Fermilab Accelerator Physics Center

Computation of Eigen-Emittances (and Optics Functions!) from Tracking Data

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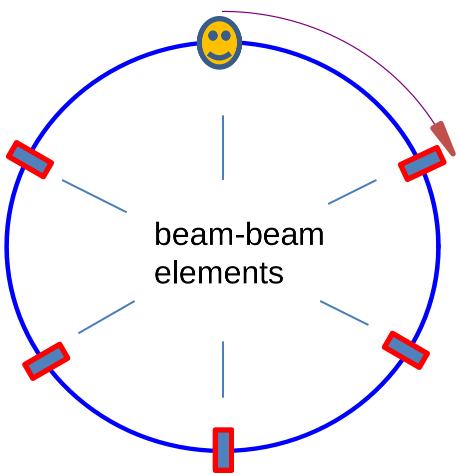
Motivation

- "Express" space-charge modeling (e.g. with MADX beam-beam elements)

observation point,
Σ_0 and/or ε_m



$$\Sigma^{(k)} = \mathrm{M}_{0\to k} \cdot \Sigma^{(0)} \cdot \mathrm{M}_{0\to k}^{\mathrm{T}}$$

does not require knowledge of Σ_k in a simple kick-rotate scheme (but does require Σ_i, i=0,…, k-1)

alternatively

$$\sigma_u^2 = \sum_m \beta_{um}\varepsilon_m, \quad u = x, y, z$$

requires knowledge of β-functions, with fixed β-s can be used as a rough approximation

- Muon cooling channel design -

http://map-docdb.fnal.gov/cgi-bin/ShowDocument?docid=4358

Outline

- How to suppress halo contribution to covariance matrix in a self-consistent way to obtain right sizes for the space charge forces computation? - Multidimensional case please!

- Iterative procedure for nonlinear fit of particle distribution in the phase space with a Gaussian or other smooth function.

- How to find the normal mode emittances (eigen-emittances) when optics functions are not known?

- Eigen-emittances as well as optics functions can be determined from the covariance matrix.

- (Extremely fast & simple!) exponential fit of particle distribution when the optics functions are known – already implemented in MAD-X, but not described anywhere.

Definitions

Phase space vector:

$$\underline{z} = \{x, P_x, y, P_y, s - c\beta_0 t, \delta\}$$

Canonical momenta in units of the reference value $p_0 = mc\beta_0\gamma_0$:

$$P_x = (p_x + \frac{e}{c} A_x) / p_0$$

Energy deviation (disguised as momentum)

$$\delta = (\gamma - \gamma_0) / \beta_0^2 \gamma_0$$

Covariance matrix (Σ- matrix)

$$\Sigma_{i,j} = \frac{1}{N} \sum_{k=1}^{N} \zeta_i^{(k)} \zeta_j^{(k)}, \quad \zeta_i^{(k)} = z_i^{(k)} - \bar{z}_i, \quad \bar{z}_i = \frac{1}{N} \sum_{k=1}^{N} z_i^{(k)}, \quad i = 1, ..., 6$$

Basic assumption: particle distribution is a function of quadratic form

$$\Phi(\underline{\zeta}) = (\underline{\zeta}, \Sigma^{-1} \underline{\zeta}) \equiv \sum_{i=1}^{6} \zeta_i (\Sigma^{-1} \underline{\zeta})_i = \sum_{i,j=1}^{6} \Sigma_{ij}^{-1} \zeta_i \zeta_j$$

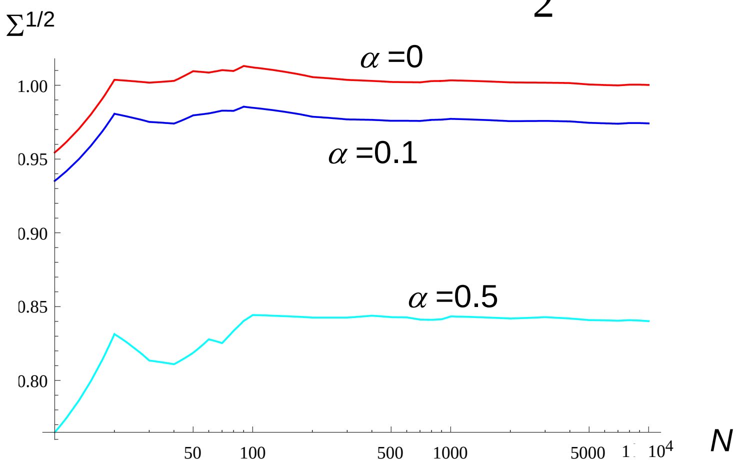
How to Suppress Halo Contribution?

And to do this in a self-consistent way?

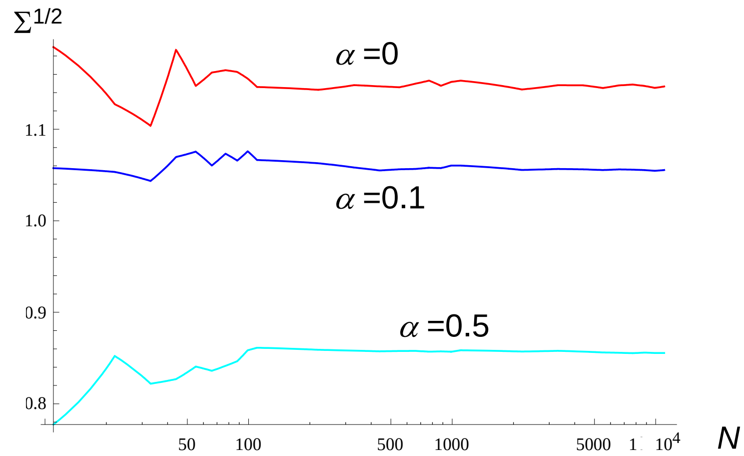
- a simple heuristic method is to introduce weights proportional to some degree of the distribution function. This leads to an iterative procedure

$$\bar{z}_i = \sum_{k=1}^{N} w_k z_i^{(k)} / \sum_{k=1}^{N} w_k, \quad \zeta_i^{(k)} = z_i^{(k)} - \bar{z}_i, \quad \Sigma_{i,j} = \sum_{k=1}^{N} w_k \zeta_i^{(k)} \zeta_j^{(k)} / \sum_{k=1}^{N} w_k, \qquad (1)$$

For Gaussian $w_k = \exp[-\frac{\alpha}{2}(\underline{\zeta}^{(k)}, \Sigma^{-1}\underline{\zeta}^{(k)})]$, α being a fitting parameter ($0<\alpha<1$)



Square root of Σ from eq.(1) averaged over 25 realizations of 1D Gaussian distribution with σ=1as function of the number of particles *N*.



Square root of Σ from eq.(1) averaged over 25 realizations of superposition of 1D Gaussian distributions with σ=1(90%) and σ=3(10%)

This method is imprecise and ambiguous ⇒ something based on a more solid foundation is needed.

Nonlinear Fit of the Klimontovich Distribution

$$G(\underline{z}) = \frac{1}{N}\sum_{k=1}^{N}\delta_{6D}(\underline{z}-\underline{z}^{(k)}) \equiv \frac{1}{N}\sum_{k=1}^{N}\prod_{i=1}^{6}\delta(z_i - z_i^{(k)})$$

We want to approximate it with a smooth function, e.g. Gaussian

$$F(\underline{\zeta}) = \frac{\eta}{(2\pi)^{n/2}\sqrt{\det\Sigma}}\exp[-\frac{1}{2}(\underline{\zeta},\Sigma^{-1}\underline{\zeta})]$$

where η is the fraction of particles in the beam core,
via the minimization problem

$$\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}|F-G|^2\,dz_1..dz_n = \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}(F^2-2FG)dz_1..dz_n + \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}G^2 dz_1..dz_n \to \min$$

or the maximization problem for the 1st term in the r.h.s. taken with the opposite sign

$$M(\bar{z},\Sigma,\eta) = \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}(2FG-F^2)dz_1..dz_n =$$

$$\frac{\eta}{(2\pi)^{n/2}\sqrt{\det\Sigma}}\left\{\frac{2}{N}\sum_{k=1}^{N}\exp[-\frac{1}{2}(\underline{\zeta}^{(k)},\Sigma^{-1}\underline{\zeta}^{(k)})] - \frac{\eta}{2^{n/2}}\right\} \to \max$$

For n=6 there is n(n+3)/2+1=28 fitting parameters – convergence too slow

Rigorous Iterative Procedure

By differentiating $M(\bar{z},\Sigma,\eta)$ w.r.t. fitting parameters we recover equations which can be solved iteratively.

For average values of coordinates the equations coincide with heuristic ones with α=1

$$\bar{z}_i = \sum_{k=1}^{N} z_i^{(k)} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)},\Sigma^{-1}\underline{\zeta}^{(k)})] / \sum_{k=1}^{N} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)},\Sigma^{-1}\underline{\zeta}^{(k)})], \quad \zeta_i^{(k)} = z_i^{(k)} - \bar{z}_i$$

$$(\frac{1}{N}\sum_{k=1}^{N}\ldots \rightarrow \sum_{k=1}^{N} w_k \ldots / \sum_{k=1}^{N} w_k$$

for weighted particles)

We can keep η fixed (i.e. set the fraction of particles taken into account)

Then for Σ- matrix we get

$$\Sigma_{ij} = \frac{1}{N}\sum_{k=1}^{N} \zeta_i^{(k)}\zeta_j^{(k)} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)},\Sigma^{-1}\underline{\zeta}^{(k)})] / \left(\frac{1}{N}\sum_{k=1}^{N} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)},\Sigma^{-1}\underline{\zeta}^{(k)})] - \frac{\eta}{2^{n/2+1}} \right)$$

For $\eta \rightarrow 1$ some damping is necessary in n=6 case to avoid oscillations:

$$\Sigma^{(i)} = (1-d)\Sigma^{(i-1)} + d\Sigma^{(formula)}, \quad d \approx 0.8$$

(Mathematics is presented in the cited MAP note)

Rigorous Iterative Procedure (cont'd)

We can try to find the optimal fraction of particles η for the fit.

From equation $\frac{d}{d\eta} M(\bar{z}, \Sigma, \eta) = 0$ we get

$$\eta = \frac{2^{n/2}}{N} \sum_{k=1}^{N} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)}, \Sigma^{-1} \underline{\zeta}^{(k)})], \quad \zeta_i^{(k)} = z_i^{(k)} - \bar{z}_i$$

Equations for average values of coordinates remain the same,

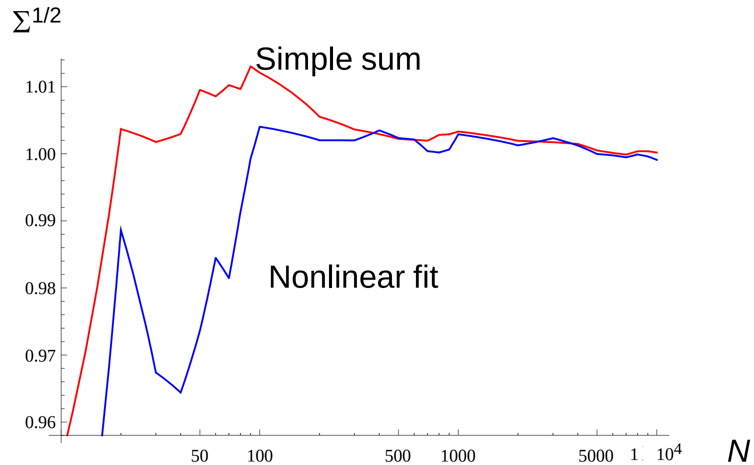
whereas for Σ- matrix we obtain expression with an extra factor of 2 (!) compared to the heuristic one

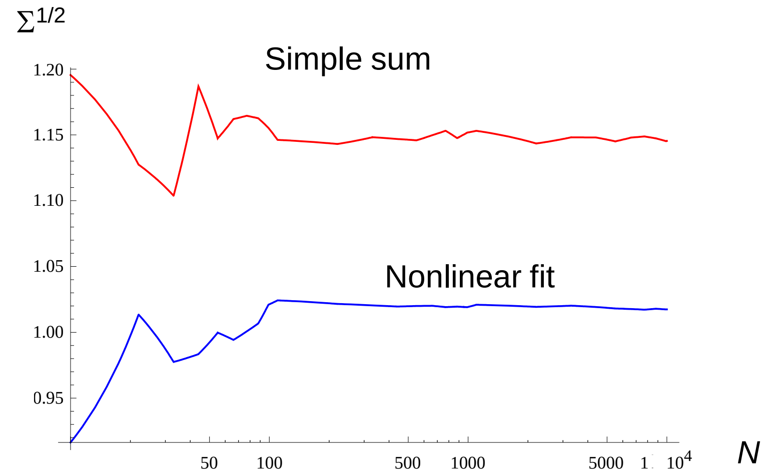
$$\Sigma_{ij} = 2 \sum_{k=1}^{N} \zeta_i^{(k)} \zeta_j^{(k)} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)}, \Sigma^{-1} \underline{\zeta}^{(k)})] / \sum_{k=1}^{N} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)}, \Sigma^{-1} \underline{\zeta}^{(k)})]$$

Damping is not necessary in this case.

For n=6 in all cases just 20-30 iterations are required to achieve precision $\leq 10^{-6}$, it takes *Mathematica* ~13 seconds with N= 10^4 on my home PC. For a Fortran or C code it will be a fraction of a second.

1D Precision Test





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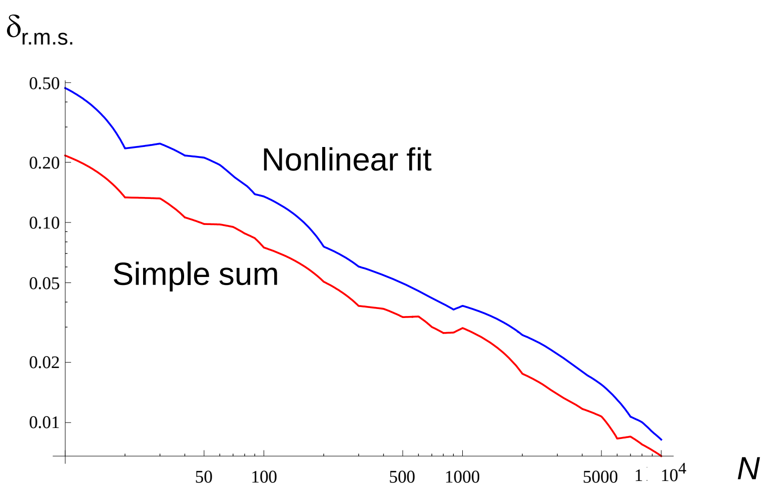
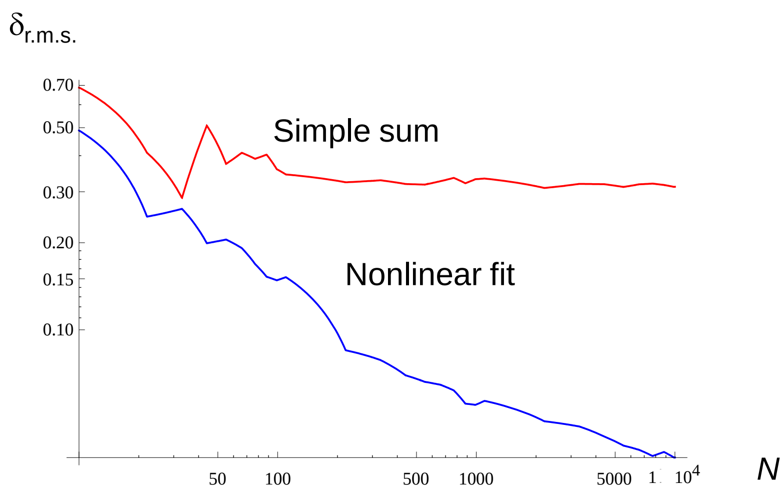
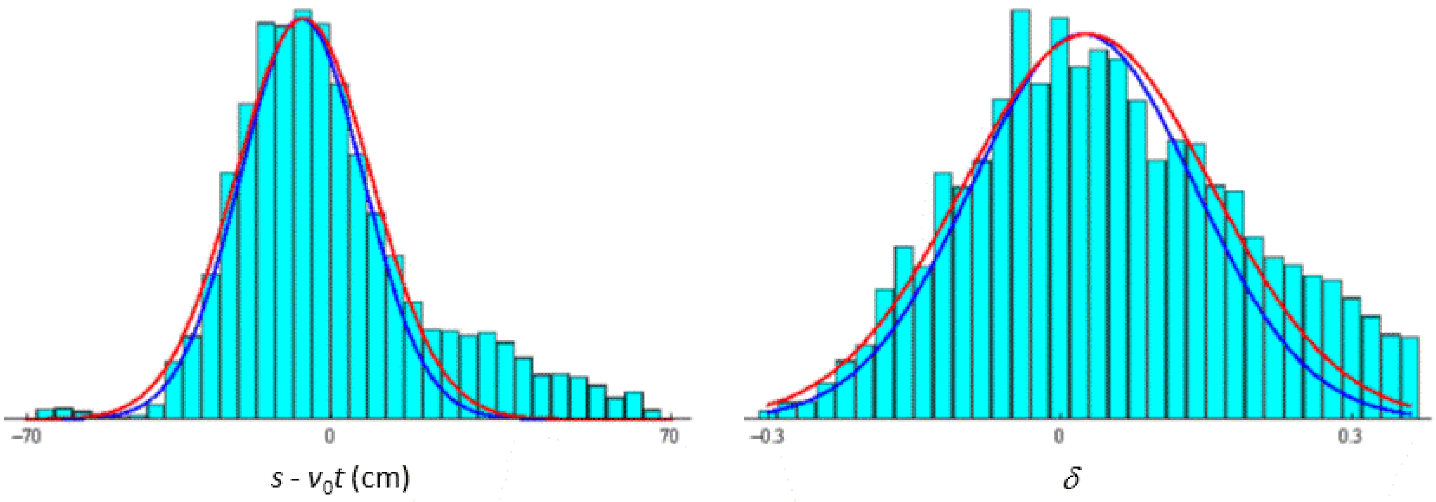




R.m.s. error in $\Sigma^{1/2}$ from above

R.m.s. error in $\Sigma^{1/2}$ from above

Realistic Example



Projections onto the longitudinal coordinate (left) and δ (right) of the original particle distribution (cyan bars) and of its Gaussian fit with η =1 and $\eta = \eta_{\text{fit}}$ (red and blue solid lines respectively).

N.B. Projection of the Gaussian distribution onto the m^{th} axis in a multidimensional case is proportional to

$$\exp\left\{-\frac{1}{2}(\Sigma^{-1})_{mm}\zeta_m^2[2-(\Sigma^{-1})_{mm}\Sigma_{mm}]\right\}$$

Eigen-Emittances from Σ- matrix

With Σ- matrix known, how to find the normal mode emittances?

- Σ- matrix has positive eigenvalues but they are useless unless the matrix of transformation to diagonal form is symplectic (generally not the case)
- solution suggested by theory developed by V.Lebedev & A.Bogacz :

Consider a product $\Omega = S\Sigma^{-1}$ of inverse Σ- matrix and symplectic unity matrix

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

Matrix Ω has purely imaginary eigenvalues which are inverse eigen-emittances :

$$\lambda_{2m-1} = -\frac{i}{\varepsilon_m}, \quad \lambda_{2m} = \frac{i}{\varepsilon_m}, \quad m = 1, 2, 3$$

(Again, mathematics is presented in the cited MAP note)

Eigen-Vectors of Matrix Ω

Using real and imaginary parts of eigen-vectors $\underline{v}_i^{/} \equiv \mathrm{Re}\,\underline{v}_i, \quad \underline{v}_i^{//} \equiv \mathrm{Im}\,\underline{v}_i$ as columns we can build a matrix:

$$\mathrm{V} = \{\underline{v}_1^{/}, -\underline{v}_1^{//}, \underline{v}_3^{/}, -\underline{v}_3^{//}, \underline{v}_5^{/}, -\underline{v}_5^{//}\}$$

which is symplectic, $V^t S V = S$, and brings Ω to diagonal form:

$$\mathrm{V}^{-1}\,\Omega\,\mathrm{V} = \mathrm{S}\,\Xi, \qquad \Xi = \mathrm{diag}\left(\frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_2}, \frac{1}{\varepsilon_2}, \frac{1}{\varepsilon_3}, \frac{1}{\varepsilon_3}\right).$$

The quadratic form Φ takes the form:

$$\Phi = (\underline{\zeta}, \Sigma^{-1}\underline{\zeta}) \to (\underline{\xi}, \Xi\underline{\xi}) = \sum_{m=1}^{3} \frac{\xi_{2m-1}^2 + \xi_{2m}^2}{\varepsilon_m} = 2\sum_{m=1}^{3} \frac{J_m}{\varepsilon_m}, \quad \underline{\xi} = \mathrm{V}^{-1}\,\underline{\zeta}$$

Eigen-vectors provide information on β- and dispersion functions :

$$\beta_{xm} = \left|(\underline{v}_{2m})_1\right|^2, \quad \beta_{ym} = \left|(\underline{v}_{2m})_3\right|^2, \quad \beta_{sm} = \left|(\underline{v}_{2m})_5\right|^2, \quad m = 1, 2, 3$$

$$D_x = \frac{x}{\delta} = \frac{V_{16}V_{55} - V_{15}V_{56}}{V_{66}V_{55} - V_{65}V_{56}}, \quad D_y = \frac{y}{\delta} = \frac{V_{36}V_{55} - V_{35}V_{56}}{V_{66}V_{55} - V_{65}V_{56}}.$$

Exponential Fit

If the optics is known – and therefore matrix V of eigen-vectors – we can find action variables of particles:

$$J_m = \frac{1}{2}(\xi_{2m-1}^2 + \xi_{2m}^2), \quad \underline{\xi} = \mathrm{V}^{-1}\underline{\zeta}$$

Ignoring the actual distribution in canonical angles we may look for distribution in the form

$$F = \frac{1}{(2\pi)^3 \varepsilon_1 \varepsilon_2 \varepsilon_3} \exp[-\sum_{m=1}^{3} J_m / \varepsilon_m] = F_1 F_2 F_3$$

The fitting is reduced to 3 one-dimensional exponential fits!

Actually it is better to fit the integrated distribution function:

$$f_m(J_m) = \int_0^{2\pi} d\varphi \int_0^{J_m} F_m(x)dx = 1 - \exp[-J_m / \varepsilon_m]$$

Exponential Fit (cont'd)

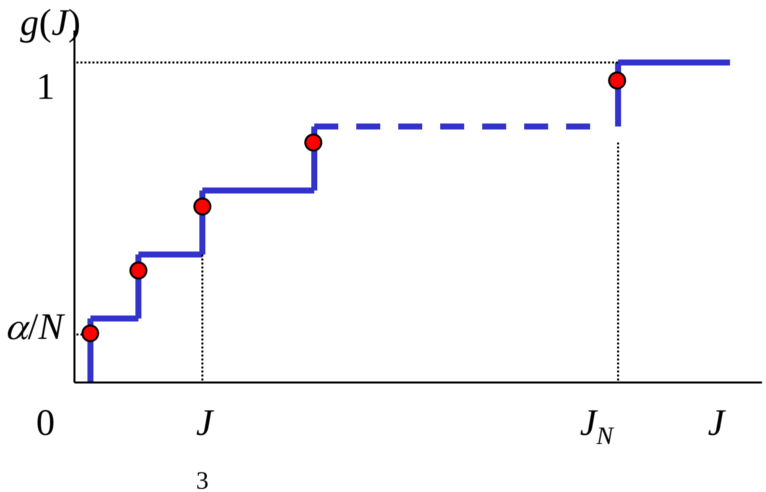
The corresponding part of the Klimontovich distribution (integrated over all other variables)

$$G(J) = \frac{1}{N}\sum_{k=1}^{N}\delta(J - J_k) \quad \rightarrow \quad g(J) = \int_0^J G(x)dx = \frac{1}{N}\sum_{k=1}^{N}\theta(J - J_k)$$

where θ(x) is an asymmetric Heaviside step-function

$$\theta(x) = \begin{cases} 0, & x < 0 \\ 0 < \alpha < 1, & x = 0 \\ 1, & x > 0 \end{cases}$$

Parameter α is empirically adjusted (α=0.1 is slightly better than 1 or 0)



Exponential Fit (cont'd)

Now let us numerate particles in the order of increasing J_k

take $\log[1-g(J)]$ at all $J=J_k$ and equiate it with $\log[1-f(J)]=-J/\varepsilon$

Taking simple average over all particles we get

$$\frac{1}{\varepsilon}=-\frac{1}{N}\sum_{k=1}^{N}\frac{1}{J_k}\log[1-\frac{k-1+\alpha}{N}]$$

The only (complicated) thing to do is to re-order the particles!

This formula gives a precise result in absence of halo, but provides only moderate (~1/J) suppression of tails.