = \langle E^2 \rangle_{\text{iso}}(T, \mu) \\
n(x) &= n_{\text{iso}}(T, \mu) \implies \langle E \rangle = \langle E \rangle_{\text{iso}}(T, \mu)
\end{align*}
\]

\( \implies \langle E \rangle_\delta = \langle E^2 \rangle_\delta = 0 \implies \) fixes \( T(x), \mu(x) \)

Viscous decomposition of \( j^\mu, T^{\mu\nu} \):

\[
\begin{align*}
  j^\mu &= j_{\text{id}}^\mu + V^\mu \\
  T^{\mu\nu} &= T_{\text{id}}^{\mu\nu} - \Pi \Delta^{\mu\nu} + \pi^{\mu\nu}
\end{align*}
\]

\[
\begin{align*}
  V^\mu &= \left\langle p^{(\mu)} \right\rangle_\delta \\
  \Pi &= -\frac{1}{3} \left\langle p^{(\alpha)} p^{(\alpha)} \right\rangle_\delta, \\
  \pi^{\mu\nu} &= \left\langle p^{(\mu)} p^{(\nu)} \right\rangle_\delta.
\end{align*}
\]

Here \( A^{(\mu\nu)} \equiv \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta} \) with \( \Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2} \left( \Delta^{\mu}_{\alpha} \Delta^{\nu}_{\beta} + \Delta^{\nu}_{\alpha} \Delta^{\mu}_{\beta} \right) - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} \)

\( \implies \pi^{\mu\nu} = T^{(\mu\nu)} \) has 5 independent components (3 for (2+1)-d, 1 for (0+1)-d)

Altogether 9 viscous flow degrees of freedom.
Israel-Stewart viscous fluid dynamics (II)

Israel-Stewart equations of motion for viscous pressures (Israel&Stewart 1979, Muronga 2002):

Define \( \dot{F} \equiv DF \equiv u^\mu \partial_\mu F, \theta \equiv \partial \cdot u, \sigma^{\mu\nu} = \partial^{\langle \mu} u^{\nu \rangle} \):

\[
\dot{\Pi} = -\frac{1}{\tau_\Pi} \left[ \Pi + \zeta \theta + \Pi \zeta T \partial_\mu \left( \frac{\tau_\Pi u^\mu}{2 \zeta T} \right) \right] \equiv -\frac{1}{\tau'_\Pi} [\Pi + \zeta' \theta],
\]

\[
\dot{\langle \mu\nu \rangle} = \Delta_{\alpha\beta}^{\mu\nu} D_\pi \pi^{\alpha\beta} = -\frac{1}{\tau_\pi} \left[ \pi^{\mu\nu} - 2 \eta \sigma^{\mu\nu} + \pi^{\mu\nu} \eta T \partial_\mu \left( \frac{\tau_\pi u^\mu}{2 \eta T} \right) \right] \equiv -\frac{1}{\tau'_\pi} [\pi^{\mu\nu} - 2 \eta' \sigma^{\mu\nu}],
\]

where \( \eta, \zeta \) are shear and bulk viscosity (first order transp. coeff.), \( \tau_\Pi, \tau_\pi \) are shear and bulk pressure relaxation times (second order transp. coeff.), and

\[
\tau'_\Pi = \frac{\tau_\Pi}{1 + \gamma_\Pi}, \quad \tau'_\pi = \frac{\tau_\pi}{1 + \gamma_\pi}, \quad \zeta' = \frac{\zeta}{1 + \gamma_\Pi}, \quad \eta' = \frac{\eta}{1 + \gamma_\pi},
\]

\[
\gamma_\Pi = \frac{1}{2} \zeta T \partial_\mu \left( \frac{\tau_\Pi u^\mu}{2 \zeta T} \right) \rightarrow \frac{4}{3} \tau_\Pi \theta,
\]

\[
\gamma_\pi = \frac{1}{2} \eta T \partial_\mu \left( \frac{\tau_\pi u^\mu}{2 \eta T} \right) \rightarrow \frac{4}{3} \tau_\pi \theta,
\]

where the arrow indicates the conformal limit.
The Israel-Stewart equations are not the most general form of second order equations of motion for the viscous pressures. For a complete set of second-order terms, together with the associated transport coefficients computed from Boltzmann theory, see Denicol, Molnar, Niemi, Rischke, EPJA 48 (2012) 170 (DMNR).

Problem with applying IS theory to heavy-ion collisions:
for early times, as $\tau \to 0$,

$$\tau^2 \sigma \eta = -(\sigma^{xx} + \sigma^{yy}) \to \frac{2}{3\tau}$$

$\implies$ very large viscous corrections! $\implies \delta f$ no longer small.

This problem is caused by the rapid self-similar longitudinal expansion.
Anisotropic hydrodynamics (\textsc{aHydro}) (I)

Martinez and Strickland 2009, 2010; Florkowski and Ryblewski 2010

A non-perturbative method to account for large shear viscous effects stemming from large difference between longitudinal and transverse expansion rates.

\[
f(x, p) = f_{\text{iso}} \left( \frac{\sqrt{p_{\mu} \Xi_{\mu\nu}(x) p_{\nu} - \tilde{\mu}(x)}}{\Lambda(x)} \right) \equiv f_{RS}(x, p)
\]

where \( \Xi_{\mu\nu}(x) = u_{\mu}(x)u_{\nu}(x) + \xi(x)z_{\mu}(x)z_{\nu}x. \) (Romatschke&Strickland 2003)

3 flow and 3 “thermodynamic” parameters: \( u_{\mu}(x); \Lambda(x), \tilde{\mu}(x), \xi(x). \)

\textsc{aHydro} decomposition:

\[
j_{RS}^{\mu} = n_{RS}u^{\mu}, \quad T_{RS}^{\mu\nu} = e_{RS}u^{\mu}u^{\nu} - P_{T}\Delta^{\mu\nu} + (P_{L} - P_{T})z^{\mu}z^{\nu},
\]

where, for massless partons (\( m = 0 \)), the effects of local momentum anisotropy can be factored out:

\[
n_{RS} = \langle E \rangle_{RS} = R_{0}(\xi)n_{\text{iso}}(\Lambda, \tilde{\mu}),
\]

\[
e_{RS} = \langle E^{2} \rangle_{RS} = R(\xi)e_{\text{iso}}(\Lambda, \tilde{\mu}),
\]

\[
P_{T,L} = \langle p_{T,L}^{2} \rangle_{RS} = R_{T,L}(\xi)P_{\text{iso}}(\Lambda, \tilde{\mu}).
\]

(See paper for \( R \)-functions.) The isotropic pressure is obtained from a locally isotropic EOS,

\[
P_{\text{iso}}(\Lambda, \tilde{\mu}) = P_{\text{iso}}(e_{\text{iso}}(\Lambda, \tilde{\mu}), n_{\text{iso}}(\Lambda, \tilde{\mu}))
\]

For massless noninteracting partons, \( P_{\text{iso}}(\Lambda, \tilde{\mu}) = \frac{1}{3}e_{\text{iso}}(\Lambda, \tilde{\mu}) \) independent of chemical composition.
Anisotropic hydrodynamics \((\text{aHydro})\) (II)

Martinez and Strickland 2009, 2010; Florkowski and Ryblewski 2010

If we want to compare \(\text{aHydro}\) with ideal and IS viscous hydro, we need to assign the locally anisotropic system an appropriate temperature \(T(x) = T\left(\xi(x), \Lambda(x), \bar{\mu}(x)\right)\) and chemical potential \(\mu(x) = \mu\left(\xi(x), \Lambda(x), \bar{\mu}(x)\right)\), and think of \(f_{\text{RS}}(\xi, \Lambda)\) as an expansion around the locally isotropic distribution \(f_{\text{iso}}(T)\). This is done by “dynamical Landau matching”: We demand that \(e_{\text{RS}}(\xi, \Lambda, \bar{\mu}) = e_{\text{iso}}(T, \mu)\) and \(n_{\text{RS}}(\xi, \Lambda, \bar{\mu}) = R_0(\xi)n_{\text{iso}}(T, \mu)\).

For example, using a Boltzmann distribution for \(f_{\text{iso}}(x, p)\) with \(\mu = \bar{\mu} = 0\), one finds (Martinez & Strickland 2010)

\[
T = \Lambda R^{1/4}(\xi)
\]

With this matching we can write

\[
T_{\text{RS}}^{\mu\nu} = T_{\text{id}}^{\mu\nu} - (\Delta P + \Pi_{\text{RS}}) \Delta^{\mu\nu} + \pi_{\text{RS}}^{\mu\nu}
\]

where

\[
\Delta P + \Pi_{\text{RS}} = -\frac{1}{3} \int p_\alpha \Delta^{\alpha\beta} p_\beta (f_{\text{RS}} - f_{\text{iso}}) \quad (= 0 \text{ for } m = 0),
\]

\[
\pi_{\text{RS}}^{\mu\nu} = \int p^{(\mu} p^{\nu)} (f_{\text{RS}} - f_{\text{iso}}) = (P_T - P_L) \frac{x^{(\mu} x^{\nu)} + y^{(\mu} y^{\nu)} - 2 z^{(\mu} z^{\nu)} }{3}
\]

We see that \(\pi_{\text{RS}}^{\mu\nu}\) has only one independent component, \(P_T - P_L\), so \(\text{aHydro}\) leaves 4 of the 5 components of \(\pi^{\mu\nu}\) unaccounted for.
Anisotropic hydrodynamics (AHYDRO) (III)

Martinez and Strickland 2009, 2010; Florkowski and Ryblewski 2010

For massless particles we have

\[
\frac{P_T - P_L}{P_{iso}(e)} = R_T(\xi) - R_L(\xi),
\]

so the EOM for \( \pi_{\mu\nu}^{RS} \) can be replaced by an EOM for \( \xi \).

For \( m \neq 0 \), to separate \( \Delta P \) from the viscous pressure \( \Pi \), we need an “anisotropic EOS” for

\[
\frac{\Delta P}{P_{iso}} = \frac{2P_T + P_L}{3P_{iso}} - 1.
\]
Viscous anisotropic hydrodynamics (vaHydro) (I)

\[ f(x, p) = f_{RS}(x, p) + \delta \tilde{f}(x, p) = f_{iso} \left( \frac{\sqrt{p_\mu \Xi^{\mu\nu}(x)p_\nu - \tilde{\mu}(x)}}{\Lambda(x)} \right) + \delta \tilde{f}(x, p) \]

Landau matching:

no contribution to \( e, n \) from \( \delta \tilde{f} \):

no contribution to \( P_T - P_L \) from \( \delta \tilde{f} \):

vaHydro decomposition:

\[ j^\mu = j^\mu_{RS} + \tilde{V}^\mu, \]

\[ T^{\mu\nu} = T^{\mu\nu}_{RS} - \tilde{\Pi} \Delta^{\mu\nu} + \tilde{\pi}^{\mu\nu}, \]

\[ u_\mu \tilde{\pi}^{\mu\nu} = 0 \]

Strategy: solve hydrodynamic equations for AHydro (which treat \( P_T - P_L \) nonperturbatively) with added viscous flows from \( \delta \tilde{f} \), together with IS-like “perturbative” equations of motion for \( \tilde{\Pi}, \tilde{V}^\mu, \tilde{\pi}^{\mu\nu} \).
Viscous anisotropic hydrodynamics \( \text{(vaHYDRO)} \) (II)

Hydrodynamic equations of motion:

\[ \partial_{\mu} j^\mu = C \equiv \int_p C(x, p) \implies \dot{n}_{\text{RS}} = -n_{\text{RS}} \theta - \partial_{\mu} \tilde{V}^\mu + \frac{n_{\text{RS}} - n_{\text{iso}}}{\tau_{\text{rel}}} \quad \text{in RTA} \]

\[ \partial_{\mu} T^\mu{}^\nu = 0 \implies \]

\[ \dot{\theta} = -(e+P_T)\theta_\perp - (e+P_L)\frac{u_0}{\tau} - \tilde{\Pi}\theta + \tilde{\pi}^\mu{}^\nu \sigma_{\mu\nu}, \]

\[ (e+P_T+\tilde{\Pi})\dot{u}_\perp = -\partial_\perp (P_T+\tilde{\Pi}) - u_\perp (\dot{P}_T+\dot{\tilde{\Pi}}) - u_\perp (P_T-P_L)\frac{u_0}{\tau} + \left( \frac{u_x \Delta_1 + u_y \Delta_2}{u_\perp} \right) \partial_{\mu} \tilde{\pi}^\mu{}^\nu, \]

\[ (e+P_T+\tilde{\Pi})u_\perp \dot{\phi}_u = -D_\perp (P_T+\tilde{\Pi}) - \frac{u_y \partial_{\mu} \tilde{\pi}^{\mu1} - u_x \partial_{\mu} \tilde{\pi}^{\mu2}}{u_\perp}, \]

where \( \theta_\perp = \partial_\tau u_0 + \nabla_\perp \cdot u_\perp \) and \( D_\perp = (u_x \partial_y - u_y \partial_x)/u_\perp \).

To derive equations of motion for \( \tilde{\Pi}, \tilde{V}^\mu, \) and \( \tilde{\pi}^\mu{}^\nu \), we follow DMNR (2012). Ignoring heat conduction by setting \( \tilde{\mu} = 0 \) and taking \( m = 0 \) we find

\[ \dot{\tilde{\pi}}^{\mu{}^\nu} = -2u_\alpha \tilde{\pi}^{\alpha(\mu} u^{\nu)} - \frac{1}{\tau_{\text{rel}}} \left[ (P-P_T)\Delta^{\mu{}^\nu} + (P_L-P_T) z^{\mu} z^{\nu} + \tilde{\pi}^{\mu{}^\nu} \right] + \mathcal{K}_{0}^{\mu{}^\nu} + \mathcal{L}_{0}^{\mu{}^\nu} + \mathcal{H}_{0}^{\mu{}^\nu} \dot{z}_\lambda \]

\[ + Q_{0}^{\mu{}^\nu} \lambda^{\alpha} \nabla_\lambda u_\alpha + \chi_{0}^{\mu{}^\nu} \lambda^{\alpha} \nabla_\lambda z_\alpha - 2\lambda_{\pi}^{0} \tilde{\pi}^{\lambda(\mu} \sigma^{\nu)} + 2\tilde{\pi}^{\lambda(\mu} \omega^{\nu)} - 2\delta_{\pi}^{0} \tilde{\pi}^{\mu{}^\nu} \theta \]

(see Bazow, UH, Strickland, arXiv:1311.6720v2 for details).
Test of **VAHYDRO**: (0+1)-dimensional expansion (I)

For (0+1)-d (longitudinally boost-invariant) expansion, the BE can be solved exactly in RTA ([Florkowski, Ryblewski, Strickland, PRC88 (2013) 024903](#)), and the solution can be used to test the various macroscopic hydrodynamic approximation schemes.

Setting homogeneous initial conditions in $r$ and $\eta/s$ and zero transverse flow, $\tilde{\pi}^{\mu\nu}$ reduces to a single non-vanishing component $\tilde{\pi}$: $\tilde{\pi}^{\mu\nu} = \text{diag}(0, -\tilde{\pi}/2, -\tilde{\pi}/2, \tilde{\pi})$ at $z = 0$.

We use the factorization $n_{RS}(\xi \Lambda) = R_0(\xi) n_{iso}(\Lambda)$ etc. to get EOMs for $\dot{\xi}$, $\dot{\Lambda}$, $\tilde{\pi}$:

$$\frac{\dot{\xi}}{1+\xi} - 6\frac{\dot{\Lambda}}{\Lambda} = \frac{2}{\tau} + \frac{2}{\tau_{\text{rel}}} \left( 1 - \sqrt{1+\xi} R^{3/4}(\xi) \right),$$

$$R'(\xi) \dot{\xi} + 4R(\xi) \frac{\dot{\Lambda}}{\Lambda} = - \left( R(\xi) + \frac{1}{3} R_L(\xi) \right) \frac{1}{\tau} + \frac{\tilde{\pi}}{e_{\text{iso}}(\Lambda) \tau},$$

$$\tilde{\pi} = - \frac{1}{\tau_{\text{rel}}} \left[ \left( R(\xi) - R_L(\xi) \right) P_{iso}(\Lambda) + \tilde{\pi} \right] - \frac{38 \tilde{\pi}}{21 \tau}$$

$$+ 12 \left[ \frac{\dot{\Lambda}}{\Lambda} \left( R_L(\xi) - \frac{1}{3} R(\xi) \right) + \left( \frac{1+\xi}{\tau} - \frac{\dot{\xi}}{2} \right) \left( R_{-1}^{zzzz}(\xi) - \frac{1}{3} R_1^{zz}(\xi) \right) \right] P_{iso}(\Lambda),$$

$\tau_{\text{rel}}$ and $\eta/s$ are related by ([Denicol, Koide, Rischke, PRL 105 (2010)](#))

$$\tau_{\text{rel}} = 5 \frac{\eta/s}{T} = 5 \frac{\eta/s}{R^{1/4}(\xi) \Lambda}$$

We solve these equations and compare with the exact solution:
Test of \texttt{vaHydro}: $(0+1)$-dimensional expansion (II)

Pressure anisotropy $P_L/P_T$ vs. $\tau$:

- $\xi_0 = 10$, $4\pi\eta/S = 3$, $T_0 = 0.6$ GeV
- $\xi_0 = 100$, $4\pi\eta/S = 100$, $T_0 = 0.6$ GeV
Test of **vaHydro**: (0+1)-dimensional expansion (III)

Total entropy (particle) production

$$\frac{n(\tau_f) \cdot \tau_f}{n(\tau_0) \cdot \tau_0} - 1$$

![Graph showing the comparison of total entropy production across different hydrodynamic approximations.](image-url)
Conclusions

- For early times and/or near the transverse edge in heavy-ion collision fireballs, rapid longitudinal expansion generates large inverse Reynolds numbers for the shear pressure, $R_{\pi}^{-1} = \sqrt{\pi^{\mu\nu}\pi_{\mu\nu}}/P_{\text{iso}}$, causing Israel-Stewart second order viscous hydrodynamics to break down.

- The large local pressure anisotropies caused by a large difference in longitudinal and transverse expansion rates can be treated efficiently by using the non-perturbative aHydro approach which is based on an expansion around a locally spheroidally deformed distribution $f_{RS}$.

- This strongly reduces the shear inverse Reynolds numbers $\tilde{R}_{\pi}^{-1} = \sqrt{\tilde{\pi}^{\mu\nu}\tilde{\pi}_{\mu\nu}}/P_{\text{iso}}$ associated with the remaining shear stress tensor $\tilde{\pi}^{\mu\nu}$ resulting from the much smaller deviation $\delta \tilde{f}$ of the local distribution function from $f_{RS}$.

- vaHydro combines the advantages of aHydro with a complete (although perturbative) second-order treatment of all remaining viscous effects à la Israel-Stewart.

- In a test of $(0+1)$-d expansion, which maximizes the difference between longitudinal and transverse expansion rates, against an exact solution of the Boltzmann equation, vaHydro outperforms all other known hydrodynamic approximation schemes by a considerable margin.

- This should open the door in $(3+1)$-d systems to match microscopic pre-equilibrium theories to viscous hydrodynamics at earlier times than possible with IS-theory and its variants.